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ON THE DIMENSIONS OF VECTOR SPACES CONCERNING HOLOMORPHIC VECTOR BUNDLES OVER ELLIPTIC ORBITS, IV

NOBUTAKA BOUMUKI

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Abstract. In this paper, we study the complex vector space of holomorphic cross-sections of a homogeneous holomorphic vector bundle over an elliptic adjoint orbit, associated with an irreducible representation. The main purpose is to give a necessary and sufficient condition for the vector space to be infinite-dimensional.

1. Introduction

In general, given a connected real semisimple Lie group G, any homogeneous pseudo-Kähler manifold G/L of G is an elliptic (adjoint) orbit of G and vice versa. Suppose that G acts on G/L effectively and $\dim_{\mathbb{C}} G/L \neq 0$. Then, the center $Z(G)$ of G is trivial and G/L is G-equivariant biholomorphic to a domain in some complex flag manifold $G_{\mathbb{C}}/Q^-$ via the mapping $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$. cf. Dorfmeister-Guan [7, 8]. From a finite-dimensional complex vector space V and a holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$, let us construct the homogeneous holomorphic vector bundle $G_{\mathbb{C}} \times_\rho \mathsf{V}$ over $G_{\mathbb{C}}/Q^-$ associated with ρ. Identifying G/L with the image $\iota(G/L)$ via ι, one can consider its restriction $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ to the domain $G/L = \iota(G/L) \subset G_{\mathbb{C}}/Q^-$, and then obtain the complex vector spaces $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$ and $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$ of holomorphic cross-sections of the bundles $G_{\mathbb{C}} \times_\rho \mathsf{V}$ and $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$, respectively. Our concern is the dimension of this vector space $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$.

$$\begin{array}{ccc} \iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V}) & & G_{\mathbb{C}} \times_\rho \mathsf{V} \\ \downarrow & & \downarrow \\ G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^- \end{array}$$

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As a side note, the same supposition enables us to decompose G into the direct product of some connected real *simple* Lie groups G_a with trivial centers, and decompose G/L into the direct product of some homogeneous pseudo-Kähler manifolds G_a/L_a of G_a such that G_a act on G_a/L_a effectively and $\dim_{\mathbb{C}} G_a/L_a \neq 0$,

$$G/L = G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n;$$

moreover, each G_a/L_a is G_a-equivariant biholomorphic to a domain in some complex flag manifold $(G_a)_{\mathbb{C}}/Q_a^-$, $1 \leq a \leq n$. This paper is a sequel to the papers [3, 4, 5]. Theorem 3.3 in [5] and its contraposition, with Remark 1.4 in [5], imply that for the direct product $G/L = G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$ above, each factor G_a/L_a is compact Kähler, non-compact Kähler or non-Kähler (pseudo-Kähler), and

(I) $\dim_{\mathbb{C}} \mathcal{O}(G/L) = 1$ (i.e., all holomorphic functions on G/L are constant) if and only if $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) < \infty$ for all finite-dimensional complex vector spaces V and holomorphic homomorphisms $\rho : Q^- \to GL(\mathsf{V})$ if and only if $\dim_{\mathbb{C}} \mathcal{O}(G_a/L_a) = 1$ for all $1 \leq a \leq n$ if and only if the direct product never includes any non-compact Kähler factors at all,

(II) $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ if and only if

$$\text{either } \dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = 0 \text{ or } \dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty \text{ holds}$$

for each finite-dimensional complex vector space V and holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$ if and only if $\dim_{\mathbb{C}} \mathcal{O}(G_b/L_b) \neq 1$ for some $1 \leq b \leq n$ if and only if the direct product includes at least one non-compact Kähler factor G_b/L_b.

These are main results in the papers. Remark that Huckleberry [10] and Kollár [14] study the dimension $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho)$, but they do not mention the equivalent condition in (II).

Now, let us suppose the representation $\rho : Q^- \to GL(\mathsf{V})$ to be irreducible. Then Lemma 2.16 in [4] implies that in the case (I) above,

$$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho) < \infty$$

and the induced representation $\tilde{\varrho}$ of the semisimple Lie group $G_{\mathbb{C}}$ on the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$ is irreducible; therefore, we completely find the dimension $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho)$ in the case (I). Indeed, one can derive a necessary and sufficient condition for the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$ to be not equal to $\{0\}$ from the Borel-Weil theorem and holomorphic induction in stages (cf. Proposition 2.16). When $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho) \neq \{0\}$, one can take the highest weight ϖ of $\tilde{\varrho}_* : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$ with respect to a positive system $\triangle^+$ of roots and obtain

$$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \frac{\prod_{\alpha \in \triangle^+} \langle \alpha, \varpi + \delta \rangle}{\prod_{\alpha \in \triangle^+} \langle \alpha, \delta \rangle}$$

from Weyl's dimension formula ($\delta = (1/2) \sum_{\alpha \in \triangle^+} \alpha$). In contrast, no one has yet given a necessary and sufficient condition for the vector space $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$ in the case (II) to be infinite-dimensional.

The main purpose of this paper is to establish Theorem 3.18 which, together with Propositions 2.16 and 2.21-(ii), supplies us with a necessary and sufficient condition for

$$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty$$

in the case where $\rho : Q^- \to GL(\mathsf{V})$ is irreducible and $G/L = G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$ includes at least one non-compact Kähler factor. For example, Theorem 3.18, Proposition 2.16 and Weyl's dimension formula enable us to deduce

- $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m}) = 0$, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = 0$ and $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m}) = 0$ if $n < 0$ and any m,
- $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m}) = n + 1$, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = \infty$ and $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m}) = 0$ if $n \geq 0$ and $m < 0$,
- $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m}) = n + 1$, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = \infty$ and $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m}) = (n + 1)(m + 1)(n + m + 2)/2$ if $n, m \geq 0$,

which shows that $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = \infty$ if and only if $n \geq 0$ (and any m), where $G/L = SU(2,1)/S(U(1) \times U(1) \times U(1))$, $\mathsf{V} = \mathbb{C}$ and $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$ is a complex flag manifold embedded into $\iota(G/L)$ (see Example 3.20 for detail).

This paper consists of three sections. In §2 we fix our setting and confirm Proposition 2.16 which gives a necessary and sufficient condition for $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho) \neq 0$ in the case where $\rho : Q^- \to GL(\mathsf{V})$ is irreducible. In addition, we prove that for $G/L = G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$,

$$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0 \text{ if and only if } \dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0 \text{ for all } 1 \leq a \leq n$$

whenever $\rho : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V})$ is the external tensor product $\rho_1 \boxtimes \rho_2 \boxtimes \cdots \boxtimes \rho_n : Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V}_1 \otimes \mathsf{V}_2 \otimes \cdots \otimes \mathsf{V}_n)$ of some representations $\rho_a : Q_a^- \to GL(\mathsf{V}_a)$ (see Proposition 2.20), and also prove that an arbitrary irreducible representation $\rho : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V})$ is isomorphic to the external tensor product of some irreducible representations $\rho_a : Q_a^- \to GL(\mathsf{V}_a)$ (see Proposition 2.21-(ii)). Finally in §3 we mainly deal with the case where G is a connected real simple Lie group of Hermitian type and G/L is a non-compact homogeneous Kähler manifold, take a new complex flag manifold $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$ into consideration, and demonstrate that $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ if and only if $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) \neq 0$ (see Proposition 3.17). Collecting Propositions 2.20, 3.6 and 3.17, one can accomplish the main purpose.

Contents

2. Preliminaries

In this section, we recall the definitions of elliptic element and elliptic (adjoint) orbit, fix our setting, and deduce Propositions 2.16, 2.20 and 2.21 from some known facts.

Notation. Throughout this paper, for a Lie group G we denote its Lie algebra by the corresponding Fraktur small letter $\mathfrak{g}$ and use the following notation:

(n1) $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural numbers, real numbers, and complex numbers, respectively, where $\mathbb{N}$ does not contain the zero,
(n2) $\mathbb{C}^* := \mathbb{C} - \{0\}$,
(n3) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,
(n4) $GL(\mathsf{V})$, $\mathfrak{gl}(\mathsf{V})$: the general linear group, and linear Lie algebra on a complex vector space V, respectively,
(n5) $Z(G)$: the center of G,
(n6) e : the unit element of G,
(n7) Ad, ad : the adjoint representations of G and $\mathfrak{g}$, respectively,
(n8) $C_G(T) := \{g \in G \mid \operatorname{Ad} g(T) = T\}$ for an element $T \in \mathfrak{g}$,
(n9) $N_G(\mathfrak{m}) := \{g \in G \mid \operatorname{Ad} g(\mathfrak{m}) \subset \mathfrak{m}\}$ for a vector subspace $\mathfrak{m} \subset \mathfrak{g}$,
(n10) σ_* : the differential homomorphism of a Lie group homomorphism σ,
(n11) $f|_S$: the restriction of a mapping $f : X \to Y$ to a subset $S \subset X$,
(n12) id_S : the identity mapping on a set S,
(n13) $i := \sqrt{-1}$,
(n14) $\mathcal{O}(M)$: the complex vector space of holomorphic functions on a complex manifold M.

2.1. **Definition of elliptic orbit.** Here is the definition of elliptic (adjoint) orbit.

Definition 2.1 (cf. Kobayashi [12])**.** Let $\mathfrak{g}$ be a real semisimple Lie algebra, and G a connected Lie group with Lie algebra $\mathfrak{g}$.

(i) An element $T \in \mathfrak{g}$ is said to be *elliptic*, if the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto [T, X]$, is semisimple and all the eigenvalues of $\operatorname{ad} T$ are purely imaginary.
(ii) The adjoint orbit $\operatorname{Ad} G(T) = G/C_G(T)$ of G through an elliptic element $T \in \mathfrak{g}$ is called an *elliptic adjoint orbit* or an *elliptic orbit* for short.

Needless to say, the dimension of the manifold $G/C_G(T)$ is zero if and only if $T = 0$.

2.2. **Setting on an elliptic orbit and a complex structure.** Let us fix our setting as follows:

- $G_{\mathbb{C}}$ is a connected complex semisimple Lie group,
- G is a connected closed subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$,
- T is a non-zero, elliptic element of $\mathfrak{g}$,
- $L := C_G(T)$, $L_{\mathbb{C}} := C_{G_{\mathbb{C}}}(T)$, $\mathfrak{g}^{\lambda} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\}$ for a $\lambda \in \mathbb{R}$,
- $\mathfrak{u}^+ := \bigoplus_{\lambda>0} \mathfrak{g}^{\lambda}$, $\mathfrak{u}^- := \bigoplus_{\lambda>0} \mathfrak{g}^{-\lambda}$,
- $U^s := \exp \mathfrak{u}^s$, $Q^s := N_{G_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^s)$ for each $s = \pm$,
- $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ containing T, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$,
- $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$,
- K and G_u are the maximal compact subgroups of G and $G_{\mathbb{C}}$ whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{g}_u$, respectively,
- $\overline{\theta}$ is the (anti-holomorphic) Cartan involution of $G_{\mathbb{C}}$ whose fixed point set coincides with G_u,
- $L_u := C_{G_u}(T)$,
- $\mathfrak{t}$ is a maximal torus of $\mathfrak{k}$ containing T,
- $\mathfrak{h} := \{X \in \mathfrak{g} \mid [X, H] = 0 \text{ for all } H \in \mathfrak{t}\}$, $\mathfrak{a} := \mathfrak{p} \cap \mathfrak{h}$, $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{t} \oplus \mathfrak{a}$,
- $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the (non-zero) root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$, where $\mathfrak{h}_{\mathbb{C}}$ is the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{h}$,
- $\mathfrak{g}_{\alpha}$ is the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for a root $\alpha \in \triangle$,
- $\triangle^+$ is a positive system of roots in $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ satisfying $\beta(-iT) \geq 0$ for all $\beta \in \triangle^+$,
- $\triangle(T, 0) := \{\gamma \in \triangle \mid \gamma(T) = 0\}$.

Here we refer to Varadarajan [15, p.280] for the definition of positive system of roots. Note that both K and G_u are connected, that the restriction $\theta := \overline{\theta}|_G$ is the Cartan involution of G whose fixed point set coincides with K, and that $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$ because $\theta_*(\mathfrak{l}) \subset \mathfrak{l}$ comes from $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(T)$ and $\theta_*(T) = T$. In addition, we note $\mathfrak{h}_{\mathbb{R}} = \{A \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(A) \in \mathbb{R} \text{ for all } \alpha \in \triangle\}$,

$$\begin{aligned} &\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}^+ \oplus \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^- = \mathfrak{h}_{\mathbb{C}} \oplus \textstyle\bigoplus_{\beta \in \triangle^+} \mathfrak{g}_{\beta} \oplus \mathfrak{g}_{-\beta}, \\ &\mathfrak{u}^+ = \textstyle\bigoplus_{\beta \in \triangle^+ - \triangle(T,0)} \mathfrak{g}_{\beta},\ \mathfrak{l}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\gamma \in \triangle(T,0)} \mathfrak{g}_{\gamma},\ \mathfrak{u}^- = \bigoplus_{\beta \in \triangle^+ - \triangle(T,0)} \mathfrak{g}_{-\beta}. \end{aligned} \tag{2.2}$$

In the setting above, one can prove

Lemma 2.3. *The following nine items hold. Here $s = \pm$.*

(i) *Both of the closed subgroups $L \subset G$ and $L_u \subset G_u$ are connected and reductive. In particular, L_u is compact.*
(ii) *The closed complex (Lie) subgroup $L_{\mathbb{C}} \subset G_{\mathbb{C}}$ is connected and reductive.*
(iii) $\overline{\theta}(L_{\mathbb{C}}) = L_{\mathbb{C}}$, $\overline{\theta}(U^s) = U^{-s}$.
(iv) *U^s is a simply connected, closed complex nilpotent subgroup of $G_{\mathbb{C}}$ whose Lie algebra coincides with $\mathfrak{u}^s$, and $\exp : \mathfrak{u}^s \to U^s$ is biholomorphic.*
(v) *Q^s is a connected, closed complex parabolic subgroup of $G_{\mathbb{C}}$ such that $Q^s = L_{\mathbb{C}} \ltimes U^s$ (semidirect) and $\mathfrak{q}^s = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^s = \bigoplus_{\mu \geq 0} \mathfrak{g}^{s\mu}$.*

(vi) *The product mapping* $U^{-s} \times Q^s \ni (u,q) \mapsto uq \in G_{\mathbb{C}}$ *is a biholomorphism of* $U^{-s} \times Q^s$ *onto a domain in* $G_{\mathbb{C}}$.
(vii) $\iota_u : G_u/L_u \to G_{\mathbb{C}}/Q^-$, $g_uL_u \mapsto g_uQ^-$, *is a* G_u*-equivariant real analytic diffeomorphism of* G_u/L_u *onto* $G_{\mathbb{C}}/Q^-$.
(viii) $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, *is a* G*-equivariant real analytic diffeomorphism of* G/L *onto a simply connected domain in* $G_{\mathbb{C}}/Q^-$.
(ix) GQ^- *is a domain in* $G_{\mathbb{C}}$.

Proof. e.g. Lemma 7.3.3, Lemma 8.0.1, Proposition 8.2.1, Lemma 11.1.2 in [6, p.71, p.75, p.78, p.117]. □

In general, the elliptic orbit G/L admits several kinds of G-invariant complex structures (cf. Example 10.5.1 in [6, p.115]). Since the complex flag manifold $G_{\mathbb{C}}/Q^-$ is a complex homogeneous space, it naturally admits a $G_{\mathbb{C}}$-invariant complex structure. Hence, one can induce the G-invariant complex structure J on the orbit G/L from $G_{\mathbb{C}}/Q^-$ by identifying G/L with the domain $\iota(G/L) \subset G_{\mathbb{C}}/Q^-$ via the mapping $\iota : gL \mapsto gQ^-$. In this way, we consider $G/L = (G/L, J)$ as a homogeneous complex manifold of G.

$$G/L \xrightarrow{\ \iota\ } G_{\mathbb{C}}/Q^-$$

2.3. **Setting on holomorphic vector bundles.** We set vector bundles $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and $G_{\mathbb{C}} \times_\rho \mathsf{V}$, and set a continuous representation ϱ (resp. $\tilde{\varrho}$) of the Lie group G (resp. $G_{\mathbb{C}}$). The setting of §§2.2 remains valid here.

Let V be a finite-dimensional complex vector space, and let $\rho : Q^- \to GL(\mathsf{V})$, $q \mapsto \rho(q)$, be a holomorphic homomorphism. Denote by $G_{\mathbb{C}} \times_\rho \mathsf{V}$ the homogeneous holomorphic vector bundle over the complex flag manifold $G_{\mathbb{C}}/Q^-$ associated with ρ, and by $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ its restriction to the domain $G/L = \iota(G/L) \subset G_{\mathbb{C}}/Q^-$.[1] Then, we consider

$$\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho) := \left\{ h : G_{\mathbb{C}} \to \mathsf{V} \;\middle|\; \begin{array}{l} (1)\ h \text{ is holomorphic,} \\ (2)\ h(aq) = \rho(q)^{-1}\big(h(a)\big) \\ \qquad \text{for all } (a,q) \in G_{\mathbb{C}} \times Q^- \end{array} \right\}, \tag{2.4}$$

$$\mathcal{V}_{G/L}(\mathsf{V},\rho) := \left\{ \psi : GQ^- \to \mathsf{V} \;\middle|\; \begin{array}{l} (1)\ \psi \text{ is holomorphic,} \\ (2)\ \psi(xq) = \rho(q)^{-1}\big(\psi(x)\big) \\ \qquad \text{for all } (x,q) \in GQ^- \times Q^- \end{array} \right\} \tag{2.5}$$

as the complex vector spaces of holomorphic cross-sections of the bundles $G_{\mathbb{C}} \times_\rho \mathsf{V}$ and $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$, respectively, and sometimes express them as $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and $\mathcal{V}_{G/L}$, respectively. Note here, GQ^- is an open submanifold of the complex manifold $G_{\mathbb{C}}$.

$$\begin{array}{ccc} \iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V}) & & G_{\mathbb{C}} \times_\rho \mathsf{V} \\ \downarrow & & \downarrow \\ G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^- \end{array}$$

[1] By abuse of language we say that this $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ is a *homogeneous holomorphic vector bundle* over the elliptic orbit G/L.

Now, let us define a homomorphism $\varrho : G \to GL(\mathcal{V}_{G/L})$, $g \mapsto \varrho(g)$ by

$$\big(\varrho(g)\psi\big)(x) := \psi(g^{-1}x) \text{ for } \psi \in \mathcal{V}_{G/L} \text{ and } x \in GQ^-, \tag{2.6}$$

and provide the vector space $\mathcal{V}_{G/L}$ with a metric topology as follows. Since the Lie group $G_{\mathbb{C}}$ is connected, it satisfies the second countability axiom. Hence the domain $GQ^- \subset G_{\mathbb{C}}$ also satisfies the same axiom, and is a locally compact Hausdorff space. Therefore there exist non-empty open subsets $O_n \subset GQ^-$ such that

- (d1) $GQ^- = \bigcup_{n=1}^{\infty} O_n$ (countable union),
- (d2) the closure $\overline{O_n}$ in GQ^- is compact for each $n \in \mathbb{N}$.

Setting $E_n := \overline{O_n}$ for an $n \in \mathbb{N}$, we construct a metric d on $\mathcal{V}_{G/L}$ from $d_{E_n}(\psi_1, \psi_2) := \sup\{\|\psi_1(y) - \psi_2(y)\| : y \in E_n\}$ and

$$d(\psi_1, \psi_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_{E_n}(\psi_1, \psi_2)}{1 + d_{E_n}(\psi_1, \psi_2)} \tag{2.7}$$

for $\psi_1, \psi_2 \in \mathcal{V}_{G/L} = \mathcal{V}_{G/L}(\mathsf{V}, \rho)$, where $\|\cdot\|$ is an arbitrary norm on the vector space V. This d is called the *Fréchet metric* on $\mathcal{V}_{G/L}$. With respect to the Fréchet metric d in (2.7), one knows

Lemma 2.8. *$\mathcal{V}_{G/L} = (\mathcal{V}_{G/L}, d)$ is a complex Fréchet space, and the mapping $\pi_\varrho : G \times \mathcal{V}_{G/L} \to \mathcal{V}_{G/L}$, $(g, \psi) \mapsto \varrho(g)\psi$, is continuous, i.e., the ϱ in* (2.6) *is a continuous representation of the Lie group G on the Fréchet space $\mathcal{V}_{G/L}$.*

Proof. Refer to Lemma 2.6.4 in [3, p.230] and references therein, for example. □

In a way similar to the way stated above, we define a homomorphism $\tilde{\varrho} : G_{\mathbb{C}} \to GL(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$, $a \mapsto \tilde{\varrho}(a)$ by

$$\big(\tilde{\varrho}(a)h\big)(a_1) := h(a^{-1}a_1) \text{ for } h \in \mathcal{V}_{G_{\mathbb{C}}/Q^-} \text{ and } a_1 \in G_{\mathbb{C}}, \tag{2.9}$$

and provide the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ with a metric topology; then $\tilde{\varrho}$ is a continuous representation of $G_{\mathbb{C}}$ on $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$. This continuous representation $\tilde{\varrho}$ is called the *induced representation* of $G_{\mathbb{C}}$ by $\rho : Q^- \to GL(\mathsf{V})$, in some papers.

Remark 2.10. Here are comments on the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-} = \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$.

- (i) Lemma 2.3-(vii) implies that $G_{\mathbb{C}}/Q^-$ is a connected compact complex manifold, so that
$$\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho) < \infty$$
for all finite-dimensional complex vector spaces V and holomorphic homomorphisms $\rho : Q^- \to GL(\mathsf{V})$. cf. Kodaira [13, Corollary, p.161].
- (ii) It follows from $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$ that $GL(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$ is a complex Lie group, and then the homomorphism $\tilde{\varrho} : G_{\mathbb{C}} \to GL(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$ is holomorphic since the mapping $\pi_{\tilde{\varrho}} : G_{\mathbb{C}} \times \mathcal{V}_{G_{\mathbb{C}}/Q^-} \to \mathcal{V}_{G_{\mathbb{C}}/Q^-}$, $(a, h) \mapsto \tilde{\varrho}(a)h$, is continuous. Conversely, given a holomorphic homomorphism $\tilde{\varsigma} : G_{\mathbb{C}} \to GL(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$, the mapping $\pi_{\tilde{\varsigma}} : G_{\mathbb{C}} \times \mathcal{V}_{G_{\mathbb{C}}/Q^-} \to \mathcal{V}_{G_{\mathbb{C}}/Q^-}$, $(a, h) \mapsto \tilde{\varsigma}(a)h$, is always continuous because it follows from $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$ that our topology for $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ is the same as the topology determined by an arbitrary norm on $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$.

2.4. **Known facts.** Obeying the setting of §§2.3, we enumerate some known facts which will play roles later.

Proposition 2.11 (cf. Ise [11, Lemma 1, p.222])**.** *Let* W *be a finite-dimensional complex vector space and* $\rho : Q^- = L_{\mathbb{C}} \ltimes U^- \to GL(\mathsf{W})$*,* $q \mapsto \rho(q)$*, a holomorphic homomorphism. Then, the following two conditions are equivalent*:

(A) *The representation* $\rho : Q^- \to GL(\mathsf{W})$ *is completely reducible.*
(B) $\rho(u) = \mathrm{id}_{\mathsf{W}}$ *for all* $u \in U^-$*.*

Lemma 2.12 (cf. Lemma 2.5.5 in [3, p.228])**.** *Let* V *be a finite-dimensional complex vector space and* $\rho : Q^- = L_{\mathbb{C}} \ltimes U^- \to GL(\mathsf{V})$*,* $q \mapsto \rho(q)$*, a holomorphic homomorphism. Suppose that*

$$\text{(s0) } \mathsf{V} \neq \{0\}, \quad \text{(s1) } \rho : Q^- \to GL(\mathsf{V}) \textit{ is irreducible.}$$

Then, (i) $\rho_* : \mathfrak{l}_{\mathbb{C}} \to \mathfrak{gl}(\mathsf{V})$ *is irreducible, and* (ii) *there exists a unique linear function* $\eta : \mathfrak{z}(\mathfrak{l}_{\mathbb{C}}) \to \mathbb{C}$ *such that*

$$\rho_*(A) = \eta(A)\,\mathrm{id}_{\mathsf{V}} \textit{ for all } A \in \mathfrak{z}(\mathfrak{l}_{\mathbb{C}});$$

moreover, $\eta(iH) \in \mathbb{R}$ *for all* $H \in \mathfrak{k} \cap \mathfrak{z}(\mathfrak{l}_{\mathbb{C}})$*. Here* $\mathfrak{z}(\mathfrak{l}_{\mathbb{C}})$ *denotes the center of the Lie algebra* $\mathfrak{l}_{\mathbb{C}}$*.*

Lemma 2.13 (cf. Corollary 3.2.2 in [3, p.241])**.** *Let* V *be a finite-dimensional complex vector space and* $\rho : Q^- \to GL(\mathsf{V})$ *a holomorphic homomorphism. Suppose that*

$$\text{(s1) } \rho : Q^- \to GL(\mathsf{V}) \textit{ is irreducible.}$$

Then, the continuous representation ϱ *of the Lie group* G *on the vector space* $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$ *is indecomposable. Here we refer to* (2.6) *and* (2.5) *for* ϱ *and* $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$*, respectively.*

Lemma 2.14 (cf. Lemma 2.16 in [4, p.9])**.** *Let* V *be a finite-dimensional complex vector space and* $\rho : Q^- \to GL(\mathsf{V})$ *a holomorphic homomorphism. Suppose that*

(s1) $\rho : Q^- \to GL(\mathsf{V})$ *is irreducible, and*
(I) $\dim_{\mathbb{C}} \mathcal{V}_{G/L} < \infty$*.*

Then, the following two items hold:

(II) ϱ *is an irreducible representation of* G *on* $\mathcal{V}_{G/L} = \mathcal{V}_{G/L}(\mathsf{V}, \rho)$*.*
(III) *The representation module* $(\mathcal{V}_{G_{\mathbb{C}}/Q^-}, \tilde{\varrho})$ *is* G*-equivariant isomorphic to* $(\mathcal{V}_{G/L}, \varrho)$ *via* $\mathcal{F} : h \mapsto h|_{GQ^-}$*.*

Here we refer to (2.5)*,* (2.6)*,* (2.4) *and* (2.9) *for* $\mathcal{V}_{G/L}$*,* ϱ*,* $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ *and* $\tilde{\varrho}$*, respectively.*

2.5. **A condition for** $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho) \neq 0$ **(Proposition 2.16).** The setting here is the same as that of §§2.3.

In §1 we have stated that one can derive a necessary and sufficient condition for the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho)$ to be not equal to $\{0\}$ from the Borel-Weil theorem and holomorphic induction in stages. For the sake of completeness, we will confirm Proposition 2.16 below.

Identifying the elliptic orbit $G_u/L_u = G_u/C_{G_u}(T)$ with the complex flag manifold $G_{\mathbb{C}}/Q^-$ via the mapping ι_u in Lemma 2.3-(vii), we consider G_u/L_u as a homogeneous complex manifold of the compact semisimple Lie group G_u, and set the complex vector space $\mathcal{V}_{G_u/L_u}(\mathsf{V},\rho)$ in a similar way to (2.5).

$$\begin{array}{ccc} \iota_u^{\sharp}(G_{\mathbb{C}} \times_\rho \mathsf{V}) & & G_{\mathbb{C}} \times_\rho \mathsf{V} \\ \downarrow & & \downarrow \\ G_u/L_u & \xrightarrow{\ \iota_u\ } & G_{\mathbb{C}}/Q^- \end{array}$$

Then, one has

$$\begin{aligned} \mathcal{V}_{G_u/L_u}(\mathsf{V},\rho) &= \left\{ h : G_u Q^- \to \mathsf{V} \;\middle|\; \begin{array}{l} (1)\ h \text{ is holomorphic,} \\ (2)\ h(xq) = \rho(q)^{-1}\big(h(x)\big) \\ \qquad \text{for all } (x,q) \in G_u Q^- \times Q^- \end{array} \right\} \\ &\overset{(2.4)}{=} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho) \end{aligned}$$

because $G_u Q^- = G_{\mathbb{C}}$ comes from Lemma 2.3-(vii). By $\mathcal{V}_{G_u/L_u}(\mathsf{V},\rho) = \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho)$ and Lemma 2.13 we conclude

Corollary 2.15. *Let* V *be a finite-dimensional complex vector space and* $\rho : Q^- \to GL(\mathsf{V})$ *a holomorphic homomorphism. Suppose* $\rho : Q^- \to GL(\mathsf{V})$ *to be irreducible. Then, the restriction* $\tilde{\varrho}|_{G_u}$ *is an irreducible (continuous) representation of the compact subgroup* $G_u \subset G_{\mathbb{C}}$ *on the vector space* $\mathcal{V}_{G_{\mathbb{C}}/Q^-} = \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho)$*; in particular, the induced representation* $\tilde{\varrho}$ *of* $G_{\mathbb{C}}$ *on* $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ *is irreducible.*

Proof. Since $\tilde{\varrho}$ is a continuous representation of $G_{\mathbb{C}}$ on $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$, it follows that the mapping $G_u \times \mathcal{V}_{G_{\mathbb{C}}/Q^-} \ni (g_u, h) \mapsto \tilde{\varrho}(g_u)h \in \mathcal{V}_{G_{\mathbb{C}}/Q^-}$ is continuous, so that $\tilde{\varrho}|_{G_u}$ is a continuous representation of G_u on $\mathcal{V}_{G_u/L_u} = \mathcal{V}_{G_{\mathbb{C}}/Q^-}$. Then, since Lemma 2.13 still holds even if we substitute $\tilde{\varrho}|_{G_u}$, G_u and $\mathcal{V}_{G_u/L_u}$ for ϱ, G and $\mathcal{V}_{G/L}$, respectively, one can assert that the representation $\tilde{\varrho}|_{G_u}$ of G_u on $\mathcal{V}_{G_u/L_u}$ is indecomposable, and furthermore is irreducible because the Lie group G_u is compact and $\dim_{\mathbb{C}} \mathcal{V}_{G_u/L_u} = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$. □

Taking the $\mathfrak{h}_{\mathbb{C}}$, $\triangle^+$ in §§2.2, we are now in a position to confirm

Proposition 2.16. *Let* V *be a finite-dimensional complex vector space and* $\rho : Q^- \to GL(\mathsf{V})$ *a holomorphic homomorphism. Suppose that* (s1) $\rho : Q^- \to GL(\mathsf{V})$ *is irreducible. Then, the following conditions* (a) *and* (b) *are equivalent for the vector space* $\mathcal{V}_{G_{\mathbb{C}}/Q^-} = \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V},\rho)$ *in* (2.4):

(a) $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} \neq 0$.

(b) *There exist a non-zero* $\mathsf{u} \in \mathsf{V}$ *and a holomorphic homomorphism* $\chi : H_{\mathbb{C}} \to \mathbb{C}^*$, $x \mapsto \chi(x)$, *such that*
  (1) χ_* *is a dominant integral form on* $\mathfrak{h}_{\mathbb{C}}$ *with respect to the positive system* $\triangle^+ \subset \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$*, and*
  (2) $\rho_*(X)\mathsf{u} = \chi_*(X)\mathsf{u}$ *for all* $X \in \mathfrak{h}_{\mathbb{C}}$.

Here $H_{\mathbb{C}}$ *denotes the connected Lie subgroup of the semisimple Lie group* $G_{\mathbb{C}}$ *corresponding to the Cartan subalgebra* $\mathfrak{h}_{\mathbb{C}}$ *of* $\mathfrak{g}_{\mathbb{C}}$.

Proof. (a)⇒(b). We shall prove (a)⇒(b) by taking Remark 2.10 and Corollary 2.15 into consideration. Suppose $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} \neq 0$. Then, since $\tilde{\varrho}_* : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(\mathcal{V}_{G_{\mathbb{C}}/Q^-})$ is irreducible and $\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$, there exist a non-zero $h_0 \in \mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and a unique linear function $\nu_0 : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ such that

$$\text{(i) } \tilde{\varrho}_*(\mathfrak{g}_\beta)h_0 = \{0\} \text{ for all } \beta \in \triangle^+, \quad \text{(ii) } \tilde{\varrho}_*(X)h_0 = \nu_0(X)h_0 \text{ for all } X \in \mathfrak{h}_{\mathbb{C}}.$$

The Cartan-Weyl theorem tells that this ν_0 is a dominant integral form on $\mathfrak{h}_{\mathbb{C}}$ with respect to $\triangle^+$. By virtue of $h_0 \neq 0$ and (ii), we can set a holomorphic homomorphism $\chi : H_{\mathbb{C}} \to \mathbb{C}^*$, $x \mapsto \chi(x)$ as

$$\tilde{\varrho}(x)h_0 = \chi(x)h_0 \text{ for an } x \in H_{\mathbb{C}},$$

and deduce that

$$\chi_*(X) = \nu_0(X) \text{ for all } X \in \mathfrak{h}_{\mathbb{C}}.$$

In view of (i) and (2.2) we see that $\tilde{\varrho}_*(Y)h_0 = 0$ for all $Y \in \mathfrak{u}^+$, so that h_0 is the constant mapping with the value $h_0(e)$ on U^+ because of Lemma 2.3-(iv). If $h_0(e) = 0$, then it follows from $h_0 \in \mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and (2.4)-(2) that $h_0 = 0$ on U^+Q^-, so that Lemma 2.3-(vi) and the theorem of identity lead to $h_0 = 0$ on the whole $G_{\mathbb{C}}$, which contradicts $h_0 \neq 0$. For this reason

$$h_0(e) \in \mathsf{V} - \{0\}.$$

Putting $\mathsf{u} := h_0(e)$, one can conclude (b); indeed, (2.4)-(2), (2.9) and (ii) yield $\rho_*(X)\mathsf{u} = \rho_*(X)(h_0(e)) = (\tilde{\varrho}_*(X)h_0)(e) = (\nu_0(X)h_0)(e) = (\chi_*(X)h_0)(e) = \chi_*(X)\mathsf{u}$ for all $X \in \mathfrak{h}_{\mathbb{C}}$.

(b)⇒(a). Suppose that there exist a non-zero $\mathsf{u}_1 \in \mathsf{V}$ and a holomorphic homomorphism $\chi_1 : H_{\mathbb{C}} \to \mathbb{C}^*$, $x \mapsto \chi_1(x)$, such that

(si) χ_{1*} is a dominant integral form on $\mathfrak{h}_{\mathbb{C}}$ with respect to $\triangle^+$, and
(sii) $\rho_*(X)\mathsf{u}_1 = \chi_{1*}(X)\mathsf{u}_1$ for all $X \in \mathfrak{h}_{\mathbb{C}}$.

By the Borel-Weil theorem and holomorphic induction in stages, we will prove

$$\text{ⓐ} \qquad \mathcal{V}_{G_{\mathbb{C}}/Q^-} \neq \{0\}.$$

The arguments below will be pretty much the same as those in the proof of Theorem 3.0.1 in [3, p.244].

Our first aim is to set a complex Borel subgroup B^- of $G_{\mathbb{C}}$ and a holomorphic homomorphism $\chi_2 : B^- \to \mathbb{C}^*$. Let R be a regular element in $\mathfrak{h}_{\mathbb{C}}$ satisfying $\alpha(R) > 0$ for all $\alpha \in \triangle^+$, and let

$$\mathfrak{n}^- := \textstyle\bigoplus_{\beta\in\triangle^+} \mathfrak{g}_{-\beta}, \quad N^- := \exp \mathfrak{n}^-, \quad B^- := N_{G_{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}^-).$$

Then, $H_{\mathbb{C}}$ accords with $C_{G_{\mathbb{C}}}(iR)$, and Lemma 2.3-(ii), (iv), (v) still holds even if we substitute $H_{\mathbb{C}}$, $\mathfrak{n}^-$, N^- and B^- for $L_{\mathbb{C}}$, $\mathfrak{u}^s$, U^s and Q^s, respectively. Here $H_{\mathbb{C}} \subset L_{\mathbb{C}}$, $U^- \subset N^-$ and $B^- \subset Q^-$, and the converse inclusions also hold whenever $B^- = Q^-$. Now, one can define a holomorphic homomorphism $\chi_2 : B^- = H_{\mathbb{C}} \ltimes N^- \to \mathbb{C}^*$ by

$$\chi_2(xn) := \chi_1(x) \text{ for a } (x, n) \in H_{\mathbb{C}} \times N^-. \tag{2.17}$$

By use of this χ_2 we define complex vector spaces $\mathcal{L}_{G_\mathbb{C}/B^-}$ and $\mathcal{L}_{Q^-/B^-}$ by

$$\mathcal{L}_{G_\mathbb{C}/B^-} := \left\{ f : G_\mathbb{C} \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ f \text{ is holomorphic,} \\ (2)\ f(ab) = \chi_2(b)^{-1} f(a) \text{ for all } (a,b) \in G_\mathbb{C} \times B^- \end{array} \right\},$$
$$\mathcal{L}_{Q^-/B^-} := \left\{ \zeta : Q^- \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ \zeta \text{ is holomorphic,} \\ (2)\ \zeta(qb) = \chi_2(b)^{-1} \zeta(q) \text{ for all } (q,b) \in Q^- \times B^- \end{array} \right\},$$

respectively, define a homomorphism $\hat{\varrho} : Q^- \to GL(\mathcal{L}_{Q^-/B^-})$, $q \mapsto \hat{\varrho}(q)$ by

$$\big(\hat{\varrho}(q)\zeta\big)(q_1) := \zeta(q^{-1}q_1) \text{ for } \zeta \in \mathcal{L}_{Q^-/B^-} \text{ and } q_1 \in Q^-,$$

and define a complex vector space $(\mathcal{V}_{G_\mathbb{C}/Q^-})'$ by

$$(\mathcal{V}_{G_\mathbb{C}/Q^-})' := \left\{ h : G_\mathbb{C} \to \mathcal{L}_{Q^-/B^-} \;\middle|\; \begin{array}{l} (1)\ h \text{ is holomorphic,} \\ (2)\ h(aq) = \hat{\varrho}(q)^{-1}\big(h(a)\big) \\ \qquad \text{for all } (a,q) \in G_\mathbb{C} \times Q^- \end{array} \right\}.$$

Here we note that the connected complex reductive Lie group $L_\mathbb{C}$ is the locally direct product $S_\mathbb{C} \cdot Z$ of a connected complex semisimple Lie subgroup $S_\mathbb{C} \subset L_\mathbb{C}$ and the identity component Z of the center $Z(L_\mathbb{C})$, that $S_\mathbb{C}/(S_\mathbb{C} \cap B^-)$ is biholomorphic to Q^-/B^- in terms of $Q^- = L_\mathbb{C}U^- = S_\mathbb{C}ZU^-$, $ZU^- \subset B^-$, and that

① $$\dim_\mathbb{C} \mathcal{L}_{Q^-/B^-} < \infty$$

because $\mathcal{L}_{Q^-/B^-}$ is the complex vector space of holomorphic cross-sections of a holomorphic line bundle over the complex flag manifold $S_\mathbb{C}/(S_\mathbb{C} \cap B^-) = Q^-/B^-$. In view of (si), (sii), (2.17) and the Borel-Weil theorem (e.g. Akhiezer [2, Theorem, p.114]), one sees that

② $$\mathcal{L}_{G_\mathbb{C}/B^-} \neq \{0\},$$

and that in case of $B^- \neq Q^-$ there exists a linear isomorphism $\mathtt{f} : \mathcal{L}_{Q^-/B^-} \to \mathsf{V}$, $\zeta \mapsto \mathtt{f}(\zeta)$ satisfying

③ $$\mathtt{f} \circ \hat{\varrho}(s) = \rho(s) \circ \mathtt{f} \text{ for all } s \in S_\mathbb{C},$$

where we remark that $\mathcal{L}_{Q^-/B^-} \neq \{0\}$ (resp. $\mathsf{V} \neq \{0\}$) follows from $\mathcal{L}_{G_\mathbb{C}/B^-} \neq \{0\}$ (resp. $\mathsf{u}_1 \in \mathsf{V}$), and that $\rho_* : \mathfrak{s}_\mathbb{C} \to \mathfrak{gl}(\mathsf{V})$ is irreducible (due to Lemma 2.12) and has $\chi_{2*}|_{\mathfrak{s}_\mathbb{C} \cap \mathfrak{h}_\mathbb{C}}$ as its highest weight with respect to $\{\gamma|_{\mathfrak{s}_\mathbb{C} \cap \mathfrak{h}_\mathbb{C}} : \gamma \in \triangle^+ \cap \triangle(T, 0)\}$. If $B^- = Q^-$, then (s1), Proposition 2.11, (sii) and (2.17) imply that $\rho(b)\mathsf{u}_1 = \chi_2(b)\mathsf{u}_1$ for all $b \in B^-$; thus, we conclude ⓐ from ② (because one can get a non-zero $h \in \mathcal{V}_{G_\mathbb{C}/B^-}$ by setting $h(a) := f(a)\mathsf{u}_1$ for $a \in G_\mathbb{C}$, $0 \neq f \in \mathcal{L}_{G_\mathbb{C}/B^-}$). For this reason we suppose $B^- \neq Q^-$ henceforth. In this setting also, we can conclude ⓐ, if one proves the following ⓑ and ⓒ:

ⓑ the vector space $\mathcal{L}_{G_\mathbb{C}/B^-}$ can be embedded into $(\mathcal{V}_{G_\mathbb{C}/Q^-})'$,

ⓒ $\mathtt{f} \circ \hat{\varrho}(q) = \rho(q) \circ \mathtt{f}$ for all $q \in Q^-$

because ⓑ and ② yield $(\mathcal{V}_{G_\mathbb{C}/Q^-})' \neq \{0\}$; moreover, ⓒ and (2.4) imply that the vector space $(\mathcal{V}_{G_\mathbb{C}/Q^-})'$ is isomorphic to $\mathcal{V}_{G_\mathbb{C}/Q^-}$. Consequently, the rest of proof is to confirm these ⓑ and ⓒ. Our second aim is to prove ⓑ. Taking any $f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}$ and $a \in G_{\mathbb{C}}$, we define a holomorphic function $f'(a) : Q^- \to \mathbb{C}$ by

$$f'(a)(q) := f(aq) \text{ for a } q \in Q^-. \tag{2.18}$$

Then, it is immediate from $f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}$ that

$$f'(a) \in \mathcal{L}_{Q^-/B^-}.$$

Hence $f' : a \mapsto f'(a)$ gives rise to a mapping of $G_{\mathbb{C}}$ into $\mathcal{L}_{Q^-/B^-}$ for every $f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}$. We want to show that f' belongs to $(\mathcal{V}_{G_{\mathbb{C}}/Q^-})'$. For any $a \in G_{\mathbb{C}}$ and $q, q_1 \in Q^-$, a direct computation with (2.18) enables us to obtain $f'(aq)(q_1) = f'(a)(qq_1) = \big(\hat{\varrho}(q)^{-1}(f'(a))\big)(q_1)$; hence

④ two elements $f'(aq)$ and $\hat{\varrho}(q)^{-1}\big(f'(a)\big)$ of $\mathcal{L}_{Q^-/B^-}$ are equal.

Let us demonstrate that the $f' : G_{\mathbb{C}} \to \mathcal{L}_{Q^-/B^-}$, $a \mapsto f'(a)$, is holomorphic. By ① there exists a complex basis $\{\zeta_i\}_{i=1}^m \subset \mathcal{L}_{Q^-/B^-}$, and $f'(a)$ is expressed as

$$f'(a) = \alpha^1(a)\zeta_1 + \alpha^2(a)\zeta_2 + \cdots + \alpha^m(a)\zeta_m \quad (a \in G_{\mathbb{C}}),$$

where $\alpha^i : G_{\mathbb{C}} \to \mathbb{C}$. Since the sequence $\zeta_1, \zeta_2, \dots, \zeta_m$ is linearly independent, one can choose m-elements $q_j \in Q^-$ so that the $m \times m$ matrix $(\zeta_i(q_j))_{1 \le i,j \le m}$ is regular. Denote by $(A^{ij})_{1 \le i,j \le m}$ its inverse matrix. Then one has

$$\alpha^i(a) = \textstyle\sum_{k=1}^m f(aq_k)A^{ki} \quad (a \in G_{\mathbb{C}}, 1 \le i \le m)$$

because (2.18) yields $f(aq_k) = f'(a)(q_k) = \sum_{j=1}^m \alpha^j(a)\zeta_j(q_k)$. This and $f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}$ assure that each $\alpha^i : G_{\mathbb{C}} \to \mathbb{C}$ is holomorphic. Accordingly $f' : G_{\mathbb{C}} \to \mathcal{L}_{Q^-/B^-}$, $a \mapsto f'(a)$, is holomorphic; furthermore, it follows from ④ that

$$f' \in (\mathcal{V}_{G_{\mathbb{C}}/Q^-})' \text{ for all } f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}.$$

Here one can get a linear mapping $\mathcal{L}_{G_{\mathbb{C}}/B^-} \ni f \mapsto f' \in (\mathcal{V}_{G_{\mathbb{C}}/Q^-})'$, which is injective; indeed, if $f' = 0$, $f \in \mathcal{L}_{G_{\mathbb{C}}/B^-}$, then $f(a) = f'(a)(e) = 0(e) = 0$ for all $a \in G_{\mathbb{C}}$. For this reason ⓑ holds. Now, our last aim is to prove ⓒ $\mathtt{f} \circ \hat{\varrho}(q) = \rho(q) \circ \mathtt{f}$ for all $q \in Q^-$. Fix an arbitrary $z \in Z$. For any $(s, z_1, u_1) \in S_{\mathbb{C}} \times Z \times U^-$ and $\zeta \in \mathcal{L}_{Q^-/B^-}$ one obtains

$$\begin{aligned}\big(\hat{\varrho}(z)\zeta\big)(sz_1u_1) &= \zeta(z^{-1}sz_1u_1) = \zeta(z^{-1}sz_1) \quad (\because u_1 \in U^- \subset N^-, (2.17)) \\ &= \zeta(sz_1z^{-1}) \quad (\because sz_1 \in L_{\mathbb{C}}, z \in Z(L_{\mathbb{C}})) \\ &= \chi_2(z)\zeta(sz_1) = \chi_2(z)\zeta(sz_1u_1).\end{aligned}$$

This and $Q^- = S_{\mathbb{C}}ZU^-$ imply that

⑤ $\hat{\varrho}(z)\zeta = \chi_2(z)\zeta$ for all $\zeta \in \mathcal{L}_{Q^-/B^-}$.

From $z \in Z \subset H_{\mathbb{C}}$, (sii) and (2.17) one deduces $\rho(z)\mathsf{u}_1 = \chi_2(z)\mathsf{u}_1$. This, together with $z \in Z(L_{\mathbb{C}})$ and $\mathsf{V} = \mathrm{span}_{\mathbb{C}}\{\rho(l)\mathsf{u}_1 : l \in L_{\mathbb{C}}\}$, implies that

⑥ $\rho(z)\mathsf{v} = \chi_2(z)\mathsf{v}$ for all $\mathsf{v} \in \mathsf{V}$.

Fix an arbitrary $u \in U^-$. Similarly, we obtain

$$\big(\hat{\varrho}(u)\zeta\big)(sz_1u_1) = \zeta(u^{-1}sz_1u_1) = \zeta\big(sz_1(sz_1)^{-1}u^{-1}(sz_1)u_1\big) = \zeta(sz_1) = \zeta(sz_1u_1)$$

(because of Lemma 2.3-(v), $sz_1 \in L_{\mathbb{C}}$, $\{(sz_1)^{-1}u^{-1}(sz_1), u_1\} \subset U^- \subset N^-$, (2.17)), and

⑦ $$\hat{\varrho}(u)\zeta = \zeta \text{ for all } \zeta \in \mathcal{L}_{Q^-/B^-}.$$

(s1) and Proposition 2.11 tell us that

⑧ $$\rho(u)\mathsf{v} = \mathsf{v} \text{ for all } \mathsf{v} \in \mathsf{V}.$$

Now, we are in a position to conclude ⓒ. Take any $q \in Q^-$ and $\zeta \in \mathcal{L}_{Q^-/B^-}$. By $Q^- = S_{\mathbb{C}}ZU^-$ there exists a $(s,z,u) \in S_{\mathbb{C}} \times Z \times U^-$ satisfying $q = szu$; and

$$\begin{aligned}\mathtt{f}\big(\hat{\varrho}(q)\zeta\big) &= \mathtt{f}\big(\hat{\varrho}(szu)\zeta\big)\\ &\overset{⑦,⑤}{=} \chi_2(z)\mathtt{f}\big(\hat{\varrho}(s)\zeta\big) \overset{③}{=} \chi_2(z)\rho(s)\big(\mathtt{f}(\zeta)\big) \overset{⑧,⑥}{=} \rho(szu)\big(\mathtt{f}(\zeta)\big) = \rho(q)\big(\mathtt{f}(\zeta)\big)\end{aligned}$$

because $\mathtt{f} : \mathcal{L}_{Q^-/B^-} \to \mathsf{V}$ is linear and $\chi_2(z) \in \mathbb{C}$. Thus ⓒ holds. This completes the proof of Proposition 2.16. □

2.6. **Direct product of elliptic orbits.** The main result in this paper (Theorem 3.18) is concerned with the direct product of elliptic orbits. Here we fix the setting of Theorem 3.18.

Let $(G_1)_{\mathbb{C}}, (G_2)_{\mathbb{C}}, \dots, (G_n)_{\mathbb{C}}$ be a finite number of connected complex semisimple Lie groups, let G_a be a connected closed subgroup of $(G_a)_{\mathbb{C}}$ such that $\mathfrak{g}_a$ is a real form of $(\mathfrak{g}_a)_{\mathbb{C}}$, and let T_a be a non-zero elliptic element of $\mathfrak{g}_a$, $1 \le a \le n$. From $(G_a)_{\mathbb{C}}, G_a, T_a$ we construct L_a, $(\mathfrak{g}_a)^{\lambda}$ and Q_a^- in a similar way to construct the L, $\mathfrak{g}^{\lambda}$, Q^- in §§2.2, and induce the complex structure J_a on $G_a/L_a = \iota_a(G_a/L_a)$ from the complex flag manifold $(G_a)_{\mathbb{C}}/Q_a^-$, where $\iota_a : G_a/L_a \to (G_a)_{\mathbb{C}}/Q_a^-$, $g_aL_a \mapsto g_aQ_a^-$. Then, one can get the *direct product*

$$G/L = G_1/L_1 \times G_2/L_2 \times \cdots \times G_n/L_n$$

by setting $G := G_1 \times G_2 \times \cdots \times G_n$ and $L := L_1 \times L_2 \times \cdots \times L_n$. We equip this elliptic orbit G/L with the complex structure $J := J_1 \times J_2 \times \cdots \times J_n$. Remark that this J accords with the complex structure on G/L induced by $\iota = \iota_1 \times \iota_2 \times \cdots \times \iota_n : G/L \to G_{\mathbb{C}}/Q^-$, where $G_{\mathbb{C}} := (G_1)_{\mathbb{C}} \times (G_2)_{\mathbb{C}} \times \cdots \times (G_n)_{\mathbb{C}}$ and $Q^- := Q_1^- \times Q_2^- \times \cdots \times Q_n^-$. In this setting, one can describe the vector space $\mathcal{V}_{G/L}(\mathsf{V},\rho)$ in (2.5) as follows:

$$\begin{aligned}(2.19)\quad \mathcal{V}_{G/L}(\mathsf{V},\rho) = \big\{\psi : G_1Q_1^- \times \cdots \times G_nQ_n^- \to \mathsf{V} \;\big|\; &(1)\ \psi \text{ is holomorphic},\\ &(2)\ \psi(x_1q_1,\dots,x_nq_n) = \rho(q_1,\dots,q_n)^{-1}\big(\psi(x_1,\dots,x_n)\big)\\ &\qquad \text{for all } (x_a,q_a) \in G_aQ_a^- \times Q_a^-,\ 1 \le a \le n\big\}.\end{aligned}$$

2.7. **Propositions on external tensor products (Propositions 2.20 and 2.21).** In the setting of §§2.6 we show Propositions 2.20 and 2.21 below.

Proposition 2.20. *Suppose that for a finite-dimensional complex vector space* V *and a holomorphic homomorphism* $\rho : Q^- \to GL(\mathsf{V})$, (S2′) *the representation* $\rho = \rho(q_1, q_2, \dots, q_n) : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V})$ *is the external tensor product of given holomorphic representations* $\rho_a : Q_a^- \to GL(\mathsf{V}_a)$,

$$\mathsf{V} = \mathsf{V}_1 \otimes \mathsf{V}_2 \otimes \cdots \otimes \mathsf{V}_n, \quad \rho = \rho_1 \boxtimes \rho_2 \boxtimes \cdots \boxtimes \rho_n.$$

Then, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ *if and only if* $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$ *for all* $1 \leq a \leq n$. *Here we refer to* (2.5) *for* $\mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a)$.

Proof. (⇒) Suppose $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$, and fix any $1 \leq a \leq n$. On the one hand, by (2.19) there exist a $\psi_0 \in \mathcal{V}_{G/L}(\mathsf{V}, \rho)$ and a $(g_1, \dots, g_n) \in G_1 \times \cdots \times G_n$ satisfying $\psi_0(g_1, \dots, g_n) \neq 0$. Putting $\psi_1 := \varrho(g_1^{-1}, \dots, g_n^{-1})\psi_0$, we have

① $$\psi_1 \in \mathcal{V}_{G/L}(\mathsf{V}, \rho), \quad \psi_1(e, \dots, e) \in \mathsf{V} - \{0\}.$$

On the other hand, let $\{\mathsf{e}_{i_d}\}_{i_d=1}^{m_d}$ be a complex basis of the vector space V_d ($1 \leq d \leq n$), let

$$\mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n} := \{\mathsf{e}_{i_1} \otimes \cdots \otimes \mathsf{e}_{i_{a-1}} \otimes \mathsf{w} \otimes \mathsf{e}_{i_{a+1}} \otimes \cdots \otimes \mathsf{e}_{i_n} \,|\, \mathsf{w} \in \mathsf{V}_a\}$$

($1 \leq i_1 \leq m_1, \dots, 1 \leq i_{a-1} \leq m_{a-1}, 1 \leq i_{a+1} \leq m_{a+1}, \dots, 1 \leq i_n \leq m_n$), and let

$$\rho_a'(q) := \rho(\underbrace{e, \dots, e}_{a-1}, q, e, \dots, e) = \mathrm{id}_{\mathsf{V}_1} \boxtimes \cdots \boxtimes \mathrm{id}_{\mathsf{V}_{a-1}} \boxtimes \rho_a(q) \boxtimes \mathrm{id}_{\mathsf{V}_{a+1}} \boxtimes \cdots \boxtimes \mathrm{id}_{\mathsf{V}_n}$$

($q \in Q_a^-$). Remark that each $\mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}$ is a $\rho_a'(Q_a^-)$-invariant complex vector subspace of $\mathsf{V} = \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_n$, and that ρ_a' is a holomorphic homomorphism of Q_a^- into $GL(\mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n})$. Let us define a linear isomorphism $f_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n} : \mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n} \to \mathsf{V}_a$ by

$$f_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}(\mathsf{e}_{i_1} \otimes \cdots \otimes \mathsf{e}_{i_{a-1}} \otimes \mathsf{w} \otimes \mathsf{e}_{i_{a+1}} \otimes \cdots \otimes \mathsf{e}_{i_n}) := \mathsf{w}.$$

Then, it turns out that

(1) the representation module $(\mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}, \rho_a')$ is Q_a^--equivariant isomorphic to (V_a, ρ_a) via $f_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}$;

(2) $\mathsf{V} = \oplus_{i_1=1}^{m_1} \cdots \oplus_{i_{a-1}=1}^{m_{a-1}} \oplus_{i_{a+1}=1}^{m_{a+1}} \cdots \oplus_{i_n=1}^{m_n} \mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}$ (direct sum).

By virtue of (2) we can get the projection $\mathrm{Pr}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}$ of V onto $\mathsf{W}_{i_1 \cdots i_{a-1} i_{a+1} \cdots i_n}$. Consequently, it is immediate from ① and (2) that the $\mathsf{W}_{j_1 \cdots j_{a-1} j_{a+1} \cdots j_n}$-component of $\psi_1(e, \dots, e)$ is not zero for some $1 \leq j_1 \leq m_1, \dots, 1 \leq j_{a-1} \leq m_{a-1}, 1 \leq j_{a+1} \leq m_{a+1}, \dots, 1 \leq j_n \leq m_n$. Defining a holomorphic mapping $\phi : G_a Q_a^- \to \mathsf{V}_a$ by

$$\phi(x) := (f_{j_1 \cdots j_{a-1} j_{a+1} \cdots j_n} \circ \mathrm{Pr}_{j_1 \cdots j_{a-1} j_{a+1} \cdots j_n} \circ \psi_1)(\underbrace{e, \dots, e}_{a-1}, x, e, \dots, e)$$

for an $x \in G_a Q_a^-$, we have $0 \neq \phi \in \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a)$. Hence $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$.

(⇐) Suppose that $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$ for all $1 \leq a \leq n$. For each $1 \leq a \leq n$ one can take a non-zero $\phi_a \in \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a)$. Setting $\psi(x_1, \dots, x_n) := \phi_1(x_1) \otimes \cdots \otimes \phi_n(x_n)$ for a $(x_1, \dots, x_n) \in G_1 Q_1^- \times \cdots \times G_n Q_n^-$, we have $0 \neq \psi \in \mathcal{V}_{G/L}(\mathsf{V}, \rho)$. □

Proposition 2.21. *The following items* (i) *and* (ii) *hold for the direct product* $Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^-$ *of complex parabolic subgroups* Q_a^- :

(i) *Given finite-dimensional irreducible holomorphic representations* $\sigma_a : Q_a^- \to GL(\mathsf{W}_a)$, $1 \le a \le n$, *the external tensor product* $(\mathsf{W}_1 \otimes \mathsf{W}_2 \otimes \cdots \otimes \mathsf{W}_n, \sigma_1 \boxtimes \sigma_2 \boxtimes \cdots \boxtimes \sigma_n)$ *is an irreducible representation module of* $Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^-$.

(ii) *For any finite-dimensional irreducible holomorphic representation* $\sigma : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{W})$, $(q_1, q_2, \ldots, q_n) \mapsto \sigma(q_1, q_2, \ldots, q_n)$, *there exist irreducible holomorphic representations* $\sigma_a : Q_a^- \to GL(\mathsf{W}_a)$, $1 \le a \le n$ *such that the representation module* (W, σ) *is* $Q_1^- \times Q_2^- \times \cdots \times Q_n^-$*-equivariant isomorphic to the external tensor product* $(\mathsf{W}_1 \otimes \mathsf{W}_2 \otimes \cdots \otimes \mathsf{W}_n, \sigma_1 \boxtimes \sigma_2 \boxtimes \cdots \boxtimes \sigma_n)$.

Proof. From the $(G_a)_{\mathbb{C}}$, G_a and T_a in §§2.6, we construct $(L_a)_{\mathbb{C}}$, U_a^- and $(L_a)_u$ in a similar way to construct the $L_{\mathbb{C}}$, U^- and L_u in §§2.2. Note that $(L_a)_u$ and L_u are maximal compact subgroups of the connected complex Lie groups $(L_a)_{\mathbb{C}}$ and $L_{\mathbb{C}}$, respectively, and that $(\mathfrak{l}_a)_u$ and $\mathfrak{l}_u$ are real forms of $(\mathfrak{l}_a)_{\mathbb{C}}$ and $\mathfrak{l}_{\mathbb{C}}$, respectively. Here $L_u = (L_1)_u \times \cdots \times (L_n)_u$, $L_{\mathbb{C}} = (L_1)_{\mathbb{C}} \times \cdots \times (L_n)_{\mathbb{C}}$.

(i) Let us only investigate the case where $\mathsf{W}_a \neq \{0\}$ for all $1 \le a \le n$ (otherwise our assertions are trivial). Since $\sigma_a : Q_a^- = (L_a)_{\mathbb{C}} \ltimes U_a^- \to GL(\mathsf{W}_a)$ is irreducible, Lemma 2.12-(i) assures that $(\sigma_a)_* : (\mathfrak{l}_a)_{\mathbb{C}} \to \mathfrak{gl}(\mathsf{W}_a)$ is irreducible, so that the complex representation $\sigma_a : (L_a)_u \to GL(\mathsf{W}_a)$ is also irreducible because $(\mathfrak{l}_a)_u$ is a real form of $(\mathfrak{l}_a)_{\mathbb{C}}$. Accordingly, Lemma 3.66 in Adams [1, p.72] enables us to prove that $\sigma_1 \boxtimes \cdots \boxtimes \sigma_n : (L_1)_u \times \cdots \times (L_n)_u \to GL(\mathsf{W}_1 \otimes \cdots \otimes \mathsf{W}_n)$ is irreducible by mathematical induction on n. Hence $(\mathsf{W}_1 \otimes \cdots \otimes \mathsf{W}_n, \sigma_1 \boxtimes \cdots \boxtimes \sigma_n)$ is an irreducible representation module of $Q_1^- \times \cdots \times Q_n^-$, since $(L_a)_u \subset Q_a^-$.

(ii) Let us only investigate the case $\mathsf{W} \neq \{0\}$. By arguments similar to those above, we see that the complex representation $\sigma : L_u \to GL(\mathsf{W})$ is irreducible since $\sigma : Q^- = L_{\mathbb{C}} \ltimes U^- \to GL(\mathsf{W})$ is irreducible (here, $U^- = U_1^- \times \cdots \times U_n^-$). Thus the proof of Lemma 3.67 in Adams [1, p.72] implies that $\sigma : L_u = (L_1)_u \times \cdots \times (L_n)_u \to GL(\mathsf{W})$ is isomorphic to the external tensor product of some irreducible complex representations $\sigma_a'' : (L_a)_u \to GL(\mathsf{W}_a)$, $1 \le a \le n$. Moreover, since the homogeneous space $(L_a)_{\mathbb{C}}/(L_a)_u$ is simply connected, for each $1 \le a \le n$ there exists a unique holomorphic homomorphism $\sigma_a' : (L_a)_{\mathbb{C}} \to GL(\mathsf{W}_a)$ such that $\sigma_a'' = \sigma_a'|_{(L_a)_u}$ by (a generalization of) Weyl's unitary trick. Then the holomorphic representation $\sigma : L_{\mathbb{C}} = (L_1)_{\mathbb{C}} \times \cdots \times (L_n)_{\mathbb{C}} \to GL(\mathsf{W})$ is isomorphic to the external tensor product of the irreducible holomorphic representations $\sigma_a' : (L_a)_{\mathbb{C}} \to GL(\mathsf{W}_a)$. Defining a holomorphic homomorphism $\sigma_a : Q_a^- = (L_a)_{\mathbb{C}} \ltimes U_a^- \to GL(\mathsf{W}_a)$ by

$$\sigma_a(lu) := \sigma_a'(l) \text{ for } l \in (L_a)_{\mathbb{C}} \text{ and } u \in U_a^-,$$

one can get the conclusion, since Proposition 2.11 tells that $\sigma(u_1, \ldots, u_n) = \mathrm{id}_{\mathsf{W}}$ for all $(u_1, \ldots, u_n) \in U_1^- \times \cdots \times U_n^-$. □

3. The main result in this paper (Theorem 3.18)

In this section we first deal with the case where $\mathfrak{g}$ is a simple Lie algebra and $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$, and clarify some properties of $\mathcal{V}_{G/L}$ (see Propositions 3.6 and

3.17), and afterwards state the main result in this paper and prove it by taking Propositions 3.6 and 3.17 into account.

3.1. **Case $\mathfrak{g}$ is simple and** $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$**.** The setting of §§3.1 is the same as that of §§2.3. Recall here that K is a maximal compact subgroup of the Lie group G whose Lie algebra $\mathfrak{k}$ contains the elliptic element T.

Let us suppose that

- the Lie algebra $\mathfrak{g}$ is simple, and
- $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$.

Then, it follows from Corollary 3.4.6-(ii) in [3, p.248] that the maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is not semisimple, so that the Lie algebra $\mathfrak{g}$ has to be of Hermitian type, and the homogeneous space G/K is an almost effective, irreducible Hermitian symmetric space of non-compact type.[2] Hence, up to sign $\pm$, there exists a unique non-zero elliptic element $W \in \mathfrak{g}$ such that $K = C_G(W)$, $\mathfrak{z}(\mathfrak{k}) = \mathrm{span}_{\mathbb{R}}\{W\}$ and the eigenvalues of $\mathrm{ad}\, W$ are $\pm i$ or zero. Setting

$$K_{\mathbb{C}} := C_{G_{\mathbb{C}}}(W), \quad \mathfrak{p}^{\pm} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}\, W(A) = \pm i A\}, \quad P^{\pm} := \exp \mathfrak{p}^{\pm}, \tag{3.1}$$

one has $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$, $[\mathfrak{p}^s, \mathfrak{p}^s] = \{0\}$ for each $s = \pm$, and can prove

Lemma 3.2. *In the setting above, the following six items hold. Here $s = \pm$.*

(i) *The closed complex subgroup $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ is connected and reductive.*
(ii) $\overline{\theta}(K_{\mathbb{C}}) = K_{\mathbb{C}}$*,* $\overline{\theta}(P^s) = P^{-s}$*.*
(iii) *P^s is a simply connected, closed complex abelian subgroup of $G_{\mathbb{C}}$ whose Lie algebra coincides with $\mathfrak{p}^s$, and $\exp : \mathfrak{p}^s \to P^s$ is biholomorphic.*
(iv) *$K_{\mathbb{C}}P^s$ is a connected, closed complex parabolic subgroup of $G_{\mathbb{C}}$ such that $K_{\mathbb{C}}P^s = K_{\mathbb{C}} \ltimes P^s$ and its Lie algebra is $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^s$.*
(v) *The product mapping $P^{-s} \times K_{\mathbb{C}}P^s \ni (p, z) \mapsto pz \in G_{\mathbb{C}}$ is a biholomorphism of $P^{-s} \times K_{\mathbb{C}}P^s$ onto a domain in $G_{\mathbb{C}}$.*
(vi) *Both $G \subset P^+K_{\mathbb{C}}P^-$ and $G \subset P^-K_{\mathbb{C}}P^+$ hold.*

Proof. We derive the items (i) through (v) from Lemma 2.3. Since $\mathfrak{g}$ is a simple Lie algebra of Hermitian type, we conclude the last item (vi) by the proof of Lemma 7.9 in Helgason [9, p.388]. □

As mentioned above, $\mathfrak{g}$ is a simple Lie algebra of Hermitian type. Accordingly $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$, the contraposition of Corollary 3.14 in [4, p.20], and Lemma 3.16 in [4, p.21] give rise to

Lemma 3.3. *Suppose that $\mathfrak{g}$ is a simple Lie algebra and $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$. Then, one of the following cases* (c1) *and* (c2) *necessarily occurs*:

(c1) $\mathfrak{p}^+ \subset \mathfrak{u}^+$*,* $L \subset K$ *and* $Q^- \subset K_{\mathbb{C}}P^-$*;*
(c2) $\mathfrak{p}^- \subset \mathfrak{u}^+$*,* $L \subset K$ *and* $Q^- \subset K_{\mathbb{C}}P^+$*.*

In particular, $L \subset K$ always holds.

[2] Here, we would like to correct errors in the paper [4]. p.18, ↓11, Read almost effective instead of effective; p.20, ↑9, Read $\mathfrak{g}_{-\alpha_p} \subset$ instead of $\mathfrak{g}_{-\alpha_p} \in$; p.20, ↑8, Read $\mathfrak{g}_{\tilde{\alpha}} \subset$ instead of $\mathfrak{g}_{\tilde{\alpha}} \in$.

Denote by $\ddot{\mathfrak{k}}$ the derived subalgebra of $\mathfrak{k}$. In view of $\mathfrak{k} = \ddot{\mathfrak{k}} \oplus \mathfrak{z}(\mathfrak{k})$ we express the elliptic element T as

$$T = T_{\mathfrak{s}} + T_{\mathfrak{z}} \tag{3.4}$$

($T_{\mathfrak{s}} \in \ddot{\mathfrak{k}} = [\mathfrak{k}, \mathfrak{k}]$, $T_{\mathfrak{z}} \in \mathfrak{z}(\mathfrak{k})$) and show

Lemma 3.5. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra and* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$. *Then, the following conditions* (a), (b) *and* (c) *are equivalent*:

(a) $\mathfrak{l} = \mathfrak{k}$.
(b) $T_{\mathfrak{s}} = 0$.
(c) *There exists a non-zero* $\lambda \in \mathbb{R}$ *satisfying* $T = \lambda W$.

Proof. (a)⇒(b). Suppose $\mathfrak{l} = \mathfrak{k}$. Then, it follows from $\mathfrak{k} \subset \mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(T)$ and $T \in \mathfrak{k}$ that $T \in \mathfrak{z}(\mathfrak{k})$, so that $T_{\mathfrak{s}} = 0$ by (3.4).

(b)⇒(c). If $T_{\mathfrak{s}} = 0$, then (3.4) yields $T = T_{\mathfrak{z}} \in \mathfrak{z}(\mathfrak{k}) = \mathrm{span}_{\mathbb{R}}\{W\}$; thus we obtain (c) from $T \neq 0$.

(c)⇒(a). Suppose that there exists a non-zero $\lambda \in \mathbb{R}$ satisfying $T = \lambda W$. Then $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{g}}(W)$ follows, and hence $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{g}}(W) = \mathfrak{k}$. □

We will investigate the case $\mathfrak{l} = \mathfrak{k}$ in §§3.1.1 (resp. $\mathfrak{l} \neq \mathfrak{k}$ in §§3.1.2) and conclude Proposition 3.6 (resp. Proposition 3.17) from Lemmas 3.2, 3.3 and 3.5.

3.1.1. *Case* $\mathfrak{l} = \mathfrak{k}$. Let us prove

Proposition 3.6. *If* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} = \mathfrak{k}$, *then it turns out that*

(1) $L = K$ *and the elliptic orbit* G/L *is an almost effective, irreducible Hermitian symmetric space of non-compact type*;
(2) $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ *if and only if* $\mathsf{V} \neq \{0\}$.

Here we refer to (2.5) *for* $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$.

Proof. By $\mathfrak{l} = \mathfrak{k}$ and Lemma 3.5, there exists a non-zero $\lambda \in \mathbb{R}$ such that $T = \lambda W$. Then one has $L = C_G(T) = C_G(W) = K$; moreover, $Q^- = K_{\mathbb{C}} P^-$ if $\lambda > 0$, and $Q^- = K_{\mathbb{C}} P^+$ if $\lambda < 0$. Here (1) holds, since $\mathfrak{g}$ is a simple Lie algebra of Hermitian type.

(2) We only prove that $\mathsf{V} \neq \{0\}$ implies $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$. Suppose $\mathsf{V} \neq \{0\}$, and take a non-zero $\mathsf{v} \in \mathsf{V}$.

(Case $\lambda > 0$) In case of $\lambda > 0$, Lemma 3.2-(vi), (iv) and $Q^- = K_{\mathbb{C}} P^-$ enable us to have $GQ^- \subset P^+ K_{\mathbb{C}} P^- (K_{\mathbb{C}} P^-) \subset P^+ K_{\mathbb{C}} P^-$. Therefore, given an $x \in GQ^-$, there exists a unique $(p_+, q) \in P^+ \times Q^-$ satisfying

$$x = p_+ q$$

by virtue of Lemma 3.2-(v) and $Q^- = K_{\mathbb{C}} P^-$. Then we can define a mapping $\psi : GQ^- \to \mathsf{V}$ as follows:

$$\psi(x) := \rho(q)^{-1} \mathsf{v}.$$

This ψ is holomorphic; furthermore, $0 \neq \psi \in \mathcal{V}_{G/L}(\mathsf{V}, \rho)$ because of $\psi(e) = \mathsf{v}$. Hence $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$.

(Case $\lambda < 0$) In case of $\lambda < 0$ also, one can conclude $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ by arguments similar to those above. □

3.1.2. *Case* $\mathfrak{l} \neq \mathfrak{k}$. Let $\ddot{K}$ be the connected Lie subgroup of the compact Lie group K corresponding to the derived subalgebra $\ddot{\mathfrak{k}}$ of $\mathfrak{k}$, let $\ddot{\mathfrak{k}}_{\mathbb{C}}$ be the complex subalgebra of $\mathfrak{k}_{\mathbb{C}}$ generated by $\ddot{\mathfrak{k}}$ (i.e., $\ddot{\mathfrak{k}}_{\mathbb{C}} = [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$), and let $\ddot{K}_{\mathbb{C}}$ be the connected Lie subgroup of the complex Lie group $K_{\mathbb{C}}$ in (3.1) corresponding to the subalgebra $\ddot{\mathfrak{k}}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$. Then we see

Lemma 3.7. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} \neq \mathfrak{k}$. *Then, it turns out that*

(1) $\ddot{K}$ *is a connected compact subgroup of* $\ddot{K}_{\mathbb{C}}$ *such that* $\ddot{\mathfrak{k}}$ *is a real form of* $\ddot{\mathfrak{k}}_{\mathbb{C}}$;
(2) $\ddot{K}_{\mathbb{C}}$ *is a connected, closed complex semisimple subgroup of* $G_{\mathbb{C}}$ *and satisfies* $\overline{\theta}(\ddot{K}_{\mathbb{C}}) = \ddot{K}_{\mathbb{C}}$.

Proof. (1) It is enough to confirm $\ddot{\mathfrak{k}} \neq \{0\}$, because it follows from $\ddot{\mathfrak{k}} \neq \{0\}$ that $\ddot{\mathfrak{k}}$ is a compact semisimple Lie algebra. However, $\ddot{\mathfrak{k}} \neq \{0\}$ is a consequence of $\mathfrak{l} \neq \mathfrak{k}$ and Lemma 3.5.

(2) Since $\overline{\theta}(K_{\mathbb{C}}) = K_{\mathbb{C}}$ yields $\overline{\theta}_*(\ddot{\mathfrak{k}}_{\mathbb{C}}) \subset [\overline{\theta}_*(\mathfrak{k}_{\mathbb{C}}), \overline{\theta}_*(\mathfrak{k}_{\mathbb{C}})] \subset [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] = \ddot{\mathfrak{k}}_{\mathbb{C}}$, we only prove that the subset $\ddot{K}_{\mathbb{C}} \subset G_{\mathbb{C}}$ is closed. Taking the Cartan decomposition of $K_{\mathbb{C}}$ by $\overline{\theta}|_{K_{\mathbb{C}}}$ into account, one knows that the mapping $\kappa : K \times \mathfrak{k} \to K_{\mathbb{C}}$, $(k, X) \mapsto k \exp(iX)$, is homeomorphic. Therefore, since $\ddot{K} \times \ddot{\mathfrak{k}}$ is closed in $K \times \mathfrak{k}$, we conclude that $\ddot{K}_{\mathbb{C}} = \kappa(\ddot{K} \times \ddot{\mathfrak{k}})$ is closed in $K_{\mathbb{C}} = \kappa(K \times \mathfrak{k})$. This assures that the subset $\ddot{K}_{\mathbb{C}} \subset G_{\mathbb{C}}$ is closed because the subset $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ is closed. □

When $\mathfrak{l} \neq \mathfrak{k}$, Lemmas 3.7 and 3.5 tell that $\ddot{K}_{\mathbb{C}}$ is a connected complex semisimple Lie group, $\ddot{K}$ is a connected closed subgroup of $\ddot{K}_{\mathbb{C}}$ such that $\ddot{\mathfrak{k}}$ is a compact real form of $\ddot{\mathfrak{k}}_{\mathbb{C}}$, and $T_{\mathfrak{s}}$ is a non-zero, elliptic element of $\ddot{\mathfrak{k}}$; moreover, $\overline{\theta}|_{\ddot{K}_{\mathbb{C}}}$ is the (anti-holomorphic) Cartan involution of $\ddot{K}_{\mathbb{C}}$ whose fixed point set coincides with $\ddot{K}$, where we remark that all elements of $\ddot{\mathfrak{k}}$ are elliptic. Then we set

$$
\left\{
\begin{array}{l}
\ddot{L} := C_{\ddot{K}}(T_{\mathfrak{s}}),\ \ddot{L}_{\mathbb{C}} := C_{\ddot{K}_{\mathbb{C}}}(T_{\mathfrak{s}}), \\
\ddot{\mathfrak{k}}^{\lambda} := \{B \in \ddot{\mathfrak{k}}_{\mathbb{C}} \mid \operatorname{ad} T_{\mathfrak{s}}(B) = i\lambda B\} \text{ for a } \lambda \in \mathbb{R}, \\
\ddot{\mathfrak{u}}^{+} := \bigoplus_{\lambda > 0} \ddot{\mathfrak{k}}^{\lambda},\ \ddot{\mathfrak{u}}^{-} := \bigoplus_{\lambda > 0} \ddot{\mathfrak{k}}^{-\lambda}, \\
\ddot{U}^{s} := \exp \ddot{\mathfrak{u}}^{s},\ \ddot{Q}^{s} := N_{\ddot{K}_{\mathbb{C}}}(\ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \ddot{\mathfrak{u}}^{s}) \text{ for each } s = \pm.
\end{array}
\right. \tag{3.8}
$$

Lemma 2.3 does hold for these $\ddot{L}$, $\ddot{L}_{\mathbb{C}}$, $\ddot{\mathfrak{k}}^{\lambda}$, $\ddot{\mathfrak{u}}^{\pm}$, $\ddot{U}^{\pm}$ and $\ddot{Q}^{\pm}$. For the sake of completeness, we show

Lemma 3.9. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} \neq \mathfrak{k}$. *Then, the following six items hold*:

(i) *The closed complex subgroup* $\ddot{L}_{\mathbb{C}} \subset \ddot{K}_{\mathbb{C}}$ *is connected and reductive.*
(ii) $\overline{\theta}(\ddot{L}_{\mathbb{C}}) = \ddot{L}_{\mathbb{C}}$, $\overline{\theta}(\ddot{U}^{s}) = \ddot{U}^{-s}$.
(iii) $\ddot{U}^{s}$ *is a simply connected, closed complex nilpotent subgroup of* $\ddot{K}_{\mathbb{C}}$ *whose Lie algebra coincides with* $\ddot{\mathfrak{u}}^{s}$, *and* $\exp : \ddot{\mathfrak{u}}^{s} \to \ddot{U}^{s}$ *is biholomorphic.*

(iv) $\ddot{Q}^s$ *is a connected, closed complex parabolic subgroup of* $\ddot{K}_{\mathbb{C}}$ *such that* $\ddot{Q}^s = \ddot{L}_{\mathbb{C}} \ltimes \ddot{U}^s$ *and* $\ddot{\mathfrak{q}}^s = \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \ddot{\mathfrak{u}}^s = \bigoplus_{\mu \geq 0} \ddot{\mathfrak{k}}^{s\mu}$.

(v) $\ddot{\iota} : \ddot{K}/\ddot{L} \to \ddot{K}_{\mathbb{C}}/\ddot{Q}^-$, $k\ddot{L} \mapsto k\ddot{Q}^-$, *is a* $\ddot{K}$*-equivariant real analytic diffeomorphism of* $\ddot{K}/\ddot{L}$ *onto* $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$.

(vi) $\ddot{K}\ddot{Q}^- = \ddot{K}_{\mathbb{C}}$.

Here $s = \pm$.

Proof. The items (i) through (v) come from Lemma 2.3. The last item (vi) is a consequence of (v). □

We will confirm Lemma 3.10, derive Corollaries 3.11 and 3.12 from the lemma, show Lemma 3.13, and then set a holomorphic vector bundle $\iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ over the complex flag manifold $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$ and the complex vector space $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$ of its holomorphic cross-sections.

Lemma 3.10. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} \neq \mathfrak{k}$. *Then, the following eight items hold*:

(1) $\mathfrak{l}_{\mathbb{C}} = \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$.
(2) $\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) \subset \mathfrak{z}(\mathfrak{l}_{\mathbb{C}})$.
(3) $\ddot{L}_{\mathbb{C}} = \ddot{K}_{\mathbb{C}} \cap L_{\mathbb{C}}$, $\ddot{L} = \ddot{K} \cap L$.
(4) $(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-) = \mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-)$.
(5) $\bigoplus_{\lambda > 0} (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+$.
(6) $\ddot{\mathfrak{k}}^\lambda = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{g}^\lambda = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda$ *for all* $\lambda \in \mathbb{R} - \{0\}$.
(7) $\ddot{\mathfrak{u}}^s = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^s = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^s$ *and* $\ddot{U}^s \subset \ddot{K}_{\mathbb{C}} \cap U^s$ *for each* $s = \pm$.
(8) $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$.

Proof. Note that $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$ comes from Lemma 3.3.

(1) Take an arbitrary $X \in \mathfrak{l}_{\mathbb{C}}$. In terms of $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}} = \ddot{\mathfrak{k}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$, there exists a unique $(X_{\mathfrak{s}}, X_{\mathfrak{z}}) \in \ddot{\mathfrak{k}}_{\mathbb{C}} \times \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ satisfying $X = X_{\mathfrak{s}} + X_{\mathfrak{z}}$. Then it follows from (3.4), $\{T, X\} \subset \mathfrak{k}_{\mathbb{C}}$ and $X \in \mathfrak{l}_{\mathbb{C}} = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T)$ that

$$[T_{\mathfrak{s}}, X_{\mathfrak{s}}] = [T - T_{\mathfrak{z}}, X - X_{\mathfrak{z}}] = [T, X] = 0,$$

so that $X_{\mathfrak{s}} \in \mathfrak{c}_{\ddot{\mathfrak{k}}_{\mathbb{C}}}(T_{\mathfrak{s}}) = \ddot{\mathfrak{l}}_{\mathbb{C}}$ by (3.8). Thus one has $X = X_{\mathfrak{s}} + X_{\mathfrak{z}} \in \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$, and

① $$\mathfrak{l}_{\mathbb{C}} \subset \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}}).$$

Now, it is immediate from $[T, \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})] \subset [\mathfrak{k}, \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})] = \{0\}$ and $\mathfrak{l}_{\mathbb{C}} = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T)$ that

② $$\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) \subset \mathfrak{l}_{\mathbb{C}}.$$

For an arbitrary $A \in \ddot{\mathfrak{l}}_{\mathbb{C}} = \mathfrak{c}_{\ddot{\mathfrak{k}}_{\mathbb{C}}}(T_{\mathfrak{s}})$, we obtain

$$0 = [T_{\mathfrak{s}}, A] = [T - T_{\mathfrak{z}}, A] = [T, A]$$

from (3.4) and $[T_{\mathfrak{z}}, A] \in [\mathfrak{z}(\mathfrak{k}), \mathfrak{k}_{\mathbb{C}}] = \{0\}$. This implies $A \in \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(T) = \mathfrak{l}_{\mathbb{C}}$; thus $\ddot{\mathfrak{l}}_{\mathbb{C}} \subset \mathfrak{l}_{\mathbb{C}}$. Consequently, we deduce $\mathfrak{l}_{\mathbb{C}} = \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ from ① and ②.

(2) is a consequence of ② and $[\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}), \mathfrak{l}_{\mathbb{C}}] \subset [\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}), \mathfrak{k}_{\mathbb{C}}] = \{0\}$.

(3) By a direct computation one has

$$\ddot{L}_{\mathbb{C}} \overset{(3.8)}{=} C_{\ddot{K}_{\mathbb{C}}}(T_{\mathfrak{s}}) = \{w \in \ddot{K}_{\mathbb{C}} \mid \operatorname{Ad} w(T_{\mathfrak{s}}) = T_{\mathfrak{s}}\}$$
$$= \{w \in \ddot{K}_{\mathbb{C}} \mid \operatorname{Ad} w(T) = T\} = \ddot{K}_{\mathbb{C}} \cap C_{G_{\mathbb{C}}}(T) = \ddot{K}_{\mathbb{C}} \cap L_{\mathbb{C}}$$

because of (3.4) and $\operatorname{Ad} w(T_{\mathfrak{z}}) = T_{\mathfrak{z}}$. Similarly, we have $\ddot{L} = \ddot{K} \cap L$.

(4) We only prove $\mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-) \subset (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-)$. Given a $Y \in \mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-)$, it follows from $Y \in \mathfrak{k}_{\mathbb{C}} = \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(W)$ that

$$[W, Y] = 0.$$

From $Y \in \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-$, there exists a unique $(Y_l, Y_u) \in \mathfrak{l}_{\mathbb{C}} \times \mathfrak{u}^-$ satisfying $Y = Y_l + Y_u$. Then, one has $0 = [W, Y_l] + [W, Y_u]$; furthermore,

$$[W, Y_l] = 0 = [W, Y_u]$$

because $W \in \mathfrak{l}$ yields $[W, \mathfrak{l}_{\mathbb{C}}] \subset \mathfrak{l}_{\mathbb{C}}$, $[W, \mathfrak{u}^-] \subset \mathfrak{u}^-$. This assures $Y_l, Y_u \in \mathfrak{c}_{\mathfrak{g}_{\mathbb{C}}}(W) = \mathfrak{k}_{\mathbb{C}}$, and $Y = Y_l + Y_u \in (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-)$. Thus $\mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-) \subset (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-)$ holds.

(5) It is clear that $\bigoplus_{\lambda>0}(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) \subset \mathfrak{k}_{\mathbb{C}} \cap (\bigoplus_{\lambda>0} \mathfrak{g}^\lambda)$. Moreover, one can prove the converse inclusion $\mathfrak{k}_{\mathbb{C}} \cap (\bigoplus_{\lambda>0} \mathfrak{g}^\lambda) \subset \bigoplus_{\lambda>0}(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda)$ by arguments similar to those above, where we remark that $[T, W] = 0$ yields $[W, \mathfrak{g}^\lambda] \subset \mathfrak{g}^\lambda$. Thus one has $\bigoplus_{\lambda>0}(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) = \mathfrak{k}_{\mathbb{C}} \cap (\bigoplus_{\lambda>0} \mathfrak{g}^\lambda)$, which enables us to conclude $\bigoplus_{\lambda>0}(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+$ from $\mathfrak{u}^+ = \bigoplus_{\lambda>0} \mathfrak{g}^\lambda$.

(6) First of all, let us confirm the following ⓐ:

$$\text{ⓐ} \qquad \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda \subset \ddot{\mathfrak{k}}^\lambda.$$

Take any $A \in \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda$. From $A \in \mathfrak{k}_{\mathbb{C}} = \ddot{\mathfrak{k}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ there exists a unique $(A_{\mathfrak{s}}, A_{\mathfrak{z}}) \in \ddot{\mathfrak{k}}_{\mathbb{C}} \times \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ satisfying $A = A_{\mathfrak{s}} + A_{\mathfrak{z}}$. From $A \in \mathfrak{g}^\lambda$ one obtains $[T, A] = i\lambda A$. Accordingly

$$i\lambda A_{\mathfrak{s}} + i\lambda A_{\mathfrak{z}} = i\lambda A = [T, A] \overset{(3.4)}{=} [T_{\mathfrak{s}} + T_{\mathfrak{z}}, A_{\mathfrak{s}} + A_{\mathfrak{z}}] = [T_{\mathfrak{s}}, A_{\mathfrak{s}}].$$

Then, it follows from $\{i\lambda A_{\mathfrak{s}}, [T_{\mathfrak{s}}, A_{\mathfrak{s}}]\} \subset \ddot{\mathfrak{k}}_{\mathbb{C}}$ and $i\lambda A_{\mathfrak{z}} \in \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ that $i\lambda A_{\mathfrak{s}} = [T_{\mathfrak{s}}, A_{\mathfrak{s}}]$ and $i\lambda A_{\mathfrak{z}} = 0$; moreover, $A_{\mathfrak{z}} = 0$ by $\lambda \neq 0$. Therefore $A = A_{\mathfrak{s}} \in \ddot{\mathfrak{k}}^\lambda$ by (3.8), and hence ⓐ holds.

By a direct computation one has

$$\ddot{\mathfrak{k}}^\lambda \overset{(3.8)}{=} \{B \in \ddot{\mathfrak{k}}_{\mathbb{C}} \mid [T_{\mathfrak{s}}, B] = i\lambda B\} = \{B \in \ddot{\mathfrak{k}}_{\mathbb{C}} \mid [T, B] = i\lambda B\} = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{g}^\lambda \subset \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda$$

since (3.4), $[T_{\mathfrak{z}}, B] = 0$ and $\ddot{\mathfrak{k}}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$. This and ⓐ imply $\ddot{\mathfrak{k}}^\lambda = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{g}^\lambda = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda$.

(7) On the one hand, it follows from $\bigoplus_{\lambda>0}(\ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) \subset \ddot{\mathfrak{k}}_{\mathbb{C}} \cap (\bigoplus_{\lambda>0} \mathfrak{g}^\lambda) = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^+$ and $\ddot{\mathfrak{k}}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$ that

$$\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+ \supset \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^+ \supset \bigoplus_{\lambda>0}(\ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) \overset{(6)}{=} \bigoplus_{\lambda>0} \ddot{\mathfrak{k}}^\lambda \overset{(3.8)}{=} \ddot{\mathfrak{u}}^+.$$

On the other hand,

$$\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+ \overset{(5)}{=} \bigoplus_{\lambda>0}(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^\lambda) \overset{(6)}{=} \bigoplus_{\lambda>0} \ddot{\mathfrak{k}}^\lambda \overset{(3.8)}{=} \ddot{\mathfrak{u}}^+.$$

Hence we conclude $\ddot{\mathfrak{u}}^+ = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^+ = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+$, and Lemma 3.9-(iii) implies $\ddot{U}^+ = \exp \ddot{\mathfrak{u}}^+ = \exp(\ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^+) \subset \ddot{K}_{\mathbb{C}} \cap U^+$.

One deduces $\ddot{\mathfrak{u}}^- = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^- = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-$, $\ddot{U}^- \subset \ddot{K}_{\mathbb{C}} \cap U^-$ from $\ddot{\mathfrak{u}}^+ = \ddot{\mathfrak{k}}_{\mathbb{C}} \cap \mathfrak{u}^+ = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^+$, $\ddot{U}^+ \subset \ddot{K}_{\mathbb{C}} \cap U^+$ and Lemmas 3.9-(ii), 3.7-(2), 2.3-(iii) and 3.2-(ii).

(8) Lemma 3.9-(iv), together with (3), (7) and Lemma 2.3-(v), assures that

$$\ddot{Q}^- = \ddot{L}_{\mathbb{C}} \ltimes \ddot{U}^- \subset (\ddot{K}_{\mathbb{C}} \cap L_{\mathbb{C}}) \ltimes (\ddot{K}_{\mathbb{C}} \cap U^-) \subset \ddot{K}_{\mathbb{C}} \cap (L_{\mathbb{C}} \ltimes U^-) = \ddot{K}_{\mathbb{C}} \cap Q^-.$$

Hence, the rest of proof is to verify that the converse inclusion $\ddot{K}_{\mathbb{C}} \cap Q^- \subset \ddot{Q}^-$ also holds. For any $p \in \ddot{K}_{\mathbb{C}} \cap Q^-$, it follows from $Q^- = N_{G_{\mathbb{C}}}(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-)$ that $\operatorname{Ad} p(\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-) \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-$, so that for any $(X, Y) \in \ddot{\mathfrak{l}}_{\mathbb{C}} \times \ddot{\mathfrak{u}}^-$ one has

$$\begin{aligned} \operatorname{Ad} p(X+Y) &\in \ddot{\mathfrak{k}}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-) \subset \mathfrak{k}_{\mathbb{C}} \cap (\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}^-) \\ &\overset{(4)}{=} (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{u}^-) \overset{(7)}{=} (\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{l}_{\mathbb{C}}) \oplus \ddot{\mathfrak{u}}^- = \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \ddot{\mathfrak{u}}^-. \end{aligned}$$

because of $\ddot{\mathfrak{l}}_{\mathbb{C}} \times \ddot{\mathfrak{u}}^- \subset \mathfrak{l}_{\mathbb{C}} \times \mathfrak{u}^-$, $\ddot{\mathfrak{k}}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$. Thus by (1) there exists a unique $(A, B, C) \in \ddot{\mathfrak{l}}_{\mathbb{C}} \times \mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) \times \ddot{\mathfrak{u}}^-$ such that $\operatorname{Ad} p(X+Y) = A + B + C$. Then, $\ddot{\mathfrak{k}}_{\mathbb{C}} \ni \operatorname{Ad} p(X+Y) - A - C = B \in \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ and therefore $\operatorname{Ad} p(X+Y) = A + C \in \ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \ddot{\mathfrak{u}}^-$. This and (3.8) provide us with $p \in N_{\ddot{K}_{\mathbb{C}}}(\ddot{\mathfrak{l}}_{\mathbb{C}} \oplus \ddot{\mathfrak{u}}^-) = \ddot{Q}^-$; thus $\ddot{K}_{\mathbb{C}} \cap Q^- \subset \ddot{Q}^-$ holds. □

$\ddot{K}_{\mathbb{C}} \cap Q^-$ is a closed complex subgroup of $G_{\mathbb{C}}$ in terms of Lemmas 3.7-(2) and 2.3-(v). Therefore Lemma 3.10-(8) leads to

Corollary 3.11. *Suppose that $\mathfrak{g}$ is a simple Lie algebra, $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ and $\mathfrak{l} \neq \mathfrak{k}$. Then, $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$ is a closed complex subgroup of Q^-.*

Corollary 3.12. *Let V be a finite-dimensional complex vector space and $\rho : Q^- \to GL(\mathsf{V})$, $q \mapsto \rho(q)$, a holomorphic homomorphism. If $\mathfrak{g}$ is a simple Lie algebra, $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ and $\mathfrak{l} \neq \mathfrak{k}$, then the following items* (i) *and* (ii) *hold, where $\ddot{\rho}$ denotes the restriction of ρ to $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$:*

(i) *$\ddot{\rho} : \ddot{Q}^- \to GL(\mathsf{V})$ is a holomorphic homomorphism.*

(ii) *Suppose that $\rho : Q^- \to GL(\mathsf{V})$ is irreducible. Then, $\ddot{\rho}_* : \ddot{\mathfrak{l}}_{\mathbb{C}} \to \mathfrak{gl}(\mathsf{V})$ is irreducible, and thus $\ddot{\rho} : \ddot{Q}^- = \ddot{L}_{\mathbb{C}} \ltimes \ddot{U}^- \to GL(\mathsf{V})$ is irreducible.*

Proof. (i) is immediate from Corollary 3.11. (ii) follows by Lemmas 2.12 and 3.10-(1), (2). □

Let us set $\iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$ after showing

Lemma 3.13. *Suppose that $\mathfrak{g}$ is a simple Lie algebra, $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ and $\mathfrak{l} \neq \mathfrak{k}$. Then, the following four items hold:*

(1) *$\ddot{K}_{\mathbb{C}} \subset \ddot{K}_{\mathbb{C}} Q^- \subset G Q^-$.*

(2) *The mapping $\iota_K : \ddot{K}_{\mathbb{C}}/\ddot{Q}^- \to G_{\mathbb{C}}/Q^-$, $w\ddot{Q}^- \mapsto wQ^-$, is a $\ddot{K}_{\mathbb{C}}$-equivariant holomorphic embedding, and $\iota_K(\ddot{K}_{\mathbb{C}}/\ddot{Q}^-) \subset \iota(G/L)$.*

(3) *$\ddot{K}_{\mathbb{C}} Q^-$ is a connected, closed complex submanifold of $G_{\mathbb{C}}$.*

(4) *The complex homogeneous space* $(\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-}$ *is biholomorphic to* $\ddot{K}_{\mathbb{C}}Q^-$ *via the mapping*

$$(\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-} \ni (w,q)\Delta_{\ddot{Q}^-} \mapsto wq^{-1} \in \ddot{K}_{\mathbb{C}}Q^-.$$

Here $\Delta_{\ddot{Q}^-} = \{(p,p) \mid p \in \ddot{Q}^-\}$.

Proof. (1) Lemmas 3.9-(vi) and 3.10-(8), combined with $\ddot{K} \subset G$ and $Q^-Q^- \subset Q^-$, imply $\ddot{K}_{\mathbb{C}} \subset \ddot{K}_{\mathbb{C}}Q^- = (\ddot{K}\ddot{Q}^-)Q^- = \ddot{K}(\ddot{K}_{\mathbb{C}} \cap Q^-)Q^- \subset \ddot{K}Q^-Q^- \subset GQ^-$.

(2) By Lemma 3.10-(8), the $\iota_K : \ddot{K}_{\mathbb{C}}/\ddot{Q}^- \to G_{\mathbb{C}}/Q^-$ is a $\ddot{K}_{\mathbb{C}}$-equivariant holomorphic embedding. Moreover, it follows from (1) that $\iota_K(\ddot{K}_{\mathbb{C}}/\ddot{Q}^-) = \ddot{K}_{\mathbb{C}}Q^-/Q^- \subset GQ^-/Q^- = \iota(G/L)$.

(3) Denote by $\pi_{\mathbb{C}}$ the projection of $G_{\mathbb{C}}$ onto $G_{\mathbb{C}}/Q^-$. On the one hand, Lemma 3.9-(v) tells that $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$ is a compact manifold. Therefore it follows from (2) that $\iota_K(\ddot{K}_{\mathbb{C}}/\ddot{Q}^-) = \ddot{K}_{\mathbb{C}}Q^-/Q^-$ is a closed subset of $G_{\mathbb{C}}/Q^-$, so that $\ddot{K}_{\mathbb{C}}Q^- = \pi_{\mathbb{C}}^{-1}(\ddot{K}_{\mathbb{C}}Q^-/Q^-)$ is closed in $G_{\mathbb{C}}$. On the other hand, the action of $\ddot{K}_{\mathbb{C}} \times Q^-$ on $G_{\mathbb{C}}$, $(\ddot{K}_{\mathbb{C}} \times Q^-) \times G_{\mathbb{C}} \ni ((w,q),a) \mapsto waq^{-1} \in G_{\mathbb{C}}$, is holomorphic, and $\ddot{K}_{\mathbb{C}}Q^-$ accords with the orbit of the connected Lie group $\ddot{K}_{\mathbb{C}} \times Q^-$ through the unit $e \in G_{\mathbb{C}}$. Consequently, $\ddot{K}_{\mathbb{C}}Q^-$ is a connected closed (regular) submanifold of $G_{\mathbb{C}}$, and in addition, it is complex because the tangent space $T_e(\ddot{K}_{\mathbb{C}}Q^-)$ is stable under the invariant complex structure on $G_{\mathbb{C}}$.

(4) The action of the complex Lie group $\ddot{K}_{\mathbb{C}} \times Q^-$ on $G_{\mathbb{C}}$, $((w,q),a) \mapsto waq^{-1}$, is holomorphic, and $\ddot{K}_{\mathbb{C}}Q^-$ accords with its orbit through the unit $e \in G_{\mathbb{C}}$. Thus

$$(\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-} \ni (w,q)\Delta_{\ddot{Q}^-} \mapsto wq^{-1} \in \ddot{K}_{\mathbb{C}}Q^-$$

is biholomorphic, since the isotropy subgroup of $\ddot{K}_{\mathbb{C}} \times Q^-$ at e is $\{(w,q) \in \ddot{K}_{\mathbb{C}} \times Q^- \mid weq^{-1} = e\} = \Delta_{\ddot{K}_{\mathbb{C}} \cap Q^-} = \Delta_{\ddot{Q}^-}$ by Lemma 3.10-(8). □

Now, we are in a position to set $\iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$. Let $\iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ be the induced bundle of the holomorphic vector bundle $G_{\mathbb{C}} \times_\rho \mathsf{V}$ by the mapping ι_K in Lemma 3.13-(2) (in other wards, the homogeneous holomorphic vector bundle over the complex flag manifold $\ddot{K}_{\mathbb{C}}/\ddot{Q}^-$ associated with $\ddot{\rho} : \ddot{Q}^- \to GL(\mathsf{V})$).

$$\begin{array}{ccc} \iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V}) & & G_{\mathbb{C}} \times_\rho \mathsf{V} \\ \downarrow & & \downarrow \\ \ddot{K}_{\mathbb{C}}/\ddot{Q}^- & \xrightarrow{\iota_K} & G_{\mathbb{C}}/Q^- \end{array}$$

Taking $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$ and $\ddot{\rho} = \rho|_{\ddot{Q}^-}$ into account, we consider

$$\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) := \left\{ \phi : \ddot{K}_{\mathbb{C}} \to \mathsf{V} \;\middle|\; \begin{array}{l} (1)\ \phi \text{ is holomorphic,} \\ (2)\ \phi(wp) = \ddot{\rho}(p)^{-1}\big(\phi(w)\big) \\ \qquad \text{for all } (w,p) \in \ddot{K}_{\mathbb{C}} \times \ddot{Q}^- \end{array} \right\} \tag{3.14}$$

as the complex vector space of holomorphic cross-sections of the bundle $\iota_K^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$, and sometimes express it as $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}$. In addition, we denote by $\ddot{\varrho}$ the induced representation of $\ddot{K}_{\mathbb{C}}$ on $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-} = \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$.

Remark 3.15. Referring to comments in Remark 2.10 and taking Lemma 3.9-(v) into consideration, one can assert that

(i) $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) < \infty$ for all finite-dimensional complex vector spaces V and holomorphic homomorphisms $\rho : Q^- \to GL(\mathsf{V})$;

(ii) the homomorphism $\ddot{\varrho} : \ddot{K}_{\mathbb{C}} \to GL(\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-})$, $w \mapsto \ddot{\varrho}(w)$, is holomorphic.

Let us prepare the following lemma for proving Proposition 3.17:

Lemma 3.16. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} \neq \mathfrak{k}$. *Then, one of the following cases* (c1′) *and* (c2′) *necessarily occurs*:

(c1′) $\ddot{K}_{\mathbb{C}} Q^- = K_{\mathbb{C}} P^-$;

(c2′) $\ddot{K}_{\mathbb{C}} Q^- = K_{\mathbb{C}} P^+$.

Proof. Recall that one of the cases (c1) and (c2) in Lemma 3.3 necessarily occurs. In view of $\mathfrak{k}_{\mathbb{C}} = \ddot{\mathfrak{k}}_{\mathbb{C}} \oplus \mathfrak{z}(\mathfrak{k}_{\mathbb{C}})$ we see that the connected Lie group $K_{\mathbb{C}}$ is the locally direct product $\ddot{K}_{\mathbb{C}} \cdot Z_k$ of the connected semisimple Lie subgroup $\ddot{K}_{\mathbb{C}}$ and the identity component Z_k of the center $Z(K_{\mathbb{C}})$. Here $Z_k \subset L_{\mathbb{C}}$ comes from Lemma 3.10-(2).

(c1′) In the case (c1) of Lemma 3.3, $\mathfrak{p}^+ \subset \mathfrak{u}^+$ and $Q^- \subset K_{\mathbb{C}} P^-$ hold. By Lemma 3.2-(ii), (iii) and Lemma 2.3-(iii), (iv), we have $\mathfrak{p}^- = \overline{\theta}_*(\mathfrak{p}^+) \subset \overline{\theta}_*(\mathfrak{u}^+) = \mathfrak{u}^-$, and $P^- \subset U^-$; then it follows from $K_{\mathbb{C}} = \ddot{K}_{\mathbb{C}} Z_k$, $Z_k \subset L_{\mathbb{C}}$, $L_{\mathbb{C}} U^- = Q^-$ and $\ddot{K}_{\mathbb{C}} \subset K_{\mathbb{C}}$ that

$$K_{\mathbb{C}} P^- \subset (\ddot{K}_{\mathbb{C}} Z_k) U^- \subset \ddot{K}_{\mathbb{C}} L_{\mathbb{C}} U^- = \ddot{K}_{\mathbb{C}} Q^- \subset \ddot{K}_{\mathbb{C}}(K_{\mathbb{C}} P^-) \subset K_{\mathbb{C}} P^-,$$

i.e., $K_{\mathbb{C}} P^- = \ddot{K}_{\mathbb{C}} Q^-$.

(c2′) In the case (c2) of Lemma 3.3, $\mathfrak{p}^- \subset \mathfrak{u}^+$ and $Q^- \subset K_{\mathbb{C}} P^+$ hold. By arguments similar to those above, we obtain $P^+ \subset U^-$ and

$$K_{\mathbb{C}} P^+ \subset (\ddot{K}_{\mathbb{C}} Z_k) U^- \subset \ddot{K}_{\mathbb{C}} L_{\mathbb{C}} U^- = \ddot{K}_{\mathbb{C}} Q^- \subset \ddot{K}_{\mathbb{C}}(K_{\mathbb{C}} P^+) \subset K_{\mathbb{C}} P^+.$$

□

Now, let us prove

Proposition 3.17. *Suppose that* $\mathfrak{g}$ *is a simple Lie algebra,* $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$ *and* $\mathfrak{l} \neq \mathfrak{k}$. *Then, it follows that*

(i) $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$ *is a connected, closed complex parabolic subgroup of* $\ddot{K}_{\mathbb{C}}$;

(ii) *the induced representation* $\ddot{\varrho}$ *of* $\ddot{K}_{\mathbb{C}}$ *on* $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$ *is irreducible, provided that* $\rho : Q^- \to GL(\mathsf{V})$ *is irreducible*;

(iii) $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ *if and only if* $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) \neq 0$.

Proof. (i) follows by Lemmas 3.9-(iv) and 3.10-(8).

(ii) Corollary 3.12 enables us to have (ii), since Corollary 2.15 still holds even if we substitute $\ddot{\rho}$, $\ddot{Q}^-$, $\ddot{\varrho}$, $\ddot{K}$, $\ddot{K}_{\mathbb{C}}$ and $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}$ for ρ, Q^-, $\tilde{\varrho}$, G_u, $G_{\mathbb{C}}$ and $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$, respectively.

(iii) ($\Rightarrow$) Suppose $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$. Then, there exists a $\psi \in \mathcal{V}_{G/L}(\mathsf{V}, \rho)$ which satisfies $\psi(e) \neq 0$. (2.5) and Lemma 3.13-(1) allow us to put $\phi := \psi|_{\ddot{K}_{\mathbb{C}}}$; then we obtain $0 \neq \phi \in \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$ from Lemma 3.7-(2), Lemma 2.3-(ix) and (3.14).

($\Leftarrow$) Suppose $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) \neq 0$. Taking a non-zero $\phi \in \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$, we define a mapping $f : \ddot{K}_{\mathbb{C}} Q^- \to \mathsf{V}$ by

$$f(wq) := \rho(q)^{-1}\bigl(\phi(w)\bigr)$$

for a $wq \in \ddot{K}_{\mathbb{C}} Q^-$ ($w \in \ddot{K}_{\mathbb{C}}$, $q \in Q^-$). Remark here that this definition is well-defined by virtue of $\ddot{Q}^- = \ddot{K}_{\mathbb{C}} \cap Q^-$ and (3.14). We want to clarify some properties of the $f : \ddot{K}_{\mathbb{C}} Q^- \to \mathsf{V}$. Needless to say, $f \neq 0$. A direct computation yields

① $$f(zq) = \rho(q)^{-1}\bigl(f(z)\bigr)$$

for all $(z, q) \in \ddot{K}_{\mathbb{C}} Q^- \times Q^-$. From now on, let us show that

ⓐ $$f : \ddot{K}_{\mathbb{C}} Q^- \to \mathsf{V} \text{ is holomorphic.}$$

The mapping $f_1 : (\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-} \to \mathsf{V}$, $(w, q)\Delta_{\ddot{Q}^-} \mapsto \rho(q)\bigl(\phi(w)\bigr)$, is well-defined by $\phi \in \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho})$ and (3.14). It is clear that the mapping $F_1 : \ddot{K}_{\mathbb{C}} \times Q^- \to \mathsf{V}$, $(w, q) \mapsto \rho(q)\bigl(\phi(w)\bigr)$, is holomorphic. For the projection $\mathrm{Pr} : \ddot{K}_{\mathbb{C}} \times Q^- \to (\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-}$ we have $F_1 = f_1 \circ \mathrm{Pr}$.

$$\begin{array}{ccc} \ddot{K}_{\mathbb{C}} \times Q^- & \xrightarrow{\quad F_1 \quad} & \\ {\scriptstyle \mathrm{Pr}}\downarrow & \searrow & \\ (\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-} & \xrightarrow[f_1]{} & \mathsf{V} \end{array}$$

Consequently $f_1 : (\ddot{K}_{\mathbb{C}} \times Q^-)/\Delta_{\ddot{Q}^-} \to \mathsf{V}$, $(w, q)\Delta_{\ddot{Q}^-} \mapsto \rho(q)\bigl(\phi(w)\bigr)$, is holomorphic. This and Lemma 3.13-(4) assure that $f : \ddot{K}_{\mathbb{C}} Q^- \to \mathsf{V}$, $wq \mapsto \rho(q)^{-1}\bigl(\phi(w)\bigr)$, is holomorphic. Thus ⓐ holds.

In the case (c1′) of Lemma 3.16, for each $x \in GQ^-$, it follows from $GQ^- \subset P^+ K_{\mathbb{C}} P^-$, $K_{\mathbb{C}} P^- = \ddot{K}_{\mathbb{C}} Q^-$ and Lemma 3.2-(v) that there exists a unique $(p_+, z) \in P^+ \times \ddot{K}_{\mathbb{C}} Q^-$ satisfying $x = p_+ z$, and then one can define a holomorphic mapping $\psi_1 : GQ^- \to \mathsf{V}$ by

$$\psi_1(x) := f(z)$$

because of ⓐ. Moreover, a direct computation, together with ①, yields $\psi_1(xq) = \rho(q)^{-1}\bigl(\psi_1(x)\bigr)$ for all $(x, q) \in GQ^- \times Q^-$. Thus $0 \neq \psi_1 \in \mathcal{V}_{G/L}(\mathsf{V}, \rho)$ follows by (2.5).

In the case (c2′) of Lemma 3.16, we can conclude $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ by taking $GQ^- \subset P^- K_{\mathbb{C}} P^+$, $K_{\mathbb{C}} P^+ = \ddot{K}_{\mathbb{C}} Q^-$ into consideration. Hence $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}) \neq 0$ always implies $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$, because one of the cases (c1′) and (c2′) necessarily occurs. □

3.2. **The statement and proof of Theorem 3.18.** The main purpose of this paper is to establish

Theorem 3.18. *In the setting of §§2.6, suppose the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n$ to be simple, and suppose further that*

(S1) *there exists an integer $1 \leq k \leq n$ satisfying $\dim_{\mathbb{C}} \mathcal{O}(G_b/L_b) \neq 1$ for all $1 \leq b \leq k$ and $\dim_{\mathbb{C}} \mathcal{O}(G_c/L_c) = 1$ for all $k+1 \leq c \leq n$,* *and that for a finite-dimensional complex vector space* V *and a holomorphic homomorphism* $\rho : Q^- \to GL(\mathsf{V})$,

(S2) *the representation* $\rho = \rho(q_1, q_2, \dots, q_n) : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V})$ *is the external tensor product of given irreducible holomorphic representations* $\rho_a : Q_a^- \to GL(\mathsf{V}_a)$,

$$\mathsf{V} = \mathsf{V}_1 \otimes \mathsf{V}_2 \otimes \cdots \otimes \mathsf{V}_n, \quad \rho = \rho_1 \boxtimes \rho_2 \boxtimes \cdots \boxtimes \rho_n.$$

Then, the following items (1), (2) *and* (3) *hold*:

(1) $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty$ *if and only if* $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$ *for all* $1 \leq a \leq n$.

(2) *For each* $1 \leq b \leq k$, *let* K_b *be a maximal compact subgroup of* G_b *whose Lie algebra* $\mathfrak{k}_b$ *contains the elliptic element* T_b. *Denote by* $(\mathfrak{k}_b)_{\mathbb{C}}$ *the complex subalgebra of* $(\mathfrak{g}_b)_{\mathbb{C}}$ *generated by* $\mathfrak{k}_b$, *denote by* $(K_b)_{\mathbb{C}}$ *the connected Lie subgroup of* $(G_b)_{\mathbb{C}}$ *corresponding to the subalgebra* $(\mathfrak{k}_b)_{\mathbb{C}}$ *of* $(\mathfrak{g}_b)_{\mathbb{C}}$, *denote by* $(\ddot{K}_b)_{\mathbb{C}}$ *the connected Lie subgroup of* $(K_b)_{\mathbb{C}}$ *corresponding to the derived subalgebra* $[(\mathfrak{k}_b)_{\mathbb{C}}, (\mathfrak{k}_b)_{\mathbb{C}}]$ *of* $(\mathfrak{k}_b)_{\mathbb{C}}$, *and set* $\ddot{Q}_b^- := (\ddot{K}_b)_{\mathbb{C}} \cap Q_b^-$, $\ddot{\rho}_b := \rho_b|_{\ddot{Q}_b^-}$.
   - (i) *Suppose* $\mathfrak{l}_b = \mathfrak{k}_b$. *Then, it turns out that*
     1) $L_b = K_b$ *and the elliptic orbit* G_b/L_b *is an almost effective, irreducible Hermitian symmetric space of non-compact type*;
     2) $\dim_{\mathbb{C}} \mathcal{V}_{G_b/L_b}(\mathsf{V}_b, \rho_b) \neq 0$ *if and only if* $\mathsf{V}_b \neq \{0\}$.
   - (ii) *Suppose* $\mathfrak{l}_b \neq \mathfrak{k}_b$. *Then, it turns out that*
     1) $(\ddot{K}_b)_{\mathbb{C}}$ *is a connected, closed complex semisimple subgroup of* $(G_b)_{\mathbb{C}}$;
     2) $\ddot{Q}_b^-$ *is a connected, closed complex parabolic subgroup of* $(\ddot{K}_b)_{\mathbb{C}}$;
     3) $\dim_{\mathbb{C}} \mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b) < \infty$;
     4) $\ddot{\rho}_b : \ddot{Q}_b^- \to GL(\mathsf{V}_b)$ *is irreducible*;
     5) *the induced representation* $\ddot{\varrho}_b$ *of the semisimple Lie group* $(\ddot{K}_b)_{\mathbb{C}}$ *on the vector space* $\mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b)$ *is irreducible*;
     6) $\dim_{\mathbb{C}} \mathcal{V}_{G_b/L_b}(\mathsf{V}_b, \rho_b) \neq 0$ *if and only if* $\dim_{\mathbb{C}} \mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b) \neq 0$.

(3) *For each* $k+1 \leq c \leq n$, $\dim_{\mathbb{C}} \mathcal{V}_{G_c/L_c}(\mathsf{V}_c, \rho_c) = \dim_{\mathbb{C}} \mathcal{V}_{(G_c)_{\mathbb{C}}/Q_c^-}(\mathsf{V}_c, \rho_c) < \infty$ *and the induced representation* $\tilde{\varrho}_c$ *of the semisimple Lie group* $(G_c)_{\mathbb{C}}$ *on the vector space* $\mathcal{V}_{(G_c)_{\mathbb{C}}/Q_c^-}(\mathsf{V}_c, \rho_c)$ *is irreducible.*

Here we refer to (2.19), (2.5) *and* (3.14) *for* $\mathcal{V}_{G/L}(\mathsf{V}, \rho)$, $\mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a)$ *and* $\mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b)$, *respectively.*

Proof. (1) On the one hand, since $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$, the contraposition of Theorem 3.3 in [5, p.303] tells that either $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = 0$ or $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty$ holds; thus, $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty$ if and only if $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$. On the other hand, Proposition 2.20 shows that $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) \neq 0$ if and only if $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$ for all $1 \leq a \leq n$. Thus we have (1).

(2)-(i) follows by Proposition 3.6.

(2)-(ii) Lemma 3.7-(2) implies 1). Proposition 3.17 implies 2), 5) and 6). Remark 3.15-(i) and Corollary 3.12 imply 3) and 4), respectively. Note that the way to set

$(K_b)_{\mathbb{C}}$ (resp. $\ddot{Q}_b^-$) here is different from the way to set the $K_{\mathbb{C}}$ in (3.1) (resp. $\ddot{Q}^-$ in (3.8)), but it does not matter due to Lemma 3.2-(i) (resp. Lemma 3.10-(8)).

(3) Theorem 3.3 in [5] and $\dim_{\mathbb{C}} \mathcal{O}(G_c/L_c) = 1$ assure $\dim_{\mathbb{C}} \mathcal{V}_{G_c/L_c}(\mathsf{V}_c, \rho_c) < \infty$. Hence, (3) comes from Lemma 2.14. □

Remark 3.19. Here are comments on Theorem 3.18.

(i) By virtue of Proposition 2.21-(ii) one may assume that the supposition (S2) holds for every finite-dimensional, irreducible holomorphic representation $\rho = \rho(q_1, q_2, \ldots, q_n) : Q^- = Q_1^- \times Q_2^- \times \cdots \times Q_n^- \to GL(\mathsf{V})$.

(ii) Proposition 2.16 gives a necessary and sufficient condition for not only the vector space $\mathcal{V}_{(G_c)_{\mathbb{C}}/Q_c^-}(\mathsf{V}_c, \rho_c)$ in (3) but also the vector space $\mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b)$ in (2)-(ii)-6) to be not equal to $\{0\}$. Therefore one can obtain a necessary and sufficient condition for $\dim_{\mathbb{C}} \mathcal{V}_{G_a/L_a}(\mathsf{V}_a, \rho_a) \neq 0$ $(1 \leq a \leq n)$ from (2)-(i)-2), (2)-(ii)-6) and (3), and this condition is the same as that for $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho) = \infty$.

(iii) One can find the dimensions $\dim_{\mathbb{C}} \mathcal{V}_{(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-}(\mathsf{V}_b, \ddot{\rho}_b)$ and $\dim_{\mathbb{C}} \mathcal{V}_{G_c/L_c}(\mathsf{V}_c, \rho_c) = \dim_{\mathbb{C}} \mathcal{V}_{(G_c)_{\mathbb{C}}/Q_c^-}(\mathsf{V}_c, \rho_c)$ by Weyl's dimension formula.

(iv) Lemma 3.9-(v) implies that K_b/L_b is real analytic diffeomorphic to the complex flag manifold $(\ddot{K}_b)_{\mathbb{C}}/\ddot{Q}_b^-$, since $\ddot{K}_b/\ddot{L}_b$ is real analytic diffeomorphic to K_b/L_b in terms of $K_b = \ddot{K}_b Z(K_b)$, $Z(K_b) \subset L_b$ and $\ddot{L}_b = \ddot{K}_b \cap L_b$. cf. Lemma 3.10-(3).

3.3. **An application of Theorem 3.18.** We end this paper with giving

Example 3.20 $(G/L = SU(2,1)/S(U(1) \times U(1) \times U(1)))$**.** Let $G_{\mathbb{C}} := SL(3, \mathbb{C})$, let $G := SU(2,1) = \{g \in G_{\mathbb{C}} \mid {}^t g I_{2,1} \overline{g} = I_{2,1}\}$, and let

$$T := \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix},$$

where $I_{2,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, this T is a non-zero elliptic element of $\mathfrak{g}$, and it follows that

$$L := C_G(T) = \left\{ \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix} \;\middle|\; \begin{array}{r} u_1, u_2, u_3 \in U(1), \\ u_1 u_2 u_3 = 1 \end{array} \right\} = S(U(1) \times U(1) \times U(1)),$$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^2 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2},$$

$$Q^- := N_{G_{\mathbb{C}}}(\mathfrak{g}^0 \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ b & d_2 & 0 \\ c & d & d_3 \end{pmatrix} \;\middle|\; \begin{array}{r} d_1, d_2, d_3, b, c, d \in \mathbb{C}, \\ d_1 d_2 d_3 = 1 \end{array} \right\}.$$

In the first place, let us confirm $\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1$. Noting that for every matrix $x \in GQ^-$ its $(3,3)$-component is non-zero, we define a holomorphic mapping f :

$GQ^- \to \mathbb{C}$ by

$$f(x) := \frac{y_3}{z_3} \text{ for an } x = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \in GQ^-.$$

Then it turns out that $f(xq) = f(x)$ for all $(x, q) \in GQ^- \times Q^-$, so that $f \in \mathcal{O}(G/L)$. Moreover, for any $\theta \in \mathbb{R}$ we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \in G, \quad f\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} = \tanh\theta.$$

This implies that f is non-constant, and thus

$$\dim_{\mathbb{C}} \mathcal{O}(G/L) \neq 1.$$

In the second place, we remark that

$$K := \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & u \end{array} \right) \;\middle|\; \begin{array}{r} A \in U(2),\ u \in U(1), \\ (\det A)u = 1 \end{array} \right\} = S(U(2) \times U(1))$$

is a maximal compact subgroup of G whose Lie algebra $\mathfrak{k}$ contains the T, that

$$\ddot{\mathfrak{k}} := [\mathfrak{k}, \mathfrak{k}] = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{X} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 0 \end{array} \right) \;\middle|\; X \in \mathfrak{su}(2) \right\}, \quad \mathfrak{z}(\mathfrak{k}) = \operatorname{span}_{\mathbb{R}}\{W\}$$

where $W := \left(\begin{array}{cc|c} i/3 & 0 & 0 \\ 0 & i/3 & 0 \\ \hline 0 & 0 & -2i/3 \end{array} \right)$, and that the $\ddot{\mathfrak{k}}$-component $T_{\mathfrak{s}}$ and the $\mathfrak{z}(\mathfrak{k})$-component $T_{\mathfrak{z}}$ of the $T \in \mathfrak{k} = \ddot{\mathfrak{k}} \oplus \mathfrak{z}(\mathfrak{k})$ are

$$T_{\mathfrak{s}} = \left(\begin{array}{cc|c} i/2 & 0 & 0 \\ 0 & -i/2 & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad T_{\mathfrak{z}} = \left(\begin{array}{cc|c} i/2 & 0 & 0 \\ 0 & i/2 & 0 \\ \hline 0 & 0 & -i \end{array} \right),$$

respectively. We set $K_{\mathbb{C}} := C_{G_{\mathbb{C}}}(W)$, denote by $\ddot{K}_{\mathbb{C}}$ the connected Lie subgroup of $K_{\mathbb{C}}$ corresponding to the derived subalgebra $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$ of $\mathfrak{k}_{\mathbb{C}}$, and then have

$$\ddot{K}_{\mathbb{C}} = \left\{ \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \;\middle|\; \begin{array}{r} a, b, c, d \in \mathbb{C}, \\ ad - bc = 1 \end{array} \right\} \cong SL(2, \mathbb{C}),$$

$$\ddot{Q}^- := \ddot{K}_{\mathbb{C}} \cap Q^- = \left\{ \left(\begin{array}{cc|c} d_1 & 0 & 0 \\ c & d_2 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \;\middle|\; \begin{array}{r} d_1, d_2, c \in \mathbb{C}, \\ d_1 d_2 = 1 \end{array} \right\}.$$

In the third place, let $\mathsf{V} := \mathbb{C}$. Given integers n and m, we define holomorphic homomorphisms $\rho_{n,m} : Q^- \to GL(\mathsf{V})$ and $\ddot{\rho}_{n,m} : \ddot{Q}^- \to GL(\mathsf{V})$ by

$$\begin{aligned} \rho_{n,m}(q) &:= (d_1)^n (d_1 d_2)^m \,\mathrm{id}_{\mathsf{V}} \text{ for a } q = \begin{pmatrix} d_1 & 0 & 0 \\ b & d_2 & 0 \\ c & d & d_3 \end{pmatrix} \in Q^-, \\ \ddot{\rho}_{n,m} &:= \rho_{n,m}|_{\ddot{Q}^-}, \end{aligned} \tag{3.21}$$

set complex Cartan subgroups $H_{\mathbb{C}} \subset G_{\mathbb{C}}$ and $\ddot{H}_{\mathbb{C}} \subset \ddot{K}_{\mathbb{C}}$ as

$$\begin{aligned} H_{\mathbb{C}} &:= \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \,\middle|\, \begin{array}{r} d_1, d_2, d_3 \in \mathbb{C}, \\ d_1 d_2 d_3 = 1 \end{array} \right\}, \\ \ddot{H}_{\mathbb{C}} &:= \ddot{K}_{\mathbb{C}} \cap H_{\mathbb{C}}, \end{aligned}$$

and define holomorphic homomorphisms $\chi_{n,m} : H_{\mathbb{C}} \to \mathbb{C}^*$ and $\chi_n : \ddot{H}_{\mathbb{C}} \to \mathbb{C}^*$ by

$$\begin{aligned} \chi_{n,m}(y) &:= (d_1)^n (d_1 d_2)^m \text{ for a } y = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in H_{\mathbb{C}}, \\ \chi_n(z) &:= (d_1)^n \text{ for a } z = \left(\begin{array}{cc|c} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in \ddot{H}_{\mathbb{C}} \quad (\text{i.e., } \chi_n := \chi_{n,m}|_{\ddot{H}_{\mathbb{C}}}), \end{aligned}$$

respectively. Needless to say, the representation $\rho_{n,m} : Q^- \to GL(\mathsf{V})$ is irreducible.

In the last place, let us give a condition for $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = \infty$. Taking $T_{\mathfrak{s}} \in \ddot{\mathfrak{h}}_{\mathbb{C}}$ into account, we denote by $\Xi = \Xi(\ddot{\mathfrak{k}}_{\mathbb{C}}, \ddot{\mathfrak{h}}_{\mathbb{C}})$ the root system of $\ddot{\mathfrak{k}}_{\mathbb{C}}$ relative to $\ddot{\mathfrak{h}}_{\mathbb{C}}$, and set $\Xi^+ := \{\xi \in \Xi \,|\, \xi(-iT_{\mathfrak{s}}) > 0\}$. In this setting, one shows that χ_{n*} is a dominant integral form on $\ddot{\mathfrak{h}}_{\mathbb{C}}$ with respect to Ξ^+ if and only if $n \geq 0$, and that $\ddot{\rho}_{n,m*}(Z) = \chi_{n*}(Z)\,\mathrm{id}_{\mathsf{V}}$ for all $Z \in \ddot{\mathfrak{h}}_{\mathbb{C}}$. Accordingly Proposition 2.16 and Theorem 3.18, combined with $\mathfrak{l} \neq \mathfrak{k}$, assure that

> $n \geq 0$ if and only if $\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m}) \neq 0$ if and only if $\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m}) = \infty$.

Here m is any integer.

Incidentally,

- In case of $n \geq 0$ (and any m), one can define mappings $\phi_1 : \ddot{K}_{\mathbb{C}} \to \mathsf{V}$, and $\psi_1 : GQ^- \to \mathsf{V}$ by

$$\begin{aligned} \phi_1(w) &:= d^n \text{ for a } w = \left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in \ddot{K}_{\mathbb{C}}, \\ \psi_1(x) &:= (y_2 z_3 - y_3 z_2)^n (z_3)^m \text{ for an } x = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \in GQ^-, \end{aligned}$$

respectively; then $0 \neq \phi_1 \in \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m})$, $0 \neq \psi_1 \in \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m})$. Here we remark that the definitions above are well-defined by virtue of $n \geq 0$ and $z_3 \neq 0$, and that $\ddot{K}_{\mathbb{C}} \subset GQ^-$, $\phi_1 = \psi_1|_{\ddot{K}_{\mathbb{C}}}$.

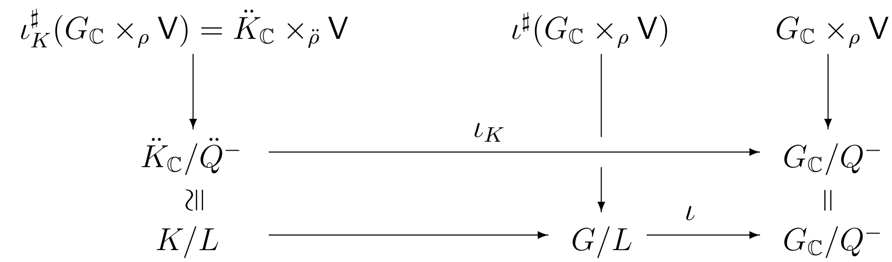
- Let $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ denote the root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$, and let $\triangle^+ := \{\alpha \in \triangle \,|\, \alpha(-iT) > 0\}$. Then, one shows that $\chi_{n,m*}$ is a dominant integral form on $\mathfrak{h}_{\mathbb{C}}$ with respect to $\triangle^+$ if and only if $n, m \geq 0$, and that $\rho_{n,m*}(Y) = \chi_{n,m*}(Y)\,\mathrm{id}_{\mathsf{V}}$ for all $Y \in \mathfrak{h}_{\mathbb{C}}$. Therefore, Proposition 2.16 assures that
$$n, m \geq 0 \text{ if and only if } \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m}) \neq 0.$$
- In case of $n, m \geq 0$, we can define a mapping $h_1 : G_{\mathbb{C}} \to \mathsf{V}$ by
$$h_1(a) := (b_2c_3 - b_3c_2)^n (c_3)^m \text{ for an } a = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in G_{\mathbb{C}},$$
and obtain $0 \neq h_1 \in \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m})$. Note here that c_3 can be zero.

By the arguments above and Weyl's dimension formula, we make Table 1 below.

Table 1

	$\dim_{\mathbb{C}} \mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m})$	$\dim_{\mathbb{C}} \mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m})$	$\dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m})$
$n < 0$ (and any m)	0	0	0
$n \geq 0$ and $m < 0$	$n+1$	∞	0
$n, m \geq 0$	$n+1$	∞	$\dfrac{(n+1)(m+1)(n+m+2)}{2}$

Here $G/L = SU(2,1)/S(U(1) \times U(1) \times U(1))$, $\mathsf{V} = \mathbb{C}$ and $\rho_{n,m}$ in (3.21). We refer to (3.14), (2.5) and (2.4) for $\mathcal{V}_{\ddot{K}_{\mathbb{C}}/\ddot{Q}^-}(\mathsf{V}, \ddot{\rho}_{n,m})$, $\mathcal{V}_{G/L}(\mathsf{V}, \rho_{n,m})$ and $\mathcal{V}_{G_{\mathbb{C}}/Q^-}(\mathsf{V}, \rho_{n,m})$, respectively.



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Nobutaka Boumuki: Division of Mathematical Sciences, Faculty of Science and Technology, Oita University, 700 Dannoharu, Oita-shi, Oita 870-1192, Japan
Email address: boumuki@oita-u.ac.jp