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Quasiconjugate duality and optimality conditions for quasiconvex optimization

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Abstract

In nonlinear optimization, conjugate functions and subdifferentials play an essential role. In particular, Fenchel conjugate is the most well known conjugate function in convex optimization. In quasiconvex optimization, extra parameters for quasiconjugate functions have been introduced in order to show duality theorems, for example $\boldsymbol{\lambda}$-quasiconjugate and $\boldsymbol{\lambda}$-semiconjugate. By these extra parameters, we can show duality results that hold for general quasiconvex objective functions. On the other hand, extra parameters usually increase the complexity of dual problems. Hence, conjugate functions without extra parameters have also been investigated, for example $\boldsymbol{H}$-quasiconjugate, $\boldsymbol{R}$-quasiconjugate, and so on. However, there are some open problems.

In this paper, we study quasiconjugate duality and optimality conditions for quasiconvex optimization without extra parameters. We investigate three types of quasiconjugate dual problems, and show sufficient conditions for strong duality. We introduce three types of quasi-subdifferentials, and study optimality conditions and characterizations of the solution set. Additionally, we give a classification of quasiconvex optimization problems in terms of quasiconjugate duality.

Keywords: quasiconvex optimization, quasiconjugate function, quasi-subdifferential, strong duality, optimality condition

1 Introduction

In quasiconvex optimization, various concepts of conjugate functions and subdifferentials have been investigated, see [6–27] and references therein. For a convex function

f, its epigraph

$$\text{epi}f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}.$$

is convex. On the other hand, for a quasiconvex function, its epigraph is not always convex, but its level set

$$\text{lev}(f, \leq, \alpha) = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is convex. This means that convex functions have $n+1$-dimensional convexity, but quasiconvex functions have only n-dimensional convexity. Hence, for general quasiconvex optimization problems, extra parameters for quasiconjugate functions have been introduced in order to compensate for this lack. Typical examples of extra parameters are λ in λ-quasiconjugate [6] and λ-semiconjugate [15, 16], and t in Q-conjugate [7]. By these extra parameters, we can show duality results that hold for general quasiconvex objective functions, for example evenly quasiconvex, lower semicontinuous (lsc) or upper semicontinuous (usc) quasiconvex functions. On the other hand, extra parameters usually increase the complexity of dual problems. Hence, conjugate functions without extra parameters have also been investigated, for example H-quasiconjugate [25], O-quasiconjugate [12], and R-quasiconjugate [26]. Additionally, in [27], we introduce conjugate dual problems without extra parameters, and show strong duality theorems. However, there are some open problems. For example, two dual problems are required for R-evenly quasiconvex objective functions, and characterizations of the solution set in terms of quasi-subdifferentials have not been investigated yet.

Hence, in this paper, we study quasiconjugate duality and optimality conditions for quasiconvex optimization without extra parameters. We investigate three types of quasiconjugate dual problems, and show sufficient conditions for strong duality. We introduce three types of quasi-subdifferentials, and study optimality conditions and characterizations of the solution set. Additionally, we give a classification of quasiconvex optimization problems in terms of quasiconjugate duality.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study quasiconjugate duality and optimality conditions for quasiconvex optimization problem. We show strong duality, optimality conditions, and characterizations of the solution set under three types of evenly quasiconvexity. In Section 4, we give a classification of quasiconvex optimization problems in terms of quasiconjugate duality.

2 Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The inner product of two vectors v and x in $\mathbb{R}^n$ is denoted by $\langle v, x\rangle$. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & otherwise. \end{cases}$$

The normal cone of A at $x_0 \in A$ is denoted by

$$N_A(x_0) = \{v \in \mathbb{R}^n : \forall x \in A, \langle v, x - x_0\rangle \leq 0\}.$$

We denote the closure, the boundary and the interior of A by $\mathrm{cl}A$, $\mathrm{bd}A$, and $\mathrm{int}A$, respectively. We define the following families of open half spaces:

$$\begin{aligned} H &= \{\mathrm{lev}(v,<,\alpha) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}\}, \\ H^+ &= \{\mathrm{lev}(v,<,\alpha) : v \in \mathbb{R}^n, \alpha > 0\}, \\ H^0 &= \{\mathrm{lev}(v,<,0) : v \in \mathbb{R}^n\}, \\ H^- &= \{\mathrm{lev}(v,<,\alpha) : v \in \mathbb{R}^n, \alpha < 0\}, \end{aligned}$$

where $\mathrm{lev}(v,<,\alpha) = \{x \in \mathbb{R}^n : \langle v, x\rangle < \alpha\}$. A subset A of $\mathbb{R}^n$ is said to be evenly (H-evenly, O-evenly, and R-evenly) convex if it is the intersection of a subfamily of H (H^+, H^0, H^-, respectively). We define the whole space and the empty set is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex by convention. It is clear that an evenly convex set is convex, and an open or closed convex set is evenly convex. Additionally, the following statements hold:

- if $A \subset \mathbb{R}^n$ is non-empty, then A is H-evenly convex if and only if A is evenly convex and contains 0,
- if A is O-evenly convex, then for each $x \in A$ and $t > 0$, $tx \in A$,
- if A is R-evenly convex, then for each $x \in A$ and $t \geq 1$, $tx \in A$.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. A function f is said to be convex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y).$$

A function f is convex if and only if $\mathrm{epi}f$ is convex. The Fenchel conjugate [4, 5] of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as

$$f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle v, x\rangle - f(x)\}.$$

A function f is said to be quasiconvex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \leq \max\{f(x), f(y)\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\mathrm{lev}(f, \diamond, \alpha) = \{x \in \mathbb{R}^n : f(x) \diamond \alpha\}$$

for each $\alpha \in \mathbb{R}$. It is well known that f is quasiconvex if and only if $\mathrm{lev}(f, \leq, \alpha)$ is convex for all $\alpha \in \mathbb{R}$. A function f is said to be upper semicontinuous (usc) if $\mathrm{lev}(f, <, \alpha)$ is open for all $\alpha \in \mathbb{R}$. A function f is said to be evenly (H-evenly, O-evenly, and R-evenly) quasiconvex if $\mathrm{lev}(f, \leq, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$. We can check easily that an evenly quasiconvex function is quasiconvex, and a lsc or usc quasiconvex function is evenly quasiconvex. In addition, the following statements hold:

- f is H-evenly quasiconvex if and only if f is evenly quasiconvex and $0 \in \mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$,
- if f is O-evenly quasiconvex or R-evenly quasiconvex, then $0 \in \mathbb{R}^n$ is a global maximizer of f in $\mathbb{R}^n$,
- if f is O-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t > 0$, $f(x) = f(tx)$,
- if f is R-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t \geq 1$, $f(x) \geq f(tx)$.

Various results for evenly convex sets and evenly quasiconvex functions have been investigated, see [2, 3, 5, 10, 17, 18, 21, 22, 26, 27] and references therein.

A function f is said to be essentially quasiconvex if f is quasiconvex and each local minimizer x of f in $\mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$. All convex functions are essentially quasiconvex, and a real-valued continuous quasiconvex function f is essentially quasiconvex if and only if f is semistrictly quasiconvex, see [1]. A function f is said to be essentially H-evenly quasiconvex if f is essentially quasiconvex and H-evenly quasiconvex. If a non-constant function f is O or R-evenly quasiconvex, then f is not essentially quasiconvex, see the following example.

Example 1. *Let $\alpha \in \mathbb{R}$ and f_α be the following function:*

$$f_\alpha(x) = \begin{cases} 0 & x \geq \alpha, \\ -1 & x < \alpha. \end{cases}$$

We can check that f_0 is O-evenly quasiconvex and f_{-1} is R-evenly quasiconvex. Let $x_0 = 1$, then x_0 is a local minimizer of f_0 and f_{-1} in $\mathbb{R}^n$ and is not a global minimizer of f_0 and f_{-1} in $\mathbb{R}^n$. Hence, f_0 and f_{-1} are not essentially quasiconvex.

Therefore, we define essential quasiconvexity for O-evenly and R-evenly quasiconvex functions in another way. A function f is essentially O-evenly (R-evenly) quasiconvex if f is O-evenly (R-evenly, respectively) quasiconvex, and each local minimizer x of f in $\mathbb{R}^n$ satisfying $f(x) < \sup_{y \in \mathbb{R}^n} f(y)$ is a global minimizer of f in $\mathbb{R}^n$. We can check that f_0 (f_{-1}) in Example 1 is essentially O-evenly (R-evenly, respectively) quasiconvex.

In quasiconvex optimization, various types of quasiconjugate functions have been investigated. Q-conjugate [7] of f is defined as follows:

$$f^Q(v,t) = -\inf\{f(x) : \langle v, x\rangle \geq t\}.$$

H-quasiconjugate [25] of f is defined as follows:

$$f^H(v) = \begin{cases} -\inf\{f(x) : \langle v, x\rangle \geq 1\}, & v \neq 0, \\ -\sup\{f(x) : x \in \mathbb{R}^n\}, & v = 0. \end{cases}$$

O-quasiconjugate [12] of f is defined as follows:

$$f^O(v) = -\inf\{f(x) : \langle v, x\rangle \geq 0\}.$$

R-quasiconjugate [26] of f is defined as follows:

$$f^R(v) = -\inf\{f(x) : \langle v, x\rangle \geq -1\}.$$

Although Q-conjugate and O-quasiconjugate are defined by other names, we denote f^Q and f^O for the sake of distinction.

In this paper, we consider the following quasiconvex optimization problem (P):

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } x\in A,\end{cases}$$

where f is an extended real-valued quasiconvex function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. In [20], we define the following dual problem (D) in terms of Q-conjugate for the primal problem (P):

$$(D)\begin{cases}\text{minimize } f^Q(v,t),\\ \text{subject to } (v,t)\in -\text{epi}\delta_A^*.\end{cases}$$

We denote

$$\text{val}(P)=\inf_{x\in A} f(x),\ \text{and}\ \text{val}(D)=\inf_{(v,t)\in-\text{epi}\delta_A^*} f^Q(v,t).$$

In [20], we show the following strong duality theorem.

Theorem 1. *[20] Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\text{val}(P)=-\text{val}(D).$$

Assume that f is an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A is a nonempty convex subset of $\mathbb{R}^n$, and $\alpha=\text{val}(P)>\inf_{x\in\mathbb{R}^n} f(x)$. We can check that $\text{lev}(f,<,\alpha)$ is nonempty open convex set, and $A\cap\text{lev}(f,<,\alpha)$ is empty. By the separation theorem, there exists $(v,t)\in\mathbb{R}^{n+1}$ such that for each $x\in\text{lev}(f,<,\alpha)$ and $y\in A$,

$$\langle v,x\rangle<t\le\langle v,y\rangle.$$

This shows that $\text{val}(P)=-\text{val}(D)=-f^Q(v,t)$. If $\alpha=\inf_{x\in\mathbb{R}^n} f(x)$, then

$$\text{val}(P)=\inf_{x\in\mathbb{R}^n} f(x)=-f^Q(0,0),$$

and $(0,0)\in-\text{epi}\delta_A^*$. This shows that $\text{val}(P)=-\text{val}(D)=-f^Q(0,0)$. Hence, the dual problem (D) always has the global minimizer if f is usc quasiconvex and A is nonempty convex.

We introduce the following three dual problems:

$$(D)_1\begin{cases}\text{minimize } f^Q(v,1),\\ \text{subject to } (v,1)\in -\text{epi}\delta_A^*,\end{cases}$$

$$(D)_0\begin{cases}\text{minimize } f^Q(v,0),\\ \text{subject to } (v,0)\in -\text{epi}\delta_A^*,\end{cases}$$

$$(D)_{-1}\begin{cases}\text{minimize } f^Q(v,-1),\\ \text{subject to } (v,-1)\in -\text{epi}\delta_A^*.\end{cases}$$

For the above dual problems, the following corollary holds.

Corollary 1. *[20] Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_1, \mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}.$$

In [6], Greenberg and Pierskalla introduce the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) := \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0)\}.$$

In [19, 23, 24], we study optimality conditions for quasiconvex optimization in terms of subdifferentials. In this paper, we need the following result in [23].

Theorem 2. *[23] Let f be an usc essentially quasiconvex function, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x \in A} f(x)$,

(ii) $0 \in \partial^{GP} f(x_0) + N_A(x_0)$.

3 Quasiconjugate duality and optimality conditions

In this section, we study quasiconjugate duality and optimality conditions in terms of three types of quasiconjugate functions. Additionally, we show characterizations of the solution set in terms of quasi-subdifferentials.

3.1 For H-evenly quasiconvex objective functions

In this subsection, we study duality results for H-evenly quasiconvex objective functions in terms of H-quasiconjugate. At first, we introduce the following duality theorem.

Theorem 3. *[27] Let f be an usc H-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Assume that $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$. Then*

$$\mathrm{val}(P) = -\mathrm{val}(D)_1.$$

By Corollary 1, $\mathrm{val}(P) = -\min\{\mathrm{val}(D)_1, \mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}$, that is, the strong duality between (P) and one of dual problems always holds. For H-evenly quasiconvex objective function f, we show strong duality for (P) and $(D)_1$ in Theorem 3.

Next, we define H-quasi-subdifferential of f at x_0 as follows:

$$\partial^H f(x_0) = \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0), \langle v, x_0\rangle = 1\}.$$

It is clear that $v \in \partial^H f(x_0)$ if and only if the following equation holds:

$$\langle v, x_0\rangle = 1 = f(x_0) + f^H(v) + 1.$$

By using H-quasi-subdifferential, we show the following necessary and sufficient optimality condition.

Theorem 4. *Let f be an usc essentially H-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$. Then, x_0 is a global minimizer of f in A if and only if*

$$0 \in \partial^H f(x_0) + N_A(x_0).$$

Proof. Assume that x_0 is a global minimizer of f in A. Since f is essentially quasiconvex,

$$0 \in \partial^{GP} f(x_0) + N_A(x_0)$$

by Theorem 2. Hence, there exists $v \in \partial^{GP} f(x_0)$ such that $-v \in N_A(x_0)$. By the definition of $\partial^{GP} f(x_0)$,

$$\mathrm{lev}(f, <, f(x_0)) \subset \{x \in \mathbb{R}^n : \langle v, x\rangle < \langle v, x_0\rangle\}.$$

Since f is H-evenly quasiconvex and $f(x_0) = \inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, $0 \in \mathrm{lev}(f, <, f(x_0))$. This shows that $\langle v, x_0\rangle > 0$. Let $\bar{v} = \frac{v}{\langle v, x_0\rangle}$, then $\bar{v} \in \partial^H f(x_0)$ and $-\bar{v} \in N_A(x_0)$. Hence, $0 \in \partial^H f(x_0) + N_A(x_0)$.

Conversely, assume that $0 \in \partial^H f(x_0) + N_A(x_0)$. Since $\partial^H f(x_0) \subset \partial^{GP} f(x_0)$, $0 \in \partial^{GP} f(x_0) + N_A(x_0)$. By Theorem 2, x_0 is a global minimizer of f in A. □

Let S be the solution set of (P), that is,

$$S = \{x \in A : f(x) = \min_{y\in A} f(y)\}.$$

In the following theorem, we show characterizations of the solution set in terms of H-quasi-subdifferential.

Theorem 5. *Let f be an usc essentially H-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $\bar{x} \in S$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$. Then, the following sets are equal:*

(i) $S = \{x \in A : f(x) = \min_{y\in A} f(y)\}$,
(ii) $S_1 = \{x \in A : \partial^H f(\bar{x}) \cap \partial^H f(x) \neq \emptyset\}$,
(iii) $S_2 = \{x \in A : \exists v \in \partial^H f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
(iv) $S_3 = \{x \in A : \exists v \in \partial^H f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \leq 0\}$.

Proof. We can check easily that

$$S_1 \subset S_2 \subset S_3.$$

Let $x \in S_3$, then there exists $v \in \partial^H f(x)$ such that $\langle v, x - \bar{x}\rangle \leq 0$. This shows that $1 = \langle v, x\rangle \leq \langle v, \bar{x}\rangle$. By the definition of $\partial^H f(x)$,

$$f(\bar{x}) \geq \inf\{f(y) : \langle v, y\rangle \geq \langle v, x\rangle\} \geq f(x).$$

Since $\bar{x}$ is a global minimizer of f in A, $x \in S$.

Let $x \in S$ and $y = \frac{x+\bar{x}}{2}$. Then, y is a global minimizer of f in A since f is quasiconvex and A is convex. By Theorem 4,

$$0 \in \partial^H f(y) + N_A(y),$$

that is, there exists $v \in \partial^H f(y)$ such that $-v \in N_A(y)$. Then, $\langle v, y - x\rangle \geq 0$ and $\langle v, y - \bar{x}\rangle \geq 0$. This shows that $\langle v, x\rangle = \langle v, \bar{x}\rangle = \langle v, y\rangle = 1$. Since $f(x) = f(\bar{x}) = f(y)$, $v \in \partial^H f(\bar{x}) \cap \partial^H f(x)$, that is, $x \in S_1$. This completes the proof. □

3.2 For O-evenly quasiconvex objective functions

In this subsection, we study duality results for O-evenly quasiconvex objective functions in terms of O-quasiconjugate. At first, we introduce the following duality theorem.

Theorem 6. *[27] Let f be an usc O-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\text{val}(P) = -\text{val}(D)_0.$$

In Theorem 6. We show that the strong duality between (P) and $(D)_0$ holds for O-evenly quasiconvex objective function.

We define O-quasi-subdifferential of f at x_0 as follows:

$$\partial^O f(x_0) = \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0), \langle v, x_0\rangle = 0\}.$$

We can check easily that $v \in \partial^O f(x_0)$ if and only if the following equation holds:

$$\langle v, x_0\rangle = 0 = f(x_0) + f^O(v) + 0.$$

By using O-quasi-subdifferential, we show the following necessary and sufficient optimality condition.

Theorem 7. *Let f be an usc essentially O-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $\inf_{x\in A} f(x) < \sup_{x\in\mathbb{R}^n} f(x)$. Then, x_0 is a global minimizer of f in A if and only if*

$$0 \in \partial^O f(x_0) + N_A(x_0).$$

Proof. Assume that x_0 is a global minimizer of f in A. If $\inf_{x\in A} f(x) = \inf_{x\in\mathbb{R}^n} f(x)$, then $0 \in \partial^O f(x_0) \cap N_A(x_0)$, hence $0 \in \partial^O f(x_0) + N_A(x_0)$. If $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, $\text{lev}(f, <, f(x_0))$ is nonempty open convex set and $\text{lev}(f, <, f(x_0)) \cap A$ is empty. By the separation theorem, there exists $(v, t) \in \mathbb{R}^{n+1}$ such that for each $x \in \text{lev}(f, <, f(x_0))$ and $y \in A$,

$$\langle v, x\rangle < t \leq \langle v, y\rangle.$$

Since f is O-evenly quasiconvex, we can assume $t = 0$ without loss of generality. Actually, for each $x \in \text{lev}(f, <, f(x_0))$, $\frac{1}{n}x \in \text{lev}(f, <, f(x_0))$ for each $n \in \mathbb{N}$. This shows that $t \geq 0$. If there exists $x \in \mathrm{lev}(f, <, f(x_0))$ such that $\langle v, x \rangle > 0$, then for sufficiently large $n \in \mathbb{N}$, $nx \in \mathrm{lev}(f, <, f(x_0))$ and $\langle v, nx \rangle > t$. Hence, for each $x \in \mathrm{lev}(f, <, f(x_0))$ and $y \in A$,

$$\langle v, x \rangle < 0 \leq \langle v, y \rangle .$$

Additionally, since f is essentially O-evenly quasiconvex and

$$\sup_{x \in \mathbb{R}^n} f(x) > f(x_0) = \inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x),$$

x_0 is not a local minimizer of f in $\mathbb{R}^n$. Hence, $x_0 \in \mathrm{cl}\, \mathrm{lev}(f, <, f(x_0)) \cap A$. This shows that $\langle v, x_0 \rangle = 0$. By the above separation inequality, $v \in \partial^O f(x_0)$ and $-v \in N_A(x_0)$. This shows that $0 \in \partial^O f(x_0) + N_A(x_0)$.

Conversely, assume that $0 \in \partial^O f(x_0) + N_A(x_0)$. Hence, there exists $v \in \partial^O f(x_0)$ such that $-v \in N_A(x_0)$. By the definition of O-quasi-subdifferential and the normal cone, we can check that x_0 is a global minimizer of f in A. □

We show characterizations of the solution set in terms of O-quasi-subdifferential. Although the proof of the following theorem is similar to the proof of Theorem 5, we show Theorem 8 precisely for the sake of self-contained.

Theorem 8. *Let f be an usc essentially O-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $\bar{x} \in S$. Assume that $\inf_{x \in A} f(x) < \sup_{x \in \mathbb{R}^n} f(x)$. Then, the following sets are equal:*

(i) $S = \{x \in A : f(x) = \min_{y \in A} f(y)\}$,
(ii) $S_1 = \{x \in A : \partial^O f(\bar{x}) \cap \partial^O f(x) \neq \emptyset\}$,
(iii) $S_2 = \{x \in A : \exists v \in \partial^O f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle = 0\}$,
(iv) $S_3 = \{x \in A : \exists v \in \partial^O f(x) \text{ s.t. } \langle v, x - \bar{x} \rangle \leq 0\}$.

Proof. We can check easily that $S_1 \subset S_2 \subset S_3$.

Let $x \in S_3$, then there exists $v \in \partial^O f(x)$ such that $\langle v, x - \bar{x} \rangle \leq 0$. This shows that $0 = \langle v, x \rangle \leq \langle v, \bar{x} \rangle$. By the definition of $\partial^O f(x)$, $f(\bar{x}) \geq \inf\{f(y) : \langle v, y \rangle \geq \langle v, x \rangle\} \geq f(x)$. Since $\bar{x}$ is a global minimizer of f in A, $x \in S$.

Let $x \in S$ and $y = \frac{x + \bar{x}}{2}$. Then, y is a global minimizer of f in A since f is quasiconvex and A is convex. By Theorem 7, $0 \in \partial^O f(y) + N_A(y)$, that is, there exists $v \in \partial^O f(y)$ such that $-v \in N_A(y)$. Then, $\langle v, y - x \rangle \geq 0$ and $\langle v, y - \bar{x} \rangle \geq 0$. This shows that $\langle v, x \rangle = \langle v, \bar{x} \rangle = \langle v, y \rangle = 0$. Since $f(x) = f(\bar{x}) = f(y)$, $v \in \partial^O f(\bar{x}) \cap \partial^O f(x)$. This shows that $x \in S_1$. This completes the proof. □

3.3 For *R*-evenly quasiconvex objective functions

In this subsection, we study duality results for R-evenly quasiconvex objective functions in terms of R-quasiconjugate. In [27], we show the following duality theorem for R-evenly quasiconvex objective functions.

Theorem 9. *[27] Let f be an usc R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}.$$

In Theorem 3 and Theorem 6, we introduce strong duality theorems for H-evenly and O-evenly quasiconvex objective functions by one of dual problems $(D)_1$ and $(D)_0$. On the other hand, in Theorem 9, we need not only $(D)_{-1}$ but also $(D)_0$. In the following theorem, we show sufficient conditions of the strong duality for (P) and $(D)_{-1}$.

Theorem 10. *Let f be a continuous R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty compact convex subset of $\mathbb{R}^n$. Assume that $\inf_{x\in\mathbb{R}^n} f(x) < \inf_{x\in A} f(x) < \sup_{x\in\mathbb{R}^n} f(x)$. Then*

$$\mathrm{val}(P) = -\mathrm{val}(D)_{-1}.$$

Proof. Let $v \in \mathbb{R}^n$ satisfying $(v, -1) \in -\mathrm{epi}\delta_A^*$, then for each $x \in A$, $\langle v, x\rangle \geq -1$. Hence,

$$\mathrm{val}(P) = \inf_{x\in A} f(x) \geq \inf\{f(x) : \langle v, x\rangle \geq -1\} = -f^Q(v, -1).$$

This shows that $\mathrm{val}(P) \geq -\mathrm{val}(D)_{-1}$.

Assume that there exists $\alpha \in \mathbb{R}$ such that

$$\mathrm{val}(P) > \alpha > -\mathrm{val}(D)_{-1} \text{ and } \alpha > \inf_{x\in\mathbb{R}^n} f(x).$$

Then $A \cap \mathrm{lev}(f, \leq, \alpha)$ is empty and $\mathrm{lev}(f, \leq, \alpha)$ is nonempty closed, R-evenly convex set since f is continuous R-evenly quasiconvex. By the separation theorem for compact convex set A and closed convex set $\mathrm{lev}(f, \leq, \alpha)$, there exists $(v, t_1, t_2) \in \mathbb{R}^{n+2}$ such that for each $y \in A$ and $z \in \mathrm{lev}(f, \leq, \alpha)$,

$$\langle v, y\rangle \geq t_1 > t_2 \geq \langle v, z\rangle$$

If $t_2 < 0$, then $\bar{v} = \frac{v}{-t_2}$ satisfies

$$\langle \bar{v}, y\rangle > -1 \geq \langle \bar{v}, z\rangle$$

for each $y \in A$ and $z \in \mathrm{lev}(f, \leq, \alpha)$. Since f is continuous, we can check that $-1 > \langle \bar{v}, x\rangle$ for each $x \in \mathrm{lev}(f, <, \alpha)$. This shows that $(\bar{v}, -1) \in -\mathrm{epi}\delta_A^*$ and $-f^Q(\bar{v}, -1) \geq \alpha$. This contradicts $\alpha > -\mathrm{val}(D)_{-1}$.

Assume that $t_2 \geq 0$. If there exists $z_0 \in \mathrm{lev}(f, \leq, \alpha)$ such that $\langle v, z_0\rangle > 0$, then $\langle v, nz_0\rangle > t_1$ for sufficiently large $n \in \mathbb{N}$. Hence, for each $y \in A$ and $z \in \mathrm{lev}(f, \leq, \alpha)$,

$$\langle v, y\rangle \geq t_1 > 0 \geq \langle v, z\rangle\,.$$

Since f is R-evenly quasiconvex and $\inf_{x\in\mathbb{R}^n} f(x) < \alpha < \sup_{x\in\mathbb{R}^n} f(x)$, $\mathrm{lev}(f, \leq, \alpha)$ is nonempty R-evenly convex and $\mathrm{lev}(f, \leq, \alpha) \neq \mathbb{R}^n$. Hence, there exists a subfamily of open halfspaces $\{H_\lambda^-\}_{\lambda\in\Lambda} \subseteq H^-$ such that $\mathrm{lev}(f,\leq,\alpha) = \bigcap_{\lambda\in\Lambda} H_\lambda^-$. This shows that there exists $w \in \mathbb{R}^n \setminus \{0\}$ such that $\mathrm{lev}(f,\leq,\alpha) \subset \mathrm{lev}(w,<,-1)$. Since A is compact, $\beta = \inf_{y\in A} \langle w, y\rangle \in \mathbb{R}$. Therefore, for sufficiently small $k > 0$,

$$\langle v + kw, y\rangle \geq t_1 + k\beta > 0 > -k > \langle v + kw, z\rangle$$

for each $y \in A$ and $z \in \mathrm{lev}(f,\leq,\alpha)$. Let $v^* = \frac{v+kw}{k}$, then for each $y \in A$ and $z \in \mathrm{lev}(f,\leq,\alpha)$,

$$\langle v^*, y\rangle > 0 > -1 > \langle v^*, z\rangle .$$

This shows that $(v^*,-1) \in -\mathrm{epi}\delta_A^*$ and $-f^Q(v^*,-1) \geq \alpha$. This is a contradiction. □

Next, we need the following evenly quasiconvexity. A function f is said to be strictly R-evenly quasiconvex if for each $\alpha \in \mathbb{R}$, $\mathrm{lev}(f,<,\alpha)$ is R-evenly convex. We can check easily that a strictly R-evenly quasiconvex function f is R-evenly quasiconvex since $\mathrm{lev}(f,\leq,\alpha) = \cap_{\varepsilon>0}\mathrm{lev}(f,<,\alpha+\varepsilon)$. Additionally, f is said to be essentially strictly R-evenly quasiconvex if f is strictly R-evenly quasiconvex and each local minimizer x of f in $\mathbb{R}^n$ satisfying $f(x) < \sup_{y\in\mathbb{R}^n} f(y)$ is a global minimizer of f in $\mathbb{R}^n$. For example, the following function f is continuous, essentially strictly R-evenly quasiconvex:

$$f(x) = \begin{cases} 1 & x \leq 1, \\ -x+2 & 1 \leq x \leq 2, \\ 0 & x \geq 2. \end{cases}$$

R-quasi-subdifferential [26] of f at x_0 is defined as follows:

$$\partial^R f(x_0) = \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0), \langle v, x_0\rangle = -1\}.$$

We can check that $v \in \partial^R f(x_0)$ if and only if the following equation holds:

$$\langle v, x_0\rangle = -1 = f(x_0) + f^R(v) - 1.$$

By using R-quasi-subdifferential, we show the following necessary and sufficient optimality condition.

Theorem 11. *Let f be an usc essentially strictly R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $\inf_{x\in\mathbb{R}^n} f(x) < \inf_{x\in A} f(x) < \sup_{x\in\mathbb{R}^n} f(x)$. Then, x_0 is a global minimizer of f in A if and only if*

$$0 \in \partial^R f(x_0) + N_A(x_0).$$

Proof. Assume that x_0 is a global minimizer of f in A. Since $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, $\mathrm{lev}(f,<,f(x_0))$ is nonempty open convex set and $\mathrm{lev}(f,<,f(x_0)) \cap A$ is empty. By the separation theorem, there exists $(v,t) \in \mathbb{R}^{n+1}$ such that for each $x \in \mathrm{lev}(f,<,f(x_0))$ and $y \in A$,

$$\langle v, x\rangle < t \leq \langle v, y\rangle .$$

Since f is essentially strictly R-evenly quasiconvex, and

$$\inf_{x\in\mathbb{R}^n} f(x) < f(x_0) = \inf_{x\in A} f(x) < \sup_{x\in\mathbb{R}^n} f(x),$$

x_0 is not a local minimizer of f in $\mathbb{R}^n$. This shows that $x_0 \in \text{cl lev}(f,<,f(x_0))$ and $\langle v, x_0\rangle = t$.

If $t > 0$, then $\langle v, x_0\rangle = t > 0$ and there exists $z \in \text{lev}(f,<,f(x_0))$ such that $\langle v, z\rangle > 0$. Since f is R-evenly quasiconvex, $kz \in \text{lev}(f,<,f(x_0))$ for each $k \in \mathbb{N}$. However, for sufficiently large k, $\langle v, kz\rangle > t$. This is a contradiction.

If $t = 0$, then $\langle v, x_0\rangle = t = 0$. Since f is strictly R-evenly quasiconvex and $\inf_{x\in\mathbb{R}^n} f(x) < f(x_0) < \sup_{x\in\mathbb{R}^n} f(x)$, $\text{lev}(f,<,f(x_0))$ is nonempty R-evenly convex and $\text{lev}(f,<,f(x_0)) \neq \mathbb{R}^n$. Hence, there exists a subfamily of open halfspaces $\{H^-_\lambda\}_{\lambda\in\Lambda} \subseteq H^-$ such that $\text{lev}(f,<,f(x_0)) = \bigcap_{\lambda\in\Lambda} H^-_\lambda$. By the definition of H^-, there exists $w_\lambda \in \mathbb{R}^n$ such that $H^-_\lambda = \text{lev}(w_\lambda,<,-1)$ for each $\lambda \in \Lambda$. Then, $2x_0 \in \text{lev}(f,<,f(x_0))$. Actually, $\langle w_\lambda, x_0\rangle \leq -1$ since $x_0 \in \text{cl lev}(f,<,f(x_0))$. This shows that $\langle w_\lambda, 2x_0\rangle \leq -2 < -1$ for each $\lambda \in \Lambda$, that is, $2x_0 \in \text{lev}(f,<,f(x_0))$. However, by the separation inequality and $\langle v, 2x_0\rangle = 2t = 0$, $2x_0 \notin \text{lev}(f,<,f(x_0))$. This is a contradiction.

Hence, $t < 0$, then $\bar{v} = \frac{v}{-t}$ satisfies

$$\langle \bar{v}, x\rangle < -1 \leq \langle \bar{v}, y\rangle .$$

for each $x \in \text{lev}(f,<,f(x_0))$ and $y \in A$ and $\langle \bar{v}, x_0\rangle = -1$. By the above separation inequality, we can show that $v \in \partial^R f(x_0)$ and $-v \in N_A(x_0)$. Hence, $0 \in \partial^R f(x_0) + N_A(x_0)$.

Conversely, assume that $0 \in \partial^R f(x_0) + N_A(x_0)$. Hence, there exists $v \in \partial^R f(x_0)$ such that $-v \in N_A(x_0)$. By the definition of R-quasi-subdifferential and the normal cone, we can check that x_0 is a global minimizer of f in A. □

Next, we show characterizations of the solution set for R-evenly objective functions.

Theorem 12. *Let f be an usc essentially strictly R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $\bar{x} \in S$. Assume that $\inf_{x\in\mathbb{R}^n} f(x) < \inf_{x\in A} f(x) < \sup_{x\in\mathbb{R}^n} f(x)$. Then, the following sets are equal:*

(i) $S = \{x \in A : f(x) = \min_{y\in A} f(y)\}$,
(ii) $S_1 = \{x \in A : \partial^R f(\bar{x}) \cap \partial^R f(x) \neq \emptyset\}$,
(iii) $S_2 = \{x \in A : \exists v \in \partial^R f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle = 0\}$,
(iv) $S_3 = \{x \in A : \exists v \in \partial^R f(x) \text{ s.t. } \langle v, x - \bar{x}\rangle \leq 0\}$.

Proof. We can check easily that $S_1 \subset S_2 \subset S_3$. Let $x \in S_3$, then there exists $v \in \partial^R f(x)$ such that $\langle v, x - \bar{x}\rangle \leq 0$. This shows that

$$-1 = \langle v, x\rangle \leq \langle v, \bar{x}\rangle .$$

By the definition of $\partial^R f(x)$,

$$f(\bar{x}) \geq \inf\{f(y) : \langle v, y\rangle \geq \langle v, x\rangle\} \geq f(x).$$

Since $\bar{x}$ is a global minimizer of f in A, $x \in S$.

Let $x \in S$ and $y = \frac{x+\bar{x}}{2}$. Then, y is a global minimizer of f in A since f is quasiconvex and A is convex. By Theorem 11,

$$0 \in \partial^R f(y) + N_A(y),$$

that is, there exists $v \in \partial^R f(y)$ such that $-v \in N_A(y)$. Then, $\langle v, y-x\rangle \geq 0$ and $\langle v, y-\bar{x}\rangle \geq 0$. This shows that $\langle v, x\rangle = \langle v, \bar{x}\rangle = \langle v, y\rangle = -1$. Since $f(x) = f(\bar{x}) = f(y)$, $v \in \partial^R f(\bar{x}) \cap \partial^O f(x)$. This shows that $x \in S_1$. This completes the proof. □

4 Classifications of quasiconvex optimization problems

In this section, we consider the following quasiconvex optimization problem $(P)_{f,A}$:

$$(P)_{f,A} \begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

where f is an extended real-valued usc quasiconvex function on $\mathbb{R}^n$, and A is a nonempty convex subset of $\mathbb{R}^n$. By Corollary 1, at least one of the following strong duality holds:

$$\text{val}(P)_{f,A} = -\text{val}(D)_1,\ \text{val}(P)_{f,A} = -\text{val}(D)_0, \text{ and } \text{val}(P)_{f,A} = -\text{val}(D)_{-1}.$$

Additionally, the dual problem always has a global minimizer. Hence, all usc quasiconvex problems can be classified into the following three sets of problems:

$$\begin{aligned} QP^H &= \{(P)_{f,A} : \exists v \in \mathbb{R}^n \text{ such that } \text{val}(P)_{f,A} = -\text{val}(D)_1 = -f^Q(v,1)\}, \\ QP^O &= \{(P)_{f,A} : \exists v \in \mathbb{R}^n \text{ such that } \text{val}(P)_{f,A} = -\text{val}(D)_0 = -f^Q(v,0)\}, \text{ and} \\ QP^R &= \{(P)_{f,A} : \exists v \in \mathbb{R}^n \text{ such that } \text{val}(P)_{f,A} = -\text{val}(D)_{-1} = -f^Q(v,-1)\}. \end{aligned}$$

In this section, we consider a classification of quasiconvex optimization problems in terms of these quasiconjugate duality results.

4.1 Quasiconvex problem in QP^H

By Theorem 3, if f is H-evenly quasiconvex and $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, then $P_{f,A} \in QP^H$. As shown in the proof, $P_{f,A} \in QP^H$ if and only if there exists $v \in \mathbb{R}^n$ such that for each $x \in \text{lev}(f,<,\alpha)$ and $y \in A$,

$$\langle v, x\rangle < 1 \leq \langle v, y\rangle \tag{1}$$

where $\alpha = \text{val}(P)_{f,A}$. We show the following sufficient conditions for $P_{f,A} \in QP^H$:

(i) f is H-evenly quasiconvex and $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$,

(ii) $0 \in \text{lev}(f,<,\alpha)$.

We can check easily that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ $P_{f,A} \in QP^H$. Additionally, (ii) is not a necessary condition for $P_{f,A} \in QP^H$, see the following example.

Example 2. *Let $A = \{(x_1, x_2) : x_1 \leq -1\}$, and f be the following function on $\mathbb{R}^2$:*

$$f(x_1, x_2) = x_1^2 + (x_2 - 1)^2.$$

Then, A is nonempty convex, f is continuous quasiconvex, and f is not H-evenly quasiconvex. We can check easily that $0 \notin \text{lev}(f, <, \alpha)$ and

$$\text{val}(P_{f,A}) = 1 = -\text{val}(D)_1 = -f^Q((-1, 0), 1).$$

If $0 \in \text{cl}A$, then $P_{f,A} \notin QP^H$ by Equation 1. For O-evenly quasiconvex and R-evenly quasiconvex objective functions, the strong duality between $P_{f,A}$ and $(D)_1$ may hold, see the following examples.

Example 3. *Let $A = \{(x_1, x_2) : x_1 \leq -1\}$, and f be the following function on $\mathbb{R}^2$:*

$$f(x_1, x_2) = \begin{cases} -\dfrac{x_1}{x_2} & x_1 > 0, x_2 > 0, \\ 0 & -x_2 < x_1 \leq 0, \\ 1 & \textit{otherwise.} \end{cases}$$

Then, A is nonempty convex and f is usc O-evenly quasiconvex. We can check easily that

$$\text{val}(P_{f,A}) = -\text{val}(D)_1 = -\text{val}(D)_0 = 0 > -\infty = -\text{val}(D)_{-1},$$

and $-f^Q((-1, 0), 1) = -f^Q((-1, 0), 0) = 0$.

Example 4. *Let $A = \{(x_1, x_2) : x_1 \leq -1\}$, and f be the following function on $\mathbb{R}^2$:*

$$f(x_1, x_2) = \begin{cases} 0 & x_1 > 0, x_1 x_2 > 1, \\ 1 & (x_1, x_2) \in \{(y_1, y_2) : x_1 + x_2 > 1\} \setminus \{(y_1, y_2) : x_1 > 0, x_1 x_2 > 1\}, \\ 2 & \textit{otherwise.} \end{cases}$$

Then, A is nonempty convex and f is usc R-evenly quasiconvex. We can check easily that

$$\text{val}(P_{f,A}) = -\text{val}(D)_1 = -\text{val}(D)_0 = 1 > 0 = -\text{val}(D)_{-1},$$

and $-f^Q((-1, 0), 1) = -f^Q((-1, 0), 0) = 0$.

4.2 Quasiconvex problem in QP^O

By Theorem 6, if f is O-evenly quasiconvex, then $P_{f,A} \in QP^O$. As shown in the proof, $P_{f,A} \in QP^O$ if and only if there exists $v \in \mathbb{R}^n$ such that for each $x \in \text{lev}(f, <, \alpha)$ and $y \in A$,

$$\langle v, x \rangle < 0 \leq \langle v, y \rangle .$$

We show the following sufficient conditions for $P_{f,A} \in QP^O$:

(i) f is O-evenly quasiconvex,

(ii) $0 \in (\text{bd}A) \cap (\text{bdlev}(f, <, \alpha))$.

If one of the following conditions holds, then $P_{f,A} \notin QP^O$:

(i) $0 \in \text{int}A$,
(ii) $0 \in \text{lev}(f, <, \alpha)$,
(iii) f is H-evenly quasiconvex and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$.

For H-evenly quasiconvex objective function, if $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$, then $0 \in \text{lev}(f, <, \alpha)$. Hence the strong duality between $P_{f,A}$ and $(D)_0$ does not hold. For R-evenly quasiconvex objective functions, by Theorem 9, if $P_{f,A} \notin QP^R$, then $P_{f,A} \in QP^O$, see Example 4.

4.3 Quasiconvex problem in QP^R

By Theorem 10, if f is continuous R-evenly quasiconvex, A is compact, and $\inf_{x \in \mathbb{R}^n} f(x) < \inf_{x \in A} f(x) < \sup_{x \in \mathbb{R}^n} f(x)$, then $P_{f,A} \in QP^R$. As shown in the proof, $P_{f,A} \in QP^R$ if and only if there exists $v \in \mathbb{R}^n$ such that for each $x \in \text{lev}(f, <, \alpha)$ and $y \in A$,

$$\langle v, x \rangle < -1 \leq \langle v, y \rangle .$$

We show the following sufficient conditions for $P_{f,A} \in QP^R$:

(i) f is continuous R-evenly quasiconvex, A is compact, and $\inf_{x \in \mathbb{R}^n} f(x) < \inf_{x \in A} f(x) < \sup_{x \in \mathbb{R}^n} f(x)$,
(ii) $0 \in \text{int}A$.

If one of the following conditions holds, then $P_{f,A} \notin QP^R$:

(i) $0 \in \text{cllev}(f, <, \alpha)$,
(ii) f is H-evenly quasiconvex and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$,
(iii) f is O-evenly quasiconvex and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$.

For H-evenly quasiconvex objective function, if $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$, then $0 \in \text{lev}(f, <, \alpha)$. Hence the strong duality between $P_{f,A}$ and $(D)_{-1}$ does not hold. For O-evenly quasiconvex objective functions, if $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$, then $0 \in \text{cl lev}(f, <, \alpha)$. Actually, there exists $x \in \mathbb{R}^n$ such that $f(x) < \alpha$ since $\alpha = \inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$. By O-evenly quasiconvexity of f, $f\left(\frac{x}{n}\right) = f(x) < \alpha$ for each $n \in \mathbb{N}$. This shows that $0 \in \text{cl lev}(f, <, \alpha)$. For R-evenly quasiconvex objective functions, we show an example of $P_{f,A} \notin QP^R$, see Example 4.

5 Conclusion

In this paper, we study quasiconjugate duality and optimality conditions for quasiconvex optimization. By using three types of quasiconjugate functions and quasi-subdifferentials, we show sufficient conditions for strong duality, optimality conditions, and characterizations of the solution set. If the objective function is H-evenly (O-evenly) quasiconvex, then $P_{f,A} \in QP^H$ ($P_{f,A} \in QP^O$, respectively). Additionally, for R-evenly quasiconvex objective function, we show Theorem 9 and Theorem 10. We show similar results for optimality conditions and the characterizations of the solution set. It is worth noting that we do not need the compactness of A in Theorem 11. We give a classification of quasiconvex optimization problems in terms of quasiconjugate duality in Section 4. We introduce some sufficient conditions for the strong duality results. Additionally, we show some sufficient conditions for the failure of strong duality.

Supplementary information. My manuscript has no associated data.

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