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Eigenphase distributions of unimodular circular ensembles

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Motivated by the study of Polyakov lines in gauge theories, Hanada and Watanabe recently presented a conjectured formula for the distribution of eigenphases of Haar-distributed random SU(N) matrices ($\beta = 2$), supported by explicit examples at small N and by numerical samplings at larger N. In this letter, I spell out a concise proof of their formula, and present its orthogonal and symplectic counterparts, i.e. the eigenphase distributions of Haar-random unimodular symmetric ($\beta = 1$) and selfdual ($\beta = 4$) unitary matrices parametrizing SU(N)/SO(N) and SU($2N$)/Sp($2N$), respectively.

Subject Index A10, A13, B83, B86

1. *Foreword and motivation.* This letter is inspired by a conjectured formula (117) in Ref. [1], presented without a proof. A major goal of that paper is to quantify the partial deconfinement of lattice gauge theories at finite temperature in terms of the statistical distribution of the eigenphases of the Polyakov line $P(\vec{n}) = U_0(0, \vec{n})U_0(1, \vec{n}) \cdots U_0(L_t - 1, \vec{n})$, treated as random SU($N$) matrices. From the gauge-theory point of view, it is crucial to consider a simple group SU(N) rather than semisimple U(N), because, obviously, the running of the coupling constant for each simple or Abelian factor of a gauge group is different. This naturally led the authors of Ref. [1] to conjecture the eigenphase distribution of *Haar-distributed* random SU(N) matrices, i.e. the circular unitary ensemble (CUE) with a unimodular constraint $\det U = 1$. Their formula is based upon explicit examples at $N = 2, 3$ and numerical samplings at larger N [2].

In the study of so-called fixed-trace random matrix ensembles [3], typically, the sum of the squared eigenvalues $\mathrm{tr}\, H^2 = \sum_{j=1}^{N} \lambda_j^2$ of random $N \times N$ Hermitian matrices H is constrained to a specific value. Although this type, or the more generic type ($\mathrm{tr}\, V(H) = \text{const.}$ [4]), of constraint respects the U(N) invariance of the unconstrained ensemble, additional interactions among multiple eigenvalues induced by the trace constraint destroy the determinantal property of their correlation functions. Due to this difficulty, one often had to be content with either a macroscopic large-N limit by the Coulomb-gas method [4] or asymptotic universality at $N \gg 1$ [5], while subtleties in the local correlations of eigenvalues still remain elusive.

In this letter, I make my tiny contribution to the field of constrained random matrices, namely a proof of the aforementioned conjecture on the density of the eigenphases of Haar-random SU(N) matrices. My proof encompasses Dyson's threefold way [6] all at once, as it automatically provides the densities of the eigenphases of Haar-random symmetric SU(N)

matrices ($U = U^T$) parametrizing the quotient SU(N)/SO(N) and selfdual SU($2N$) matrices ($U = U^D := JU^TJ^{-1}$, $J = i\sigma_2 \otimes \mathbb{I}_N$) parametrizing the quotient SU($2N$)/Sp($2N$), i.e. the circular orthogonal and symplectic ensembles (COE, CSE) with unimodular constraints. It would be my pleasure for this letter to serve as a useful appendix to Ref. [1].

2. *Theorem and proof*

. **Theorem**

Let $\{e^{i\theta_1}, \ldots, e^{i\theta_{N-1}}, e^{i\theta_N}(= e^{-i(\theta_1+\cdots+\theta_{N-1})})\}$ be the set of N eigenphases of either SU(N) matrices ($\beta = 2$), symmetric SU(N) matrices ($\beta = 1$), or selfdual SU($2N$) matrices ($\beta = 4$) that are Haar-distributed. Then the probability density of these eigenphases is given by[1]

$$\rho_{\beta,N}(\theta) = \frac{N}{2\pi} \times \left\{ \begin{array}{ll} 1 - (-1)^N \dfrac{2}{N} \cos N\theta & (\beta = 2) \\ 1 - (-1)^N \dfrac{\sqrt{\pi}(N-1)!}{2^{N-1}\Gamma(N/2+3/2)\Gamma(N/2+1)} \cos N\theta & (\beta = 1) \\ 1 - (-1)^N \dfrac{(2N)!!}{(2N-1)!!N} \cos N\theta + \dfrac{2}{(2N-1)N} \cos 2N\theta & (\beta = 4) \end{array} \right\}. \quad (1)$$

Proof. The normalized joint distributions of N eigenphases $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$ of Haar-distributed U(N) matrices ($\beta = 2$), symmetric U(N) matrices ($\beta = 1$), and selfdual U($2N$) matrices ($\beta = 4$) [denoted as CβE(N), respectively] are well known to be [7]

$$d\mu_{\mathrm{C}\beta\mathrm{E}(N)}(\theta_1, \ldots, \theta_N) = \frac{1}{C_{\beta,N}} \prod_{j=1}^{N} \frac{d\theta_j}{2\pi} \cdot |\Delta_N(\vec{\theta})|^\beta, \quad C_{\beta,N} = \frac{\Gamma(\beta N/2+1)}{\Gamma(\beta/2+1)^N}. \quad (2)$$

Here $\Delta_N(\vec{\theta}) := \prod_{1\le j<k\le N}(e^{i\theta_j} - e^{i\theta_k})$ stands for the Vandermonde determinant. Upon imposing the unimodular constraint $\det U = \prod_{j=1}^N e^{i\theta_j} = 1$, the joint distribution of $(N-1)$ *independent* eigenphases is given by

$$\begin{aligned} d\mu_{\beta,N}(\theta_1, \ldots, \theta_{N-1}) &= \frac{1}{C_{\beta,N}} \prod_{j=1}^{N-1} \frac{d\theta_j}{2\pi} \cdot |\Delta_N(\vec{\theta})|^\beta \Big|_{\theta_N = -\sum_{j=1}^{N-1}\theta_j} \\ &= \frac{1}{C_{\beta,N}} \int_{\theta_N} \prod_{j=1}^{N} \frac{d\theta_j}{2\pi} \cdot |\Delta_N(\vec{\theta})|^\beta \cdot 2\pi\delta\Big(\sum_{k=1}^{N} \theta_k \ (\mathrm{mod}\ 2\pi)\Big) \\ &= \sum_{n=-\infty}^{\infty} \int_{\theta_N} d\mu_{\mathrm{C}\beta\mathrm{E}(N)}(\theta_1, \ldots, \theta_N) \prod_{k=1}^{N} e^{in\theta_k}. \end{aligned} \quad (3)$$

Here $\int_{\theta_N}$ denotes an integral $\int_{-\pi}^{\pi}$ over the variable θ_N, and use is made of the Fourier expansion of the periodic delta function, $\delta(\theta \bmod 2\pi) = (2\pi)^{-1}\sum_n e^{in\theta}$.

[1] Excluding an exceptional case with $\beta = 1$, $N = 2$, for which $\rho_{1,2}(\theta) = |\sin\theta|/2$ trivially follows.

The probability distribution of a single eigenphase of unimodular matrices U is

$$\begin{aligned}\rho_{\beta,N}(\theta) &= \mathbb{E}[\mathrm{tr}\,\delta(\theta + i\log U)] \\ &= \int\!\dots\!\int_{-\pi}^{\pi} d\mu_{\beta,N}(\theta_1,\dots,\theta_{N-1}) \sum_{j=1}^{N} \delta(\theta-\theta_j)\Big|_{\theta_N=-\sum_{j=1}^{N-1}\theta_j} \\ &= \sum_{n=-\infty}^{\infty} \int\!\dots\!\int_{-\pi}^{\pi} d\mu_{\mathrm{C}\beta\mathrm{E}(N)}(\theta_1,\dots,\theta_N) \prod_{k=1}^{N} e^{in\theta_k}\cdot N\delta(\theta-\theta_N). \end{aligned} \tag{4}$$

We used the permutation symmetry of θ_j. After performing an integration over θ_N and a constant shift of the variables $\theta_j \mapsto \theta_j + \theta$ ($j = 1, \dots, N-1$), it reads

$$\rho_{\beta,N}(\theta) = \frac{N}{2\pi}\frac{1}{C_{\beta,N}} \sum_{n=-\infty}^{\infty} e^{inN\theta} \int\!\dots\!\int_{-\pi}^{\pi} \prod_{j=1}^{N-1} \left(\frac{d\theta_j}{2\pi} e^{in\theta_j}|1-e^{i\theta_j}|^{\beta}\right) |\Delta_{N-1}(\vec{\theta})|^{\beta}. \tag{5}$$

The integral in Eq. (5) is known as the Selberg integral [8] in Morris's trigonometric form [9] (see Ref. [10], p.134):

$$\begin{aligned}&\int\!\dots\!\int_{-\pi}^{\pi} \prod_{j=1}^{N} \left(\frac{d\theta_j}{2\pi} e^{i\frac{a-b}{2}\theta_j}|1-e^{i\theta_j}|^{a+b}\right) |\Delta_N(\vec{\theta})|^{2\lambda} \\ &= (-1)^{\frac{a-b}{2}N} \prod_{j=0}^{N-1} \frac{\Gamma(\lambda j + a + b + 1)\Gamma(\lambda j + \lambda + 1)}{\Gamma(\lambda j + a + 1)\Gamma(\lambda j + b + 1)\Gamma(\lambda + 1)}. \end{aligned} \tag{6}$$

Upon substituting $N \mapsto N-1$, $a = \beta/2 + n$, $b = \beta/2 - n$, $\lambda = \beta/2$ into Eq. (6), the LHS matches the integral in Eq. (5) and the RHS is equal to

$$\begin{cases} N!\,\delta_{n,0} + (-1)^{N-1}(N-1)!\,\delta_{n,\pm1} & (\beta=2) \\ \dfrac{\Gamma(N/2+1)}{\Gamma(3/2)^N}\delta_{n,0} + (-1)^{N-1}\dfrac{(N-1)!}{\Gamma(N/2+3/2)\Gamma(1/2)^{N-1}}\delta_{n,\pm1} & (\beta=1) \\ \dfrac{(2N)!}{2^N}\delta_{n,0} + (-2)^{N-1}N!(N-1)!\delta_{n,\pm1} + \dfrac{(2N-2)!}{2^{N-1}}\delta_{n,\pm2} & (\beta=4) \end{cases}, \tag{7}$$

except for a special case $\beta = 1$, $N = 2$. Substitution of Eq. (7) into Eq. (5) yields the theorem in Eq. (1). □

The theorem in Eq. (1) for $\beta = 2$ was conjectured in Eq. (117) of Ref. [1]. To the best of our knowledge, neither a proof nor even a conjecture of the theorem for $\beta = 1$ and 4 have been spotted anywhere in the literature. The above procedure is obviously applicable for imposing a constraint $\sum_{j=1}^{N}\theta_j = 0$ on the circular β-ensemble involving $|\Delta_N(\vec{\theta})|^{\beta}$ at a generic integer β.

Conflict of interest statement. None declared.

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References

[1] M. Hanada and H. Watanabe, [arXiv:2310.07533 [hep-th]] [Search inSPIRE].

[2] M. Fasi and L. Robol, Linear Algebra Appl. **620**, 297 (2021).
[3] N. Rosenzweig, Statistical mechanics of equally likely quantum systems, in Statistical Physics (Brandeis Summer Institute, 1962) (Benjamin, New York, 1963), Lectures in Theoretical Physics, Vol. 3, p. 91.
[4] G. Akemann, G. M. Cicuta, L. Molinari, and G. Vernizzi, Phys. Rev. E **59**, 1489 (1999).
[5] F. Götze and M. Gordin, Commun. Math. Phys. **281**, 203 (2008).
[6] F. J. Dyson, J. Math. Phys. **3**, 1199 (1962).
[7] F. J. Dyson, J. Math. Phys. **3**, 140 (1962).
[8] A. Selberg, Norsk Mat. Tidsskr. **26**, 71 (1944).
[9] W. G. Morris, Constant term identities for finite and affine root systems: conjectures and theorems, Ph.D. Thesis, University of Wisconsin-Madison (1982).
[10] P. J. Forrester, Log-Gases and Random Matrices (Princeton University Press, Princeton, NJ, 2010).