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Optimality Conditions for Quasiconvex Programming in Terms of Quasiconjugate Functions

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We study optimality conditions for quasiconvex programming in terms of quasiconjugate functions. By using Q-conjugate, we show a necessary and sufficient optimality condition for quasiconvex programming. Additionally, by using H-quasiconjugate, O-quasiconjugate, and R-quasiconjugate, we introduce optimality conditions for some kind of evenly quasiconvex objective functions. We investigate evenly convex sets and evenly quasiconvex functions, and continuity of quasiconvex functions.

Keywords: Quasiconvex programming, optimality condition, quasiconjugate function, subdifferential.

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1. Introduction

In this paper, we consider the following optimization problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

where f is an extended real-valued quasiconvex function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. In the research of optimization theory, optimality conditions play a central role. In particular, optimality conditions in terms of subdifferentials have been studied extensively, see [1, 5, 6, 7, 8, 9, 10, 13, 14, 15, 18, 19, 20, 21, 24, 27, 28, 31, 32, 33, 35] and references therein. One of the most important optimality conditions is the following condition for convex programming in terms of the subdifferential:

$$0 \in \partial f(x_0) + N_A(x_0),$$

where $\partial f(x_0)$ is the subdifferential of f at x_0, and $N_A(x_0)$ is the normal cone of A at x_0. By the convexity of the objective function, the above condition is a necessary and sufficient optimality condition for convex programming. The condition is a powerful tool for solving the problem. On the other hand, in quasiconvex programming, there are not many results on necessary and sufficient optimality conditions. Recently, the author introduce some necessary and sufficient optimality conditions for quasiconvex programming in terms of subdifferentials in [27, 28, 32, 33].

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In particular, by using Martínez-Legaz subdifferential in [13] and the epigraph of the support function, we show necessary and sufficient optimality conditions for general quasiconvex programming, see [33].

Other important concepts in the research of optimization theory are several types of conjugate functions, see [3, 4, 11, 12, 13, 15, 16, 17, 18, 21, 22, 23, 25, 26, 29, 30, 34, 35]. Among them, Fenchel conjugate in [3] plays an essential role in convex analysis and optimization. It is also closely related to the subdifferential. For example, the following characterization is well-known:

$$v \in \partial f(x_0) \text{ if and only if } f(x_0) + f^*(v) = \langle v, x_0 \rangle .$$

In quasiconvex programming, there is no conjugate that plays a decisive role like Fenchel conjugate, and some conjugates have been introduced and investigated. The idea of λ-quasiconjugate in [4] is closely related to Fenchel conjugate, and surrogate duality results in terms of λ-quasiconjugate are investigated. H-quasiconjugate in [34] and R-quasiconjugate in [35] are defined as special cases of λ-quasiconjugate, and duality problems in terms of these conjugates are introduced. For these conjugates, some subdifferentials are defined with reference to the relation between Fenchel conjugate and the subdifferential. In particular, Greenberg-Pierskalla subdifferential [4] and Martínez-Legaz subdifferential are related to Q-conjugate in [13] and λ-quasiconjugate. Hence, we can characterize these subdifferentials by using conjugates. In other words, optimality conditions in terms of subdifferentials can be characterized by these conjugates.

In this paper, we study optimality conditions for quasiconvex programming in terms of quasiconjugate functions. By using Q-conjugate, we show a necessary and sufficient optimality condition for quasiconvex programming. Additionally, by using H-quasiconjugate, O-quasiconjugate, and R-quasiconjugate, we introduce optimality conditions for some kind of evenly quasiconvex objective functions. We investigate evenly convex sets and evenly quasiconvex functions, and continuity of quasiconvex functions.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and fundamental results in previous studies. We show some properties of evenly quasiconvex functions. In Section 3, we investigate optimality conditions for quasiconvex programming in terms of quasiconjugate functions. We show necessary and sufficient optimality conditions by using four types of quasiconjugate functions. In Section 4, we discuss about our results. In particular, we study optimality conditions for R-evenly quasiconvex functions precisely.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The *inner product* of two vectors v and x in $\mathbb{R}^n$ is denoted by $\langle v, x \rangle$. Given a nonempty set A, we denote the *interior* of A by int$A$. The *normal cone* of A at $x \in A$ is denoted by

$$N_A(x) = \{v \in \mathbb{R}^n : \forall y \in A, \langle v, y - x \rangle \leq 0\}.$$

The *indicator function* δ_A of A is defined by

$$\delta_A(x) = \begin{cases} 0, & x \in A, \\ \infty, & \text{otherwise.} \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. The *epigraph* of f is defined as

$$\operatorname{epi} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\},$$

and f is said to be convex if $\operatorname{epi} f$ is convex. The *Fenchel conjugate* $f^* \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of $f \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined as

$$f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle v, x \rangle - f(x)\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\operatorname{lev}(f, \diamond, \alpha) = \{x \in \mathbb{R}^n : f(x) \diamond \alpha\}$$

for each $\alpha \in \mathbb{R}$. A function f is said to be quasiconvex if $\operatorname{lev}(f, \leq, \alpha)$ is convex for all $\alpha \in \mathbb{R}$. Any convex function is quasiconvex, but the converse is not true. For each $v \in \mathbb{R}^n$, we define a linear function $v(\cdot) := \langle v, \cdot \rangle$ from $\mathbb{R}^n$ to $\mathbb{R}$. We define the following families of open half spaces:

$$\begin{aligned} H &= \{\operatorname{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}\}, \\ H^+ &= \{\operatorname{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha > 0\}, \\ H^0 &= \{\operatorname{lev}(v, <, 0) : v \in \mathbb{R}^n\}, \\ H^- &= \{\operatorname{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha < 0\}. \end{aligned}$$

Clearly, $H = H^+ \cup H^0 \cup H^-$. A subset A of $\mathbb{R}^n$ is said to be evenly (H-evenly, O-evenly, and R-evenly) convex if it is the intersection of a subfamily of H (H^+, H^0, H^-, respectively). We define the whole space and the empty set is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex by convention. We can check easily that the following statements hold:

- an evenly convex set is convex,
- an open or closed convex set is evenly convex,
- a nonempty set A is H-evenly convex if and only if A is evenly convex and contains 0,
- if A is O-evenly convex, then for each $x \in A$ and $t > 0$, $tx \in A$,
- if A is R-evenly convex, then for each $x \in A$ and $t \geq 1$, $tx \in A$.

A function f is said to be evenly (H-evenly, O-evenly, and R-evenly) quasiconvex if $\operatorname{lev}(f, \leq, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$. We can check easily that the following statements hold:

- an evenly quasiconvex function is quasiconvex,
- a lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex,
- a function f is H-evenly quasiconvex if and only if f is evenly quasiconvex and $f(0) = \min_{x \in \mathbb{R}^n} f(x)$, see [30],
- if f is O-evenly quasiconvex or R-evenly quasiconvex, then, $f(0) = \max_{x \in \mathbb{R}^n} f(x)$.

Additionally, we show the following proposition on evenly quasiconvexity.

Proposition 2.1. *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Then the following statements hold:*

(i) *if f is upper semicontinuous (usc) quasiconvex, then f is evenly quasiconvex,*

(ii) *if f is O-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t > 0$, $f(x) = f(tx)$,*

(iii) *if f is R-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t \geq 1$, $f(x) \geq f(tx)$.*

Proof. (i) Assume that f is usc quasiconvex. Then, $\mathrm{lev}(f, <, \alpha)$ is open convex for each $\alpha \in \mathbb{R}$. By using the following equation,

$$\mathrm{lev}(f, \leq, \alpha) = \bigcap_{\varepsilon > 0} \mathrm{lev}(f, <, \alpha + \varepsilon),$$

$\mathrm{lev}(f, \leq, \alpha)$ is evenly convex for each $\alpha \in \mathbb{R}$.

(ii) Assume that f is O-evenly quasiconvex. Take $x \in \mathbb{R}^n$ and let $t > 0$. Since $\mathrm{lev}(f, \leq, f(x))$ is O-evenly convex, $tx \in \mathrm{lev}(f, \leq, f(x))$, that is, $f(tx) \leq f(x)$. Similarly, "$tx \in \mathrm{lev}(f, \leq, f(tx))$ and $\frac{1}{t} > 0$" implies that $x \in \mathrm{lev}(f, \leq, f(tx))$. This shows that $f(x) = f(tx)$.

(iii) Assume that f is R-evenly quasiconvex, and let $x \in \mathbb{R}^n$ and $t \geq 1$. Since $\mathrm{lev}(f, \leq, f(x))$ is R-evenly convex, $tx \in \mathrm{lev}(f, \leq, f(x))$, that is, $f(tx) \leq f(x)$. This completes the proof. □

For more details on evenly convex sets and evenly quasiconvex functions, see Section 4 and [2, 25, 26, 30, 35].

In quasiconvex analysis, various types of subdifferentials have been introduced, see [4, 5, 9, 13, 14, 18, 19, 20, 27, 28, 31, 32, 33, 35]. In this paper, we use the following Martínez-Legaz subdifferential:

$$\partial^M f(x_0) = \{(v, t) \in \mathbb{R}^{n+1} : \inf\{f(x) : \langle v, x \rangle \geq t\} \geq f(x_0), \langle v, x_0 \rangle \geq t\}.$$

It is introduced by Martínez-Legaz in [13] as a special case of c-subdifferential in Moreau's generalized conjugation in [15]. In [33], we show the following necessary and sufficient optimality condition in terms of Martínez-Legaz subdifferential.

Theorem 2.2. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x \in A} f(x)$,

(ii) $0 \in \partial^M f(x_0) + \mathrm{epi}\, \delta_A^*$.

In this paper, we use the following four quasiconjugates for quasiconvex functions. The Q-conjugate of f in [13] is the function $f^Q \colon \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$ such that for each $(v, t) \in \mathbb{R}^n \times \mathbb{R}$ we have $f^Q(v, t) = -\inf\{f(x) \mid \langle v, x \rangle \geq t\}$.

The H-quasiconjugate of f in [34] is the function $f^H : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^H(v) = \begin{cases} -\inf\{f(x) \mid \langle v, x \rangle \geq 1\}, & v \neq 0, \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\}, & v = 0. \end{cases}$$

The O-quasiconjugate of f in [16] is the function $f^O : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that for each $v \in \mathbb{R}^n$ we have

$$f^O(v) = -\inf\{f(x) \mid \langle v, x \rangle \geq 0\}.$$

The R-quasiconjugate of f in [35] is the function $f^R:\mathbb{R}^n \to \overline{\mathbb{R}}$ such that we have for each $v\in\mathbb{R}^n$,

$$f^R(v) = -\inf\{f(x) \mid \langle v, x\rangle \geq -1\}.$$

Although Q-conjugate and O-quasiconjugate are defined by other names and notations, f^c and f^*, in these literature, we denote f^Q and f^O for the sake of distinction.

3. Optimality conditions in terms of quasiconjugate functions

In this section, we show optimality conditions for quasiconvex programming in terms of quasiconjugate functions. By using four types of quasiconjugates, we show necessary and sufficient optimality conditions.

At first, we show a necessary and sufficient optimality condition in terms of Q-conjugate.

Theorem 3.1. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x\in A} f(x)$,

(ii) *there exists $(v,t) \in \mathbb{R}^{n+1}$ such that $f(x_0) + f^Q(v,t) = 0$ and $\inf_{x\in A} \langle v, x\rangle \geq t$.*

Proof. Assume that x_0 is a global minimizer of f in A. By Theorem 2.2,

$$0 \in \partial^M f(x_0) + \operatorname{epi} \delta_A^*.$$

Hence, there exists $(v,t)\in\partial^M f(x_0)$ such that $-(v,t)\in\operatorname{epi}\delta_A^*$. Since $(v,t)\in\partial^M f(x_0)$,

$$\inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0),$$

and $\langle v, x_0\rangle \geq t$. Hence $-f^Q(v,t) = \inf\{f(x) : \langle v, x\rangle \geq t\} = f(x_0)$.

Furthermore, $\langle -v, x\rangle \leq -t$ for each $x \in A$ since $-(v,t) \in \operatorname{epi}\delta_A^*$. This shows that (ii) holds.

Conversely, assume that (ii) holds. Then, we can check that

$$f(x_0) = -f^Q(v,t) = \inf\{f(x) : \langle v, x\rangle \geq t\}, \quad \langle v, x_0\rangle \geq t,$$
$$\delta_A^*(-v) = \sup_{x\in A} \langle -v, x\rangle \leq -t.$$

This shows that $(v,t) \in \partial^M f(x_0)$ and $-(v,t) \in \operatorname{epi}\delta_A^*$, that is, $0 \in \partial^M f(x_0) + \operatorname{epi}\delta_A^*$. By Theorem 2.2, (i) holds. This completes the proof. □

In the following theorem, we show a necessary and sufficient optimality condition for H-evenly quasiconvex functions in terms of H-quasiconjugate. The assumption "$\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$" will be mentioned in Section 4.4.

Theorem 3.2. *Let f be an usc H-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x\in A} f(x)$,

(ii) *there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^H(v) = 0$ and $\inf_{x\in A} \langle v, x\rangle \geq 1$.*

Proof. Assume that x_0 is a global minimizer of f in A. By Theorem 2.2,

$$0 \in \partial^M f(x_0) + \operatorname{epi} \delta_A^*.$$

Hence, there exists $(v,t) \in \partial^M f(x_0)$ such that $-(v,t) \in \operatorname{epi} \delta_A^*$. If $v = 0$, then

$$0 = \langle v, x_0 \rangle \geq t,$$

$$\inf_{x \in \mathbb{R}^n} f(x) = \inf\{f(x) : \langle v, x \rangle \geq t\} \geq f(x_0) = \inf_{x \in A} f(x).$$

This is a contradiction. Hence $v \neq 0$.

By the definition of Martínez-Legaz subdifferential, for each $x \in \mathbb{R}^n$,

$$f(x_0) > f(x) \text{ implies } \langle v, x \rangle < t.$$

Since f is H-evenly quasiconvex, 0 is a global minimizer of f in $\mathbb{R}^n$. By assumption,

$$f(x_0) = \inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x) = f(0).$$

This shows that $t > 0$.

By a similar way as in the proof of Theorem 3.1,

$$\begin{aligned} f(x_0) &= -f^Q(v,t) = \inf\{f(x) : \langle v, x \rangle \geq t\} \\ &= \inf\left\{f(x) : \left\langle \frac{v}{t}, x \right\rangle \geq 1\right\} = -f^H\left(\frac{v}{t}\right). \end{aligned}$$

Since $-(v,t) \in \operatorname{epi} \delta_A^*$ we have $\inf_{x \in A} \left\langle \frac{v}{t}, x \right\rangle \geq 1$.

Hence, $f(x_0) + f^H\left(\frac{v}{t}\right) = 0$ and $\inf_{x \in A} \left\langle \frac{v}{t}, x \right\rangle \geq 1$, that is, (ii) holds.

Assume that (ii) holds. Since $\inf_{x \in A} \langle v, x \rangle \geq 1$, $v \neq 0$. We can easily check that $f^H(v) = f^Q(v, 1)$. Hence, by Theorem 3.1, (i) holds. This completes the proof. □

In the following theorem, we show a necessary and sufficient optimality condition for O-evenly quasiconvex functions in terms of O-quasiconjugate.

Theorem 3.3. *Let f be an usc O-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x \in A} f(x)$,

(ii) *there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^O(v) = 0$ and $\inf_{x \in A} \langle v, x \rangle \geq 0$.*

Proof. Assume that x_0 is a global minimizer of f in A. By Theorem 2.2,

$$0 \in \partial^M f(x_0) + \operatorname{epi} \delta_A^*.$$

Hence, there exists $(v,t) \in \partial^M f(x_0)$ such that $-(v,t) \in \operatorname{epi} \delta_A^*$. Then,

$$\langle v, x \rangle \geq t \text{ implies } f(x_0) \leq f(x), \text{ and } \inf_{x \in A} \langle v, x \rangle \geq t.$$

If $v = 0$, then it is clear that $\inf_{x \in A} \langle v, x\rangle \geq 0$. Additionally, $0 = \langle v, x_0\rangle \geq t$, and

$$\begin{aligned}\inf_{x \in \mathbb{R}^n} f(x) &= \inf\{f(x) : \langle v, x\rangle \geq 0\} = -f^O(v) \\ &\geq \inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0) \geq \inf_{x \in \mathbb{R}^n} f(x),\end{aligned}$$

that is, $f(x_0) = -f^O(v)$. This shows that (ii) holds.

Assume that $v \neq 0$ and $t < 0$. Let $x \in \mathbb{R}^n$ satisfy $f(x_0) > f(x)$, then $\langle v, x\rangle < t < 0$. By Proposition 2.1, for each $\lambda > 0$, we have $f(x) = f(\lambda x)$. For sufficiently small λ, $\langle v, \lambda x\rangle \geq t$. This implies

$$f(x_0) \leq f(\lambda x) = f(x) < f(x_0).$$

This is a contradiction. Hence $t \geq 0$.

Let $x \in \mathbb{R}^n$ satisfying $\langle v, x\rangle > 0$. Then, for sufficiently large λ, $\langle v, \lambda x\rangle \geq t$. This shows that $f(x) = f(\lambda x) \geq f(x_0)$. Let $x \in \mathbb{R}^n$ satisfying $\langle v, x\rangle = 0$. Then $x + \frac{1}{k}v$ converges to x and $f(x + \frac{1}{k}v) \geq f(x_0)$. By the upper semicontinuity of f we deduce $f(x) \geq f(x_0)$. This shows that

$$-f^O(v) = \inf\{f(x) : \langle v, x\rangle \geq 0\} = f(x_0).$$

Additionally, $\inf_{x \in A} \langle v, x\rangle \geq t \geq 0$. Hence (ii) holds.

Assume that (ii) holds. We can easily check that $f^O(v) = f^Q(v, 0)$. Hence, by Theorem 3.1, (i) holds. This completes the proof. □

In the following theorem, we show a necessary and sufficient optimality condition for R-evenly quasiconvex functions in terms of O-quasiconjugate and R-quasiconjugate.

Theorem 3.4. *Let f be an usc R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statement* (i) *holds if and only if one of* (ii) *or* (iii) *holds:*

(i) $f(x_0) = \min_{x \in A} f(x)$,

(ii) *there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^O(v) = 0$ and $\inf_{x \in A} \langle v, x\rangle \geq 0$,*

(iii) *there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^R(v) = 0$ and $\inf_{x \in A} \langle v, x\rangle \geq -1$.*

Proof. Assume that x_0 is a global minimizer of f in A. By Theorem 2.2,

$$0 \in \partial^M f(x_0) + \operatorname{epi} \delta_A^*.$$

Hence, there exists $(v, t) \in \partial^M f(x_0)$ such that $-(v, t) \in \operatorname{epi} \delta_A^*$. Then,

$$\langle v, x\rangle \geq t \text{ implies } f(x_0) \leq f(x),$$

and $\inf_{x \in A} \langle v, x\rangle \geq t$. If $v = 0$, then we can prove that (ii) holds by the similar way in the proof of Theorem 3.3. Additionally, if $v \neq 0$ and $t < 0$, then we can easily check that (iii) holds for $-\frac{v}{t}$.

Assume that $v \neq 0$ and $t \geq 0$. If there exists $x \in \mathbb{R}^n$ such that $\langle v, x\rangle > 0$ and $f(x_0) > f(x)$, then for sufficiently large λ, $\langle v, \lambda x\rangle \geq t$ and

$$f(x_0) > f(x) \geq f(\lambda x) \geq f(x_0),$$

by Proposition 2.1. This is a contradiction. Hence,

$$\langle v, x\rangle > 0 \text{ implies } f(x_0) \leq f(x).$$

By the similar way in the proof of Theorem 3.3, we can show that

$$\langle v, x\rangle \geq 0 \text{ implies } f(x_0) \leq f(x)$$

since f is usc and $v \neq 0$. This shows that (ii) holds.

Assume that (ii) or (iii) holds. We can easily check that $f^O(v) = f^Q(v, 0)$ and $f^R(v) = f^Q(v, -1)$. Hence, by Theorem 3.1, (i) holds. This completes the proof. □

4. Discussion

In this section, we discuss our results. We investigate evenly convex sets and evenly quasiconvex functions, and continuity of quasiconvex functions. Additionally, we study optimality conditions for R-evenly quasiconvex functions in Theorem 3.4 precisely.

4.1. Continuity and evenly quasiconvexity

Let f be a continuous function, then $\mathrm{lev}(f, <, \alpha)$ is open and $\mathrm{lev}(f, \leq, \alpha)$ is closed for each $\alpha \in \mathbb{R}$. In Section 2, we show that an usc or lsc quasiconvex function is evenly quasiconvex. Hence, continuity and evenly quasiconvexity are closely related. In this section, we define other evenly quasiconvexity of functions. A function f is said to be strictly evenly (H-evenly, O-evenly, and R-evenly) quasiconvex if $\mathrm{lev}(f, <, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$. We can check easily that the following statements hold:

(i) a strictly evenly (H-evenly, O-evenly, and R-evenly) quasiconvex function is evenly (H-evenly, O-evenly, and R-evenly, respectively) quasiconvex,

(ii) an usc quasiconvex function is strictly evenly quasiconvex.

The converse of the above statement (i) is not generally true. Actually, there is an evenly quasiconvex function f which is not strictly evenly quasiconvex, see [30]. We summarize relations between continuity and evenly quasiconvexity.

$$f \text{ usc quasiconvex}$$
$$\Longrightarrow f \text{ strictly evenly quasiconvex} \Longrightarrow f \text{ evenly quasiconvex} \Longrightarrow f \text{ quasiconvex}.$$

$$f \text{ lsc quasiconvex}$$
$$\Longrightarrow f \text{ evenly quasiconvex} \Longrightarrow f \text{ quasiconvex}.$$

4.2. Open halfspace and evenly convexity

Evenly convex sets are characterized by open halfspaces. Conversely, we investigate evenly convexity of open half spaces in this section. It is clear that the following statements hold:

- if $t > 0$, $\mathrm{lev}(v, <, t)$ and $\mathrm{lev}(v, \leq, t)$ are H-evenly convex,
- $\mathrm{lev}(v, <, 0)$ is O-evenly convex and $\mathrm{lev}(v, \leq, 0)$ is H-evenly convex,
- if $t < 0$, $\mathrm{lev}(v, <, t)$ and $\mathrm{lev}(v, \leq, t)$ are R-evenly convex.

It is worth noting that $\mathrm{lev}(v, \leq, 0)$ is H-evenly convex. Actually,

$$\mathrm{lev}(v, \leq, 0) = \bigcap_{\varepsilon > 0} \mathrm{lev}(v, <, \varepsilon).$$

Hence $\mathrm{lev}(v, \leq, 0)$ is H-evenly convex.

4.3. R-evenly quasiconvexity

In Section 4.1, we define strictly R-evenly quasiconvex functions. In this section, we show the following examples.

Example 4.1. The following function f_1 on $\mathbb{R}$ is R-evenly quasiconvex, but not strictly R-evenly quasiconvex:

$$f_1(x) = \begin{cases} 0, & x \leq 0, \\ -x, & x > 0. \end{cases}$$

Actually,
$$\mathrm{lev}(f_1, \leq, \alpha) = \begin{cases} \mathbb{R}, & \alpha \geq 0, \\ \{x : x \geq -\alpha\}, & \alpha < 0. \end{cases}$$

Hence, $\mathrm{lev}(f_1, \leq, \alpha)$ is R-evenly convex for each $\alpha \in \mathbb{R}$. However,

$$\mathrm{lev}(f_1, <, 0) = \{x : x > 0\}$$

is not R-evenly convex. Hence f_1 is not strictly R-evenly quasiconvex. □

Example 4.2. The following function f_2 on $\mathbb{R}$ is strictly R-evenly quasiconvex:

$$f_2(x) = \begin{cases} 0, & x \leq 1, \\ -x + 1, & x > 1. \end{cases}$$

Actually,

$$\mathrm{lev}(f_1, <, \alpha) = \begin{cases} \mathbb{R}, & \alpha > 0, \\ \{x : x > -\alpha + 1\}, & \alpha \leq 0. \end{cases}$$

Since

$$\{x : x > -\alpha + 1\} = \{x : \langle -1, x \rangle < \alpha - 1\},$$

$\mathrm{lev}(f_1, <, \alpha)$ is R-evenly convex for each $\alpha \leq 0$. This shows that f_2 is strictly R-evenly quasiconvex. □

4.4. Optimality condition for H-evenly quasiconvex objective function

In Theorem 3.2, we assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$. If the assumption does not hold, then the statement (ii) in Theorem 3.2 is not a necessary optimality condition. Actually, we need "$\inf_{x\in A} f(x) > f(0)$", see the following example.

Example 4.3. Let f on $\mathbb{R}$ be the following function:

$$f(x) = \begin{cases} \sqrt{x-1}, & x \geq 1, \\ 0, & -1 \leq x \leq 1, \\ \sqrt{|x+1|}, & x \leq -1, \end{cases}$$

and $A = [-1, 1]$. Then, f is H-evenly quasiconvex, A is convex, $\inf_{x\in A} f(x) = \inf_{x\in\mathbb{R}^n} f(x)$, and $x_0 = -1$ is a global minimizer of f in A.
Since $0 \in A$, there does not exist $v \in \mathbb{R}^n$ such that $\inf_{x\in A} \langle v, x\rangle \geq 1$.
On the other hand, let $A' = [1, 2]$ and $v = 1$, then $x_0' = 1$ is a global minimizer of f in A', $f(x_0) + f^H(v) = 0$ and $\inf_{x\in A} \langle v, x\rangle \geq 1$. □

4.5. Optimality condition for R-evenly quasiconvex objective function

In Theorem 3.4, we show that x_0 is a global minimizer if and only if one of the following statements holds:

(O) there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^O(v) = 0$ and $\inf_{x\in A} \langle v, x\rangle \geq 0$,

(R) there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^R(v) = 0$ and $\inf_{x\in A} \langle v, x\rangle \geq -1$.

It is difficult to prove that x_0 is a global minimizer if and only if (R) holds for R-evenly quasiconvex objective function. We show the following examples.

Example 4.4. Consider a minimization problem of the following f_1 on $(-\infty, 0]$.

$$f_1(x) = \begin{cases} 0, & x \leq 0, \\ -x, & x > 0. \end{cases}$$

Then $x_0 = 0$ is a global minimizer and (O) holds for $v = -1$. Additionally, (R) does not hold. Actually, assume that $\inf_{x\in A} \langle v, x\rangle \geq -1$, then and $v \leq 0$. If $v < 0$, then

$$-f^R(v) = \inf\{f(x) : \langle v, x\rangle \geq -1\} = \frac{1}{v} < 0 = f(x_0).$$

If $v = 0$, then

$$-f^R(v) = \inf\{f(x) : \langle 0, x\rangle \geq -1\} = \inf_{x\in\mathbb{R}^n} f(x) = -\infty < f(x_0).$$

This shows that $f(x_0) + f^R(v) \neq 0$. □

Example 4.5. Consider a minimization problem of the following f_2 on $(-\infty, 1]$.

$$f_2(x) = \begin{cases} 0, & x \le 1, \\ -x+1, & x > 1. \end{cases}$$

Then $x_0 = 1$ is a global minimizer and (R) holds for $v = -1$. □

By these examples, it can be expected that (R) may be a necessary optimality condition if the objective function is strictly R-evenly quasiconvex. However, we show the following counterexample.

Example 4.6. Let f_3 be the following function on $\mathbb{R}^2$:

$$f_3(x) = \begin{cases} 0, & x \in S = \{(y_1, y_2) : y_1 > 0, y_1 y_2 > 1\}, \\ 1, & x \notin S, x_1 + x_2 > \frac{1}{2}, \\ 2, & \text{otherwise}. \end{cases}$$

We can check easily that f_3 is usc, and strictly R-evenly quasiconvex. Consider a minimization problem of f_3 on $A = \{(x_1, x_2) : x_2 \le 0\}$. Then $x_0 = (1, 0)$ is a global minimizer, (O) holds for $v = (0, -1)$ and (R) does not hold. Actually, let $v = (v_1, v_2) \in \mathbb{R}^2$ satisfying $\inf_{x \in A} \langle v, x \rangle \ge -1$, then $v_1 = 0$ and $v_2 \le 0$. If $v_2 < 0$, then

$$-f^R(v) = \inf\{f(x) : \langle v, x \rangle \ge -1\} = 0 < 1 = f(x_0).$$

If $v = 0$, then

$$-f^R(v) = \inf\{f(x) : \langle 0, x \rangle \ge -1\} = \inf_{x \in \mathbb{R}^n} f(x) = 0 < 1 = f(x_0).$$

This shows that $f(x_0) + f^R(v) \ne 0$. □

On the other hand, if $0 \in \text{int}A$ and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$, then (R) is a necessary and sufficient optimality condition, see the following theorem.

Theorem 4.7. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $0 \in \text{int}A$ and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x \in A} f(x)$,

(R) *there exists $v \in \mathbb{R}^n$ such that $f(x_0) + f^R(v) = 0$ and $\inf_{x \in A} \langle v, x \rangle \ge -1$.*

Proof. Assume that x_0 is a global minimizer of f in A. By Theorem 2.2, there exists (v, t) such that

$$\langle v, x \rangle \ge t \text{ implies } f(x_0) \le f(x),$$

and $\inf_{x \in A} \langle v, x \rangle \ge t$. If $v = 0$, then $0 = \langle v, x_0 \rangle \ge t$. Hence

$$\inf_{x \in \mathbb{R}^n} f(x) = \inf\{f(x) : \langle v, x \rangle \ge t\} = f(x_0) = \inf_{x \in A} f(x).$$

This is a contradiction. Since $0 \in \mathrm{int}A$ and $v \neq 0$, $t < 0$. This shows that (R) holds for $-\frac{v}{t}$.

Conversely, since $f^Q(v,-1) = f^R(v)$, we can show that (R) implies (i). This completes the proof. □

We do not need R-evenly quasiconvexity of f in Theorem 4.7. Sufficient conditions for equivalency of (i) and (R) for R-evenly quasiconvex objective functions may be investigated in future research.

5. Conclusion

In this paper, we study optimality conditions for quasiconvex programming in terms of four types of quasiconjugate functions. By using the Q-conjugate, we show a necessary and sufficient optimality condition in Theorem 3.1. In Theorems 3.2–3.4 we introduce optimality conditions for three types of evenly quasiconvex objective functions. In Section 4, we discuss our results and evenly convexity. In particular, we study optimality conditions for R-evenly quasiconvex functions precisely in Subsection 4.5. Even if the objective function is R-evenly quasiconvex, we cannot always characterize a global minimizer by the R-quasiconjugate. In this case, we need not only the R-quasiconjugate but also the O-quasiconjugate.

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