Jánossy densities, Tracy-Widom DEs & random matrix spacing distributions

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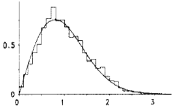
2024/9/25 BMS RMT Seminar

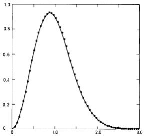
1. **Introduction/Motivation**

World of Random Matrices

Energy, Entanglement Spectra
of quantum chaotic systems
- Gutzwiller's Tr formula

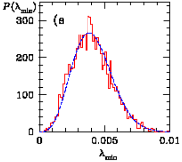


Arithmatic Chaos
- high zeroes of ζ function
- low zeroes of L functions

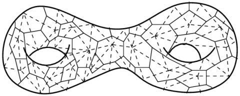


RMT

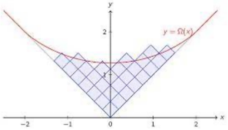
Pionic effective theory of QCD
- Dirac operator spectra



Quantum Gravity, String Theory
- random surfaces = $d \leq 2$ strings
- SYK = JT gravity $\simeq$ intersection #



Rep theory, Combinatorics
- longest increasing subseq.
- 2D, 3D tiling



Gap probability

Unitary ensemble as DetPP

$$\mathrm{Pr}\,(\text{interval } I \text{ is vacant}) = \mathrm{Det}(\mathbb{I} - \mathbf{K}|_I) \quad , \quad (\mathbf{K}|_I f)(\cdot) = \int_I dx\, K(\cdot, x) f(x)$$

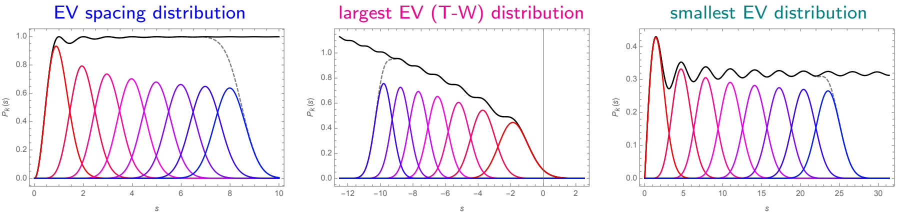
orth.poly. $\{\varphi_n(\lambda)\} \xrightarrow{\text{local asympt.}} \sin x\,, \cos x$ bulk, $\mathrm{Ai}(x)$ band edge, $J_\nu(\sqrt{x})$ reflex pt.

Jimbo-Miwa-Môri-Sato 1980 : $\mathbf{K}_{\mathrm{Sin}}|_{(0,s)} \Rightarrow$ Painlevé V for $s\partial_s \log \mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(0,s)})$

Tracy-Widom 1993 : $\mathbf{K}_{\mathrm{Airy}}|_{(s,\infty)} \Rightarrow$ Painlevé II for $\partial_s^2 \log \mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(s,\infty)})$

Tracy-Widom 1993 : $\mathbf{K}_{\mathrm{Bessel}}|_{(0,s)} \Rightarrow$ Painlevé III$'$

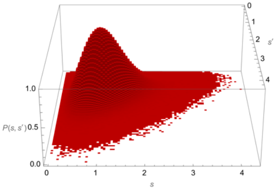
Tracy-Widom 1994 : $\mathbf{K}_{\mathrm{CUE}_N}|_{(0,s)} \Rightarrow$ Painlevé VI, $\mathbf{K}_{\mathrm{GUE}_N}|_{(s,\infty)} \Rightarrow$ Painlevé IV



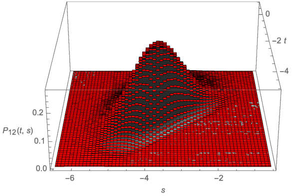
Joint distribution

What about **joint** distributions?

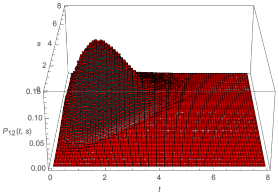
2 consecutive LSs



2 largest EVs



2 smallest EVs



$N = 128$, #sample $= 10^7$

Forrester-Odlyzko 1996 : bulk, $\min(s_n, s_{n+1}) \to$ Painlevé V

Forrester-Witte 2007 (70pp!) : hard edge → isomonodromic system for Painlevé III'

Witte-Bornemann-Forrester 2013 : soft edge → isomonodromic system for Painlevé II

Perret-Schehr 2014 : soft edge → solution of Lax pair for Painlevé XXXIV

Joint distribution

Revisit this problem for

- More **user-friendly** analytical formulation!
- Universally applicable to $^{\forall}K(x,y) = \dfrac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}$ or $\dfrac{\Psi(x)^t J\,\Psi(y)}{x-y}$
- Generalizable to $P_{12\cdots p}(s_1, \ldots, s_p)$

Solution: T-W method

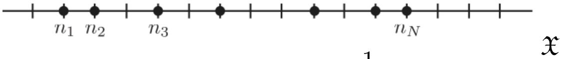
- if $m(x)\dfrac{d}{dx}\Psi(x) = \mathcal{A}(x)\Psi(x)$ with **polynomial** $\mathcal{A}, m$ and $\mathrm{tr}\,\mathcal{A} = 0$
- then $\mathrm{Det}(\mathbb{I} - \mathbf{K}|_I)$ satisfy a system of PDEs containing **coefficients** of $\mathcal{A}, m$

applied to the kernel with conditioned EVs

2 Jánossy densities & T-W DEs

Det point process

Discrete DPP



Joint prob of N pts : $\Pr(n_1, \ldots, n_N) = \dfrac{1}{N!} \det \left[K(n_i, n_j)\right]_{i,j=1}^{N}$

with kernel matrix $\mathbf{K} = \left[K(n, n')\right]_{n,n' \in \mathfrak{X}} = \mathbf{K}^t = \mathbf{K} \cdot \mathbf{K}, \quad \mathrm{tr}\, \mathbf{K} = N$

$$\Downarrow$$

Joint prob of k pts : $\rho_k(n_1, \ldots, n_k) = \det \left[K(n_i, n_j)\right]_{i,j=1}^{k}$

Gap probability $= \det(\mathbb{I} - \mathbf{K}_I), \quad \mathbf{K}_I = \left[K(n, n')\right]_{n,n' \in I}$

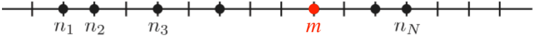
everything below carries over to DPPs on continuum or Pfaff PPs by:

- matrix $\mathbf{K} \to$ operator $\mathbf{K}$, $\det \to \mathrm{Det}$, $\rho_k(\{n\}) \to \rho_k(\{x\}) dx_1 \cdots dx_k$ (UE)
- matrix $\mathbf{K} \to$ quaternion matrix $\mathbf{K}$, $\det \to \mathrm{qdet}$ (OE, SE, OE→UE, SE→UE)

Conditional probability

fix a point m and define 'conditioned kernel'

$$\tilde{K}(n,n') = K(n,n') - \frac{K(n,m)K(m,n')}{K(m,m)}$$



- satisfies $\tilde{\mathbf{K}} = \left[\tilde{K}(n,n')\right]_{n,n'\in\mathfrak{X}} = \tilde{\mathbf{K}}^t = \tilde{\mathbf{K}}\cdot\tilde{\mathbf{K}}, \quad \mathrm{tr}\,\tilde{\mathbf{K}} = N-1$
- conditional joint prob of k pts, with m already occupied:

$$\tilde{\rho}_1(n|m) = \frac{\rho_2(n,m)}{\rho_1(m)} = \frac{K(n,n)K(m,m)-K(n,m)K(m,n)}{K(m,m)} = \tilde{K}(n,n)$$

$$\tilde{\rho}_2(n_1,n_2|m) = \frac{\rho_3(n_1,n_2,m)}{\rho_1(m)}$$

$$= \frac{K(n_1,n_1)K(n_2,n_2)K(m,m) \pm \text{(5 terms)}}{K(m,m)} = \det\left[\tilde{K}(n_i,n_j)\right]_{i,j=1}^{2}, \text{ etc.}$$

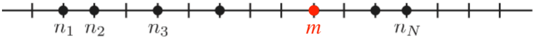
Conditional probability

thus the conditioned kernel

$$\tilde{K}(n,n') = K(n,n') - \frac{K(n,m)K(m,n')}{K(m,m)}$$

corresponds to another DPP governing the conditional joint prob

$$\tilde{\rho}_k(n_1,\ldots,n_k|m) = \det\left[\tilde{K}(n_i,n_j)\right]_{i,j=1}^{k}$$



now fix more points one by one. by induction it generalizes to ...

Conditional probability

Lemma (1)

fix p distinct points $m_1, \ldots, m_p$ and let

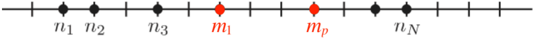
$$\boldsymbol{\kappa} = [K(m_i, m_j)]_{i,j=1}^{p}\,,\ \mathbf{k} = [K(m_i, n)]_{i=1,\ldots,p}^{n\in\mathfrak{X}}\,,\ \mathbf{K} = [K(n, n')]_{n,n'\in\mathfrak{X}}$$

then

$$\tilde{\mathbf{K}} = \mathbf{K} - \mathbf{k}^t \boldsymbol{\kappa}^{-1} \mathbf{k}$$

governs the conditional joint prob with $m_1, \ldots, m_p$ already occupied:

$$\tilde{\rho}_k(n_1, \ldots, n_k | m_1, \ldots, m_p) = \det\left[\tilde{K}(n_i, n_j)\right]_{i,j=1}^{k}$$



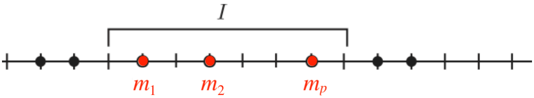
it immediately leads to...

Jánossy density

Lemma (2)

Probability that a subset $I \subset \mathfrak{X}$ is otherwise empty under the condition that $m_1, \ldots, m_p \in I$ are already occupied is given by

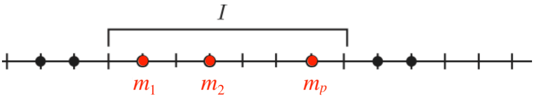
$$\tilde{J}_p(I|m_1, \ldots, m_p) = \det(\mathbb{I} - \tilde{\mathbf{K}}_I) \ , \quad \tilde{\mathbf{K}}_I = \left[\tilde{K}(n, n')\right]_{n,n' \in I}$$



Jánossy density = (unconditional) Probability that a subset $I \subset \mathfrak{X}$ contains exactly p points at $m_1, \ldots, m_p$ is given by

$$J_p(I; m_1, \ldots, m_p) = \det \boldsymbol{\kappa} \cdot \det(\mathbb{I} - \tilde{\mathbf{K}}_I)$$

Jánossy density



Obviously, this fact is nothing new.

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{D}| \cdot \left|\mathbf{A} - \mathbf{C}\mathbf{D}^{-1}\mathbf{B}\right| = |\mathbf{A}| \cdot \left|\mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{C}\right| \Rightarrow \text{Jánossy in 3 disguises}$$

$$\begin{aligned} J_p(I; m_1, \ldots, m_p) &= \det \boldsymbol{\kappa} \cdot \det \left(\mathbb{I} - (\mathbf{K} - \mathbf{k}^t \boldsymbol{\kappa}^{-1} \mathbf{k})_I\right) \\ &= (-1)^p \det \begin{vmatrix} -\boldsymbol{\kappa} & -\mathbf{k}_I \\ -\mathbf{k}_I^t & \mathbb{I} - \mathbf{K}_I \end{vmatrix} \\ &= \det(\mathbb{I} - \mathbf{K}_I) \cdot \det \left[\left(\mathbf{K}_I (\mathbb{I} - \mathbf{K}_I)^{-1}\right)_{m_i, m_j} \right]_{i,j=1}^p \end{aligned}$$

- 3rd line is found e.g. in Daley-Vere Jones (1988), p.140
- 1st line is suited for applying T-W method to $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}|_I)$

T-W method for $\tilde{\mathrm{K}}$

Tracy-Widom criteria (1994) :

1. kernel is of Christoffel-Darboux form $K(x,y) = \dfrac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}$
2. 2-component function satisfies a linear DE

$$m(x)\frac{d}{dx}\begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} A(x) & B(x) \\ -C(x) & -A(x) \end{bmatrix}\begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix}$$ with **polynomials** m, A, B, C,

then $\mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(a_1,a_2)})$ is determined by a system of PDEs in a_1, a_2

a punchline observation:

Theorem (Nishigaki, 2021)

If a kernel K satisfies the T-W criteria, so does the conditioned kernel $\tilde{K}$

T-W method for $\tilde{\mathrm{K}}$

Proof.

T-W criteria: $K(x,y) = \dfrac{\Psi(x)^t J \Psi(y)}{x-y}$ $\quad \Psi(x) = \begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\mathcal{D}_x \Psi(x) = (\partial_x + \mathcal{A}(x))\Psi(x) = 0\ ,\ \mathcal{A}(x) \in \mathfrak{sl}(2), \text{rational in } x$$

---

gauge transformation $\Psi(x) \mapsto \tilde{\Psi}(x)$:

$$\tilde{\Psi}(x) = \Psi(x) - \overbrace{\Psi(t)\frac{K(t,x)}{K(t,t)}}^{\text{avoid the point } t} = U(x)\Psi(x)\ ,\ U(x) = \mathbb{I} - \frac{\Psi(t)\Psi(t)^t J}{K(t,t)(t-x)} \in \mathrm{SL}(2), \text{rational}$$

$\Rightarrow$ conditioned kernel: $\tilde{K}(x,y) = K(x,y) - \dfrac{K(x,t)K(t,y)}{K(t,t)} \overset{!}{=} \dfrac{\tilde{\Psi}(x)^t J \tilde{\Psi}(y)}{x-y}$

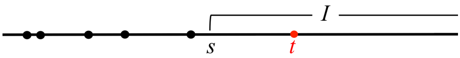
$\Rightarrow$ gauge-transformed $\tilde{\Psi}$ satisfies $\tilde{\mathcal{D}}_x \tilde{\Psi}(x) = (\partial_x + \tilde{\mathcal{A}}(x))\tilde{\Psi}(x) = 0$,

$$\tilde{\mathcal{A}}(x) = U(x)\mathcal{A}(x)U(x)^{-1} + U(x)\partial_x U(x)^{-1} \in \mathfrak{sl}(2), \text{rational in } x \qquad \square$$

generalizable to n-component $\Psi(x)$, J: $n \times n$ skew, $\mathcal{A}(x) \in \mathfrak{sl}(n)$, $U(x) \in \mathrm{SL}(n)$

T-W method for $\tilde{\mathrm{K}}_{\mathrm{Airy}}$

$$\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{\mathrm{Airy}}|_{(s,\infty)}) = J_1((s,\infty);t)$$



$\varphi(x) = \mathrm{Ai}(x)$, $\psi(x) = \mathrm{Ai}'(x)$ satisfy LDEs with
$m(x) = 1$, $A(x) = 0$, $B(x) = 1$, $C(x) = -x$

$$\Downarrow$$

$\tilde{\varphi}(x)$, $\tilde{\psi}(x)$ satisfy LDEs with $\quad a = \frac{\psi(t)}{\sqrt{K(t,t)}}, \quad b = \frac{\varphi(t)}{\sqrt{K(t,t)}}$

$$\tilde{m}(x) = (x-t)^2$$

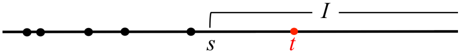
$$\tilde{A}(x) = -ab(a^2-1) - a^2 t + \left(a^2 + ab^3 - b^2 t\right) x + b^2 x^2 := \sum_{j=0}^{2} \alpha_j(t) x^j$$

$$\tilde{B}(x) = b^2(a^2-1) + 2abt + t^2 - \left(2ab + b^4 + 2t\right) x + x^2 := \sum_{j=0}^{2} \beta_j(t) x^j$$

$$\tilde{C}(x) = a^2(a^2-1) - (ab-t)^2 x - 2(ab-t)x^2 - x^3 := \sum_{j=0}^{3} \gamma_j(t) x^j$$

T-W method for $\tilde{\mathrm{K}}_{\mathrm{Airy}}$

$\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{\mathrm{Airy}}|_{(s,\infty)}) = J_1((s,\infty);t)$



$$R(s) = \partial_s \log \mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}|_{(s,\infty)})$$
$$= ((\mathbb{I} - \tilde{\mathbf{K}}|_{(s,\infty)})^{-1}\tilde{K})(s,s) = p_0(s)q_0'(s) - q_0(s)p_0'(s)$$
$$q_k(s) = ((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}x^k\varphi)(s), \quad p_k(s) = ((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}x^k\psi)(s)$$
$$u_k(s) = \int_I dx\, \varphi(x)\, x^k((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}\varphi)(x), \quad v_k(s) = \int_I dx\, \psi(x)\, x^k((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}\varphi)(x)$$
$$\tilde{v}_k(s) = \int_I dx\, \varphi(x)\, x^k((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}\psi)(x), \quad w_k(s) = \int_I dx\, \psi(x)\, x^k((\mathbb{I} - \tilde{\mathbf{K}}|_I)^{-1}\psi)(x)$$

satisfy a closed system of ODEs in s, containing coefficients $\alpha_{0,1,2}, \beta_{0,1,2}, \gamma_{0,1,2,3}(t)$:
[$' = \partial_s$, parametric dependence on t is implicit]

T-W method for $\tilde{\mathrm{K}}_{\mathrm{Airy}}$

System of ODEs for $J_1((s,\infty);t)$

$$(s-t)^2 q_0' = \sum_{j=0}^{2}\Big(\alpha_j + \sum_{k=0}^{1}\alpha_{j+k+1}v_k + \sum_{k=0}^{2}\gamma_{j+k+1}u_k\Big)q_j - v_0 q_0$$
$$+\sum_{j=0}^{2}\Big(\beta_j + \sum_{k=0}^{1}\alpha_{j+k+1}u_k + \sum_{k=0}^{1}\beta_{j+k+1}v_k\Big)p_j + u_0 p_0$$

$$(s-t)^2 p_0' = \sum_{j=0}^{3}\Big(-\gamma_j + \sum_{k=0}^{1}\alpha_{j+k+1}w_k + \sum_{k=0}^{2}\gamma_{j+k+1}\tilde{v}_k\Big)q_j - w_0 q_0$$
$$+\sum_{j=0}^{2}\Big(-\alpha_j + \sum_{k=0}^{1}\alpha_{j+k+1}\tilde{v}_k + \sum_{k=0}^{1}\beta_{j+k+1}w_k\Big)p_j + \tilde{v}_0 p_0$$

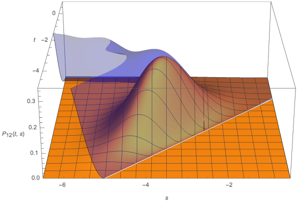
$$u_0' = -q_0q_0,\ u_1' = -q_0q_1,\ u_2' = -q_0q_2,\ v_0' = -q_0p_0,\ v_1' = -q_0p_1,\ v_2' = -q_0p_2$$
$$w_0' = -p_0p_0,\ w_1' = -p_0p_1$$
$$q_1 = s\,q_0 - v_0q_0 + u_0p_0,\ q_2 = s^2q_0 - v_0q_1 - v_1q_0 + u_0p_1 + u_1p_0$$
$$q_3 = s^3q_0 - v_0q_2 - v_1q_1 - v_2q_0 + u_0p_2 + u_1p_1 + u_2p_0$$
$$p_1 = s\,p_0 - w_0q_0 + \tilde{v}_0p_0,\ p_2 = s^2p_0 - w_0q_1 - w_1q_0 + \tilde{v}_0p_1 + \tilde{v}_1p_0$$

$\tilde{v}_0 = v_0,\ \tilde{v}_1 = v_1 - v_0\tilde{v}_0 + u_0w_0,\ \tilde{v}_2 = v_2 - v_0\tilde{v}_1 - v_1\tilde{v}_0 + u_0w_1 + u_1w_0$, with BCs at $s \gg 1$

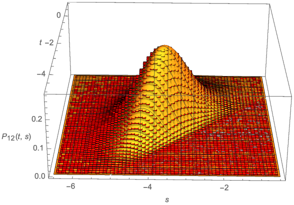
T-W method for $\tilde{\mathrm{K}}_{\mathrm{Airy}}$

Joint distribution of 1st & 2nd largest EVs

$$P_{12}(t,s) = \rho_1(t)\cdot R(s)\exp\left(-\int_s^\infty ds' R(s')\right)$$

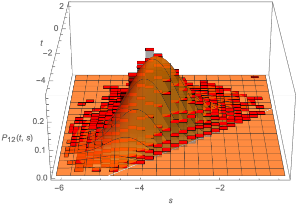


$P_{12}(t,s), \rho_2(t,s)$



#1, 2 largest EVs of GUE$_N$

$N = 128$

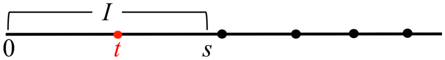


#1, 2 longest incr. subseq. of $\mathfrak{S}_N$

$N = 8096$

T-W method for $\tilde{\mathrm{K}}_{\mathrm{Bessel}}$

$$\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{\mathrm{Bessel}}|_{(0,s)}) = J_1((0,s);t)$$



$$\varphi(x) = J_\nu(\sqrt{x}), \quad \psi(x) = \frac{\sqrt{x}}{4}\left(J_{\nu-1}(\sqrt{x}) - J_{\nu+1}(\sqrt{x})\right)$$

$$m(x) = x, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = \frac{1}{4}(x - \nu^2)$$

$$\Downarrow$$

$$\tilde{m}(x) = x(x-t)^2$$

$$\tilde{A}(x) = -ab(a^2-1) - a^2 t + \frac{\nu^2 b^2}{4}(ab - t) + \left(a^2 - \frac{ab^3}{4} + \frac{b^2 t}{4} + \frac{\nu^2 b^2}{4}\right)x - \frac{b^2}{4}x^2$$

$$\tilde{B}(x) = b^2(a^2-1) + 2abt + t^2 - \frac{\nu^2 b^4}{4} + \left(-2ab + \frac{b^4}{4} - 2t\right)x + x^2$$

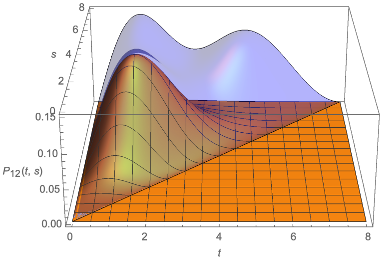
$$\tilde{C}(x) = a^2(a^2-1) - \frac{\nu^2}{4}(ab-t)^2 + \left(\frac{1}{4}(ab-t)^2 - \frac{\nu^2}{2}(ab-t)\right)x + \left(\frac{1}{2}(ab-t) - \frac{\nu^2}{4}\right)x^2 + \frac{x^3}{4}$$

$\Rightarrow$ repeat the same tedious procedure ...

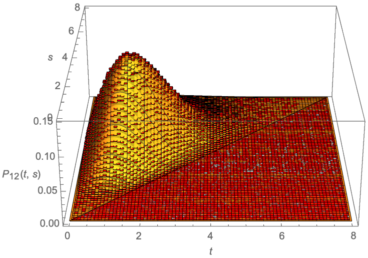
T-W method for $\tilde{\mathrm{K}}_{\mathrm{Bessel}}$

Joint distribution of 1st & 2nd smallest EVs

$$P_{12}^{(\nu)}(t,s) = \rho_1(t) \cdot R(s) \exp\left(-\int_0^s ds' R(s')\right)$$



$P_{12}^{(0)}(t,s), \rho_2^{(0)}(t,s)$



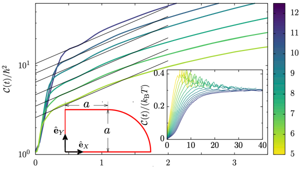
#1, 2 smallest EVs of chGUE$_N^{(0)}$

3 Gap-ratio distribution

Measure of Quantum chaoticity

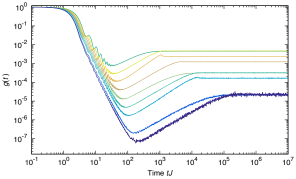
Quantization of classically nonintegrable systems $|\Psi(t)\rangle = \mathrm{e}^{-it\hat{H}}|\Psi_0\rangle$, $\hat{H}|n\rangle = E_n|n\rangle$

- Out-of-time-ordered correlation $\mathcal{C}(t) = \mathbb{E}\left(\left|[\hat{q}(t), \hat{p}(0)]\right|^2\right)$
- Krylov complexity $C(t) = \sum_n n|\langle K_n|\Psi(t)\rangle|^2$, $\{|K_n\rangle\} :=$ GS ortho. of $\{\hat{H}^n|\Psi_0\rangle\}$
- Level statistics : Wigner $\leftrightarrow$ Poisson as ETH $\leftrightarrow$ (many-body)AL
  - Form factor $g(t) = \int d\epsilon\, e^{i\epsilon t}\rho_2(E, E+\epsilon)$
  - Spacing distribution $P(s)$, $s_n := \bar{\rho}(E_n)(E_{n+1} - E_n)$



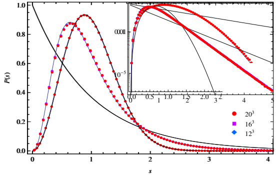
Sinai billiard: OTOC

García-Mata et al. 2022



SYK model: form factor

Cotler et al. 2016



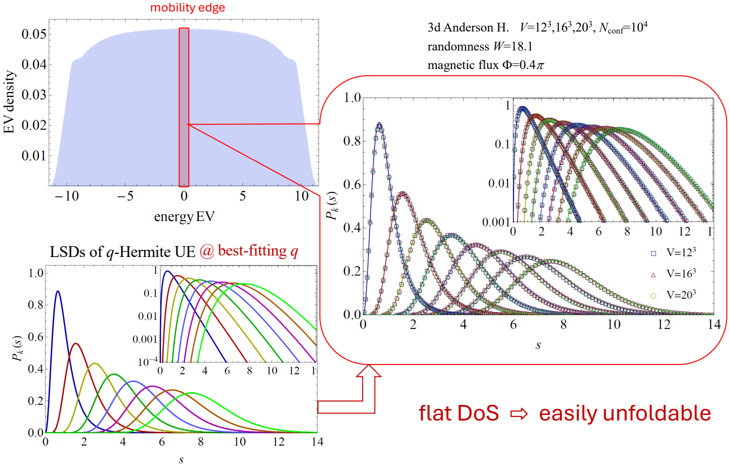
Anderson Hamiltonian: level spacing

Nishigaki 1999

Measure of Quantum chaoticity

Level spacings: Anderson Hamiltonian

Nishigaki et al. 2017 (unpub)

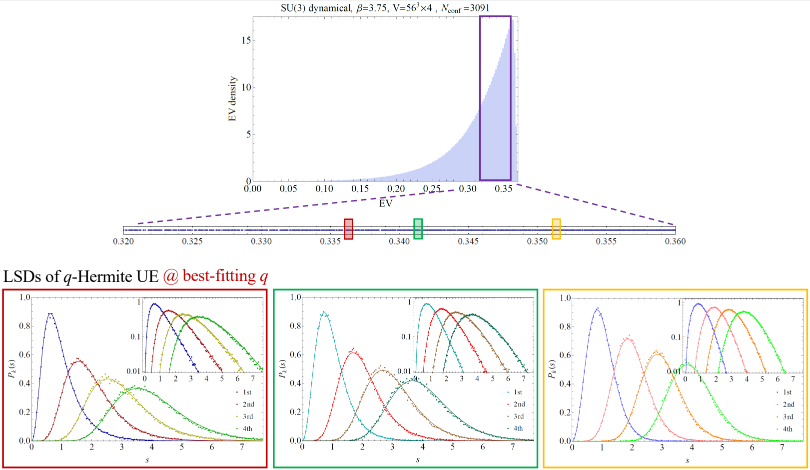


flat DoS ⇨ easily unfoldable

Measure of Quantum chaoticity

Level spacings: Dirac operator in high-T QCD

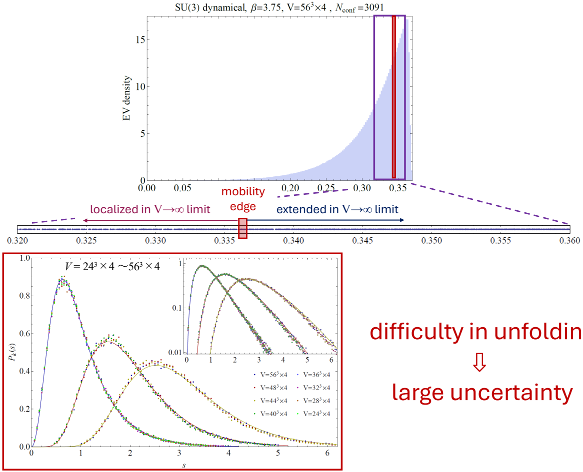
Nishigaki et al. 2017 (unpub)



Measure of Quantum chaoticity

Level spacings: Dirac operator in high-T QCD

Nishigaki et al. 2017 (unpub)



difficulty in unfolding

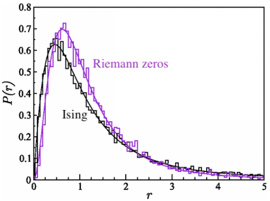
⇩

large uncertainty

Gap-ratio distribution

Quantization of classically nonintegrable systems $|\Psi(t)\rangle = \mathrm{e}^{-it\hat{H}}|\Psi_0\rangle$, $\hat{H}|n\rangle = E_n|n\rangle$

- Level statistics : Wigner ↔ Poisson as ETH ↔ (many-body)AL
  - Form factor $g(t) = \int d\epsilon\, e^{i\epsilon t} R_2(E, E+\epsilon)$
  - Spacing distribution $P(s)$, $s_n := \bar{\rho}(E_n)(E_{n+1} - E_n)$ ↙ no unfolding needed
  - Gap-ratio distribution $P_{\mathrm{r}}(r)$, $r_n := \dfrac{E_{n+1} - E_n}{E_n - E_{n-1}}$ or $\tilde{r}_n := \min(r_n, r_n^{-1}) \leq 1$



Ising model, Riemann ζ zeros: gap-ratio

Oganesyan-Huse 2007, Atas et al. 2013

Gap-ratio distribution

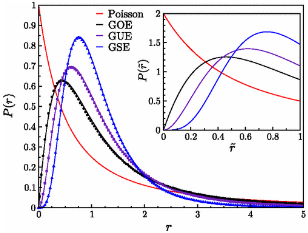
Atas-Bogomolny-Giraud-Roux 2013

$$\mathrm{G}\beta\mathrm{E}_{N=3} \quad : \quad \mathsf{JPD}(\lambda_1,\lambda_2,\lambda_3) \propto \mathrm{e}^{-\lambda_1^2-\lambda_2^2-\lambda_3^2}|(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)|^\beta$$

$$\Rightarrow \quad P_{\mathrm{r}}^{\mathrm{W}}(r) = \iiint_{\lambda_1<\lambda_2<\lambda_3} d\lambda_1 d\lambda_2 d\lambda_3\, \mathsf{JPD}(\lambda_1,\lambda_2,\lambda_3)\delta\Big(r-\frac{\lambda_3-\lambda_2}{\lambda_2-\lambda_1}\Big)$$

$$= C_\beta \frac{(r+r^2)^\beta}{(1+r+r^2)^{1+\frac{3}{2}\beta}} \qquad \text{[+ heuristic correction towards numerical data]}$$

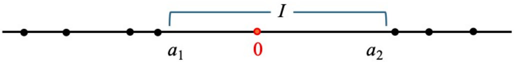
$$\Rightarrow \quad \langle\tilde{r}\rangle = 0.5307(1), 0.5996(1), 0.6744(1)\ (\beta=1,2,4) \Rightarrow \ \#\text{citation} \simeq 1000$$



most refs cite $P_{\mathrm{r}}^{\mathrm{W}}(r)$ as "outcome of RMT" $\Rightarrow$ compute exact $P_{\mathrm{r}}(r)$ & earn citations!

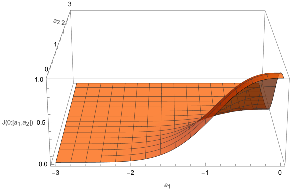
T-W method for $\tilde{\mathrm{K}}_{\mathrm{Sin}}$

$J_1([a_1, a_2]; 0) = \mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{\mathrm{Sin}}|_{[a_1, a_2]})$ **by T-W method, as before**

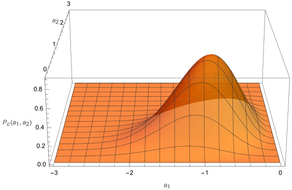


distribution of consecutive gaps **gap-ratio distribution** **expectation of $\tilde{r}_n$**

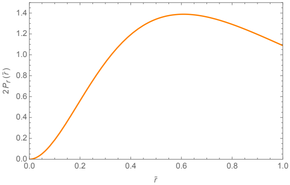
$$P_{\mathrm{c}}(a_1, a_2) = -\frac{\partial^2 J_1([a_1, a_2]; 0)}{\partial a_1 \partial a_2} \Rightarrow P_{\mathrm{r}}(r) = \int_0^\infty da\, a\, P_{\mathrm{c}}(-ra, a) \Rightarrow \langle \tilde{r} \rangle = 0.5997504209..$$



$J_1([a_1, a_2]; 0)$



$P_{\mathrm{c}}(a_1, a_2)$



$P_{\mathrm{r}}(r)$

T-W method for $\tilde{\mathrm{K}}_{\mathrm{Sin}}$

T-W criteria

$$\underbrace{\begin{bmatrix} \tilde\varphi(x) \\ \tilde\psi(x) \end{bmatrix}}_{\tilde\Psi(x)} = \underbrace{\begin{bmatrix} 1 & 0 \\ -x^{-1} & 1 \end{bmatrix}}_{U(x)} \underbrace{\begin{bmatrix} \sin x \\ \cos x \end{bmatrix}}_{\Psi(x)} \quad \text{satisfy} \quad x\frac{d}{dx}\begin{bmatrix} \tilde\varphi \\ \tilde\psi \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x \\ -x & -1 \end{bmatrix}}_{x\tilde{\mathcal{A}}(x)} \begin{bmatrix} \tilde\varphi \\ \tilde\psi \end{bmatrix}$$

System of ODEs for $J_1([a,b];0)$

integrate TW PDEs in a_1, a_2 for J_1, q_j, p_j, U, V along $(a_1(s), a_2(s)) = (sa, sb)$, $s : \epsilon \to 1$

$$s\frac{d}{ds}\log J_1([a_1,a_2];0) = a_1(q_1^2+p_1^2) - a_2(q_2^2+p_2^2) - (q_1p_2 - p_1q_2)^2 + 2U(q_1p_1 - q_2p_2) - V(q_1^2 - p_1^2 - q_2^2 + p_2^2)$$

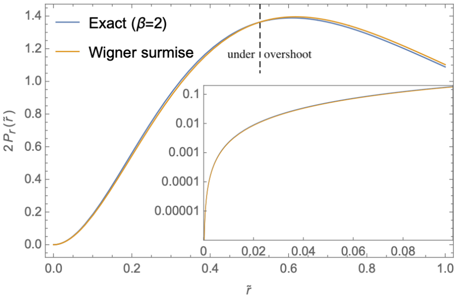
$$s\frac{d}{ds}\begin{bmatrix} q_j \\ p_j \end{bmatrix} = \begin{bmatrix} U & V + a_j \\ V - a_j & -U \end{bmatrix}\begin{bmatrix} q_j \\ p_j \end{bmatrix}$$

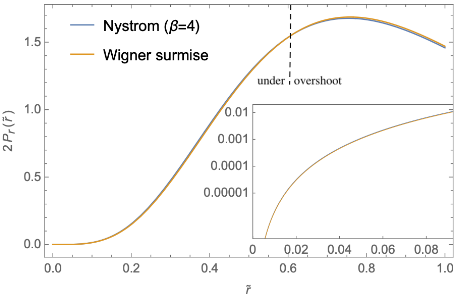
$$s\frac{dU}{ds} = -a_1(q_1^2 - p_1^2) + a_2(q_2^2 - p_2^2) \quad , \quad s\frac{dV}{ds} = -2a_1q_1p_1 + 2a_2q_2p_2 \quad \text{[BCs at } s = \epsilon]$$

Wigner-like surmise

Gap-ratio distribution for $\mathrm{G}\beta\mathrm{E}_{N=3}$

$$P_{\mathrm{r}}^{\mathrm{W}}(r) = C_\beta \frac{(r+r^2)^\beta}{(1+r+r^2)^{1+\frac{3}{2}\beta}}$$: (chances to be a) good approximation





GUE, GSE$_{N=3}$: rel. errors $\lesssim 4\%$ for $\tilde{r} \in [0,1]$ Atas et al. 2013

$\Rightarrow$ practically OK for most physical spectra (but not for ζ zeros)

4 Riemann ζ zeros

Spectral interpretation

Hilbert-Pólya conjecture

$$\zeta(\tfrac{1}{2}+it) \stackrel{?}{=} \text{``}\det(\mathbb{I}-t^{-1}\hat{H})\text{''} \quad \text{s.t.} \quad \hat{H}=\hat{H}^\dagger$$

local: **Montgomery conjecture** 1972

correlations of unfolded ζ zeros $x_n=\bar{\rho}(\gamma_n)\gamma_n=\frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}$ approach GUE as $n\to\infty$

local: **Katz-Sarnak philosophy** 1997

correlations of low zeros of $L(\frac{1}{2}+it,\chi_d)$ averaged over d approach C, D, or B as $d\to\infty$

global: **Keating-Snaith conjecture** 2000

moments of ζ are essentially identical to characteristic polynomials of CUE$_N$

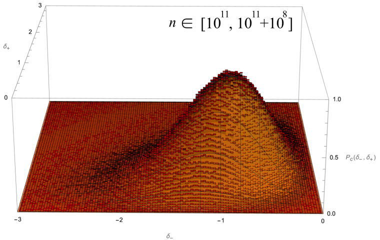
$$\lim_{T\to\infty}\int_0^T\frac{dt}{T}\left|\zeta(\frac{1}{2}+it)\right|^{2k}=a_k\frac{G(k+1)^2}{G(2k+1)}\left(\log\frac{T}{2\pi}\right)^{k^2}\Leftrightarrow\lim_{N\to\infty}\int_{\mathrm{U}(N)}dU\left|\det(\mathbb{I}-U)\right|^{2k}=\frac{G(k+1)^2}{G(2k+1)}N^{k^2}$$

$r_n=\dfrac{\gamma_{n+1}-\gamma_n}{\gamma_n-\gamma_{n-1}}$ is unaffected by the extra unfolding factor $1+\dfrac{C}{\log(\gamma_n/2\pi)}$ for the finite-size correction (Bogomolny et al 2006) $\Rightarrow$ suited to quantify deviation from GUE

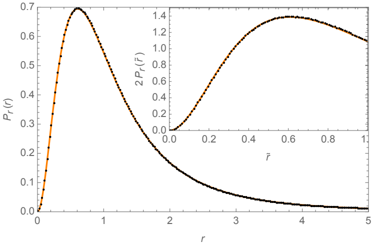
Consecutive spacings of zeros

windows of zeros: $W_N = \{\frac{1}{2} + i\gamma_n \mid n \in [N, 1.001N]\}$

N	γ_N	$\langle \tilde{r}_n \rangle$	$\langle \tilde{r}_n^2 \rangle$	$\langle \tilde{r}_n^3 \rangle$	$\langle \tilde{r}_n^4 \rangle$
10^8	$4.265354 \cdot 10^7$	.6032357	.4168926	.3133507	.2489623
10^9	$3.718702 \cdot 10^8$	.6021928	.4158748	.3125019	.2482868
10^{10}	$3.293531 \cdot 10^9$	.6014386	.4149925	.3116161	.2474310
$1.037 \cdot 10^{11}$	$3.058187 \cdot 10^{10}$	.6010277	.4145862	.3112812	.2471641
	GUE	.5997504	.4132049	.3100223	.2460560



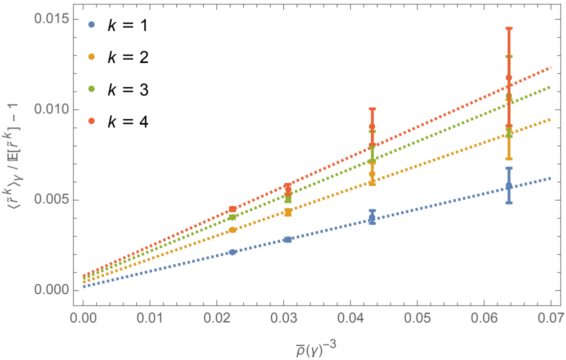
joint distribution of consecutive unfolded spacings



spacing-ratio distribution

Finite-size correction

Relative deviation of $\langle \tilde{r}_n^k \rangle_{W_N}$ from GUE



error bar: statistical fluctuation estimated by 10-bin Jackknife method

$\langle \tilde{r}_n^k \rangle_{W_N} / \mathbb{E}[\tilde{r}^k]_{\mathrm{GUE}} - 1$ asymptotes to 0, heuristically in proportion to $\bar{\rho}(\gamma_N)^{-3}$

Summary

- T-W criteria: covariant constancy $(\partial_x + \mathcal{A}(x))\Psi(x) = 0$ for rational $\mathfrak{sl}(2)$ conn. $\mathcal{A}$
- Fixing of EVs: $K(x,y) = \dfrac{\Psi(x)^T J \Psi(y)}{x-y} \mapsto \tilde{K}(x,y) = \dfrac{\tilde{\Psi}(x)^T J \tilde{\Psi}(y)}{x-y}$
  is rational $\mathrm{SL}(2)$ gauge transformation $\Psi(x) \mapsto \tilde{\Psi}(x) = U(x)\Psi(x)$
- Applicability of T-W method is inherited from K to $\tilde{K}$ (from Gap prob. to Jánossy)
- Jánossy densities for K_{Airy}, K_{Bessel} are determined by T-W system of ODEs
- Gap ratio $r_n = \dfrac{E_{n+1} - E_n}{E_n - E_{n-1}}$ is suited as local measure of quantum chaoticity
- Jánossy density ($\to$ Gap-ratio distribution) for K_{Sin} is determined by T-W system
- Convergence of $\langle \tilde{r}_n^k \rangle$ of ζ-zeros $\overset{\gamma_N \to \infty}{\longrightarrow}$ GUE is quantified, evading unfolding

To-do list: q-, Pearcey, finite-T fermions; SE, OE, Brownian motion; (elliptic)Ginibre,...;
relate T-W DEs for Jánossy densities to Painlevé/τ-functions/Hamiltonian systems