Georgian Mathematical Journal
Volume 7 (2000), Number 3, 577-584

OSCILLATION OF THE RIEMANN-WEBER VERSION OF EULER DIFFERENTIAL EQUATIONS WITH DELAY

JITSURO SUGIE AND MITSURU IWASAKI

Abstract. Our concern is to consider delay differential equations of Euler type. Necessary and sufficient conditions for the oscillation of solutions are given. The results extend some famous facts about Euler differential equations without delay.

2000 Mathematics Subject Classification: Primary: 34C10, 34K15; secondary 34A30
Key words and phrases: Delay differential equations, oscillation

1. Introduction

We consider the second order differential equation

$$y''(t) + \frac{1}{4t^2}y(t) + \frac{\delta}{(t\log t)^2}y(ct) = 0, \quad t > 0, \tag{1}$$

where $' = d/dt$ and δ and c are the parameters satisfying

$$\delta > 0 \quad \text{and} \quad 0 < c \leq 1.$$

Let $t_0 > 0$. By a *solution* of (1) at t_0 is meant a function $y : [ct_0, t_1) \to \mathbf{R}$ for some t_1, which satisfies (1) for all $t \in [t_0, t_1)$. Since equation (1) is linear, all solutions of (1) are continuable in the future. We may therefore assume that $t_1 = \infty$. A solution $y(t)$ of (1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $y(t_n) = 0$. Otherwise, $y(t)$ is said to be *nonoscillatory*.

We can find many results on the oscillation of delay differential equations of Euler type (see [1–3], etc.). The purpose of this paper is to obtain a necessary and sufficient condition in terms of δ and c under which all nontrivial solutions of (1) are oscillatory.

In case $c = 1$, equation (1) can be rewritten as

$$y'' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\delta}{(\log t)^2}\right)y = 0. \tag{2}$$

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

This linear differential equation without delay is called the Riemann–Weber version of the Euler differential equation (see [4]) and has a general solution

$$y(t) = \begin{cases} \sqrt{t}\{K_1(\log t)^z + K_2(\log t)^{1-z}\} & \text{if } \delta \neq 1/4, \\ \sqrt{t \log t}\{K_3 + K_4 \log(\log t)\} & \text{if } \delta = 1/4, \end{cases}$$

where K_i $(i = 1, 2, 3, 4)$ are arbitrary constants and z is the root of

$$z(1-z) = \delta.$$

Hence we can divide equation (2) into two types as follows:

Proposition. *All nontrivial solutions of* (2) *are oscillatory when* $\delta > 1/4$ *and are nonoscillatory when* $\delta \leq 1/4$.

In the theory of oscillations, the number 1/4 very often appears as a critical value. The question now arises whether the critical value for equation (1) is also 1/4 or not. If not so, then what condition is necessary for all nontrivial solutions of (1) to be oscillatory? We give a complete answer to this question. Our main result is the following

Theorem 1. *All nontrivial solutions of* (1) *are oscillatory if and only if*

$$\delta > \frac{1}{4\sqrt{c}}. \tag{3}$$

Clearly, Theorem 1 is a generalization of the proposition above. The proof of the theorem is given in the next section.

By Theorem 1 we see that solutions of (1) have a tendency to be nonoscillatory as the delay grows larger, i.e., the parameter c becomes smaller. In other words, the delay has an adverse effect on the oscillation of solutions of (1). This is a point of difference between equation (1) and the first order Euler differential equation

$$y'(t) + \frac{\delta}{t} y(ct) = 0. \tag{4}$$

In fact, changing variables $t = e^s$, we can transform equation (4) into the equation

$$\dot{x}(s) + \delta\, x(s - r) = 0, \tag{5}$$

where $\dot{} = d/ds$; $r = -\log c$ and $x(s) = y(e^s) = y(t)$. Hence, as is well known, the condition

$$\delta\, r > \frac{1}{e}$$

is necessary and sufficient for all nontrivial solutions of (5) (or (4)) to be oscillatory. We therefore see that the delay has a positive effect on the oscillation of solutions of (4).

In the final section, we compare Theorem 1 with a previous oscillation result concerning the second order Euler differential equation with delay

$$y''(t) + \frac{\delta}{t^2}y(ct) = 0$$

and discuss the effect of the delayed term. We also consider another Euler delay differential equation, which is different from equation (1), and give a necessary and sufficient condition for the oscillation of solutions. This result bears a dual relation to Theorem 1.

2. Proof of the Main Theorem

To prove Theorem 1, we need Brands' result. Brands [5] gave some comparison theorems on the oscillation behaviour of solutions of second order differential equations with delay. Mahfoud [6] extended his result by using a famous lemma of Kiguradze [7]. The following result is a prototype of them.

Theorem A. *Let τ_1 and τ_2 be nonnegative numbers and $p(s)$ is a nonnegative continuous function. Then all nontrivial solutions of*

$$\ddot{x}(s) + p(s)x(s-\tau_1) = 0$$

are oscillatory if and only if all nontrivial solutions of

$$\ddot{x}(s) + p(s)x(s-\tau_2) = 0$$

are oscillatory.

Theorem A shows that the constant delay τ has no effect on the oscillation of solutions of the equation $\ddot{x}(s) + p(s)x(s-\tau) = 0$. Let $\tau_1 = r$ and $\tau_2 = 0$ in Theorem A. Then we see that for any $r > 0$, all nontrivial solutions of

$$\ddot{x}(s) + \frac{\gamma}{s^2}x(s-r) = 0$$

are oscillatory if and only if

$$\gamma > 1/4,$$

which is a necessary and sufficient condition for all nontrivial solutions of the corresponding differential equation without delay

$$\ddot{x} + \frac{\gamma}{s^2}x = 0$$

to be oscillatory.

Proof of Theorem 1. Let $s = \log t$ and put $r = -\log c$ and $u(s) = y(e^s)$. Then

$$y(ct) = y(e^{s-r}) = u(s-r),$$

and therefore equation (1) is transformed into the equation

$$\ddot{u}(s) - \dot{u}(s) + \frac{1}{4}u(s) + \frac{\delta}{s^2}u(s-r) = 0. \tag{6}$$

By setting $x(s) = u(s)\exp(-s/2)$ this equation becomes

$$\ddot{x}(s) + \frac{\delta \exp(-r/2)}{s^2} x(s-r) = 0 \tag{7}$$

because

$$\ddot{x}(s) = \left\{ \ddot{u}(s) - \dot{u}(s) + \frac{1}{4} u(s) \right\} \exp(-s/2)$$

and

$$x(s-r) = u(s-r)\exp((r-s)/2).$$

Now, we consider the second order Euler differential equation without delay

$$\ddot{x} + \frac{\delta \exp(-r/2)}{s^2} x = 0. \tag{8}$$

Then, if assumption (3) holds, we have

$$\delta \exp(-r/2) = \delta\sqrt{c} > \frac{1}{4}.$$

Hence all nontrivial solutions of (8) are oscillatory. From Theorem A we conclude that all nontrivial solutions of (7) are oscillatory, and so are all nontrivial solutions of (1). Conversely, if assumption (3) is not satisfied, then by means of Theorem A again we see that equation (1) has a nonoscillatory solution. □

3. Discussion

Kulenović [2] investigated the oscillation problem for the second order Euler differential equation

$$y''(t) + \frac{\delta}{t^2} y(ct) = 0, \tag{9}$$

which has a simpler form than equation (1). Transforming equation (9) into the differential equation with delay

$$\ddot{x}(s) - \dot{x}(s) + \delta\, x(s-r) = 0 \tag{10}$$

and using the fact that all nontrivial solutions of (10) are oscillatory if and only if the corresponding characteristic equation

$$\lambda^2 - \lambda + \delta \exp(-\lambda r) = 0$$

has no real roots, he gave the following result.

Theorem B. *Every solution of* (9) *with* $0 < c < 1$ *oscillates if and only if*

$$\delta > \frac{\sqrt{r^2+4}-2}{r^2} \exp\left\{ \frac{r-2+\sqrt{r^2+4}}{2} \right\}. \tag{11}$$

It is well known that the oscillation of solutions is determined by the roots of a characteristic equation (see, for example, [8–13] and the references cited therein). Theorem B extends the following results in [14] and [3]: if $\delta > 1/(4c)$, then every solution of (9) is oscillatory; if $0 < \delta \leq 1/4$, then equation (7) has a nonoscillatory solution. However Theorem B cannot be applied to equation (1).

Let us now look at the oscillation of solutions of Euler differential equations from a different angle. Equation (10) is equivalent to the system

$$\begin{aligned} \dot{u}(s) &= v(s) + u(s), \\ \dot{v}(s) &= -\delta\, u(s-r). \end{aligned} \tag{12}$$

As is customary, we say that a solution $(u(s), v(s))$ is *oscillatory* if $u(s)$ has zeros for arbitrarily large s. By virtue of Theorem B we can decide whether all nontrivial solutions of (12) are oscillatory or not. Using L'Hospital's rule, we see that the right-hand side of (11) tends to $1/4$ as $c \to 1$ (or $r \to 0$). Hence Theorem B is a generalization of the elementary result that $\delta = 1/4$ is a critial value for the oscillation of solutions of the system

$$\begin{aligned} \dot{u} &= v + u, \\ \dot{v} &= -\delta\, u. \end{aligned}$$

We shall exmaine the critical value in more detail. Judging from the oscillation results above, it seems to be natural to consider the system

$$\begin{aligned} \dot{u}(s) &= v(s) + u(s), \\ \dot{v}(s) &= -\frac{1}{4}u(s) - \delta\, u(s-r). \end{aligned} \tag{13}$$

Nevertheless, since the characteristic equation of (13)

$$\lambda^2 - \lambda + \frac{1}{4} + \delta \exp(-\lambda r) = 0$$

has no real roots when $\delta > 0$, all nontrivial solutions of (13) are oscillatory. This means that the effect of the delayed term $u(s-r)$ is the same as that of the nondelayed term $u(s)$ on the oscillation of solutions of (13). We can interpret that the quantity $-u(s)/4$ in the second equation of (13) is the limit for nonoscillation of solutions and if we add any delayed term, then system (13) has no longer a nonoscillatory solution.

Since the second equation of (12) is rewritten as

$$\dot{v}(s) = -0\, u(s) - \delta\, u(s-r),$$

the term without delay is ignored, and therefore the parameter δ of the delayed term must satisfy estimate (11) in Theorem B.

To weaken the effect of the delayed term, we consider the system

$$\begin{aligned} \dot{u}(s) &= v(s) + u(s), \\ \dot{v}(s) &= -\frac{1}{4}u(s) - \frac{\delta}{s^2}u(s-r). \end{aligned} \tag{14}$$

Since system (14) is equivalent to equation (6), by Theorem 1 we see that all nontrivial solutions of (14) are oscillatory if and only if

$$\delta > \frac{1}{4}\sqrt{e^r}.$$

Finally, we consider the system

$$\begin{aligned} \dot{u}(s) &= v(s) + k\,u(s), \\ \dot{v}(s) &= -\frac{k^2}{4}u(s) - \frac{\delta}{s^2}u(s-r) \end{aligned} \tag{15}$$

with $k \neq 0$ and its equivalent equation

$$\ddot{u}(s) - k\,\dot{u}(s) + \frac{k^2}{4}u(s) + \frac{\delta}{s^2}u(s-r) = 0, \tag{16}$$

which are more general forms than (14) and (6), respectively. Then, as in the proof of Theorem 1, we obtain the following result.

Theorem 2. *All nontrivial solutions of* (15) *or* (16) *are oscillatory if and only if*

$$\delta > \frac{1}{4}\sqrt{e^{kr}}.$$

In case $k > 0$, putting $t = e^{ks}$ and $y(t) = y(e^{ks}) = u(s)$, we can transform equation (16) into the Riemann–Weber version of the Euler differential equation with delay (1) where $c = e^{-kr}$. Hence Theorems 1 and 2 are essentially the same in this case.

On the other hand, if $k < 0$, then we can give an alternative expression instead of Theorem 2. Let $t = e^{-ks}$ and put $y(t) = y(e^{-ks}) = u(s)$ and $c = e^{kr}$. Then equation (16) becomes

$$y''(t) + \frac{2}{t}y'(t) + \frac{1}{4t^2}y(t) + \frac{\delta}{(t\log t)^2}y(ct) = 0. \tag{17}$$

Hence we have the following

Theorem 3. *All nontrivial solutions of* (17) *are oscillatory if and only if*

$$\delta > \frac{\sqrt{c}}{4}.$$

The delay in equation (17) has a positive effect on the oscillation of solutions. Although equations (1) and (17) are of Euler type, the oscillation properties are completely different.

ACKNOWLEDGEMENT

The first author was supported in part by Grant-in-Aid for Scientific Research 10874028.

REFERENCES

1. J. Jaroš, Necessary and sufficient conditions for bounded oscillations of higher order delay differential equations of Euler's type. *Czechoslovak Math. J.* **39**(1989), 701–710.

2. M. R. S. Kulenović, Oscillation of the Euler differential equation with delay. *Czechoslovak. Math. J.* **45**(1995), 1–6.

3. J. S. W. Wong, Second order oscillation with retarded arguments. *Ordinary Differential Equations,* 581–596, *Academic Press, New York,* 1972.

4. E. Hille, Non-oscillation theorems. *Trans. Amer. Math. Soc.* **64**(1948), 234–252.

5. J. J. A. M. Brands, Oscillation theorems for second-order functional differential equations. *J. Math. Anal. Appl.* **63**(1978), 54–64.

6. W. E. Mahfoud, Comparison theorems for delay differential equations. *Pacific J. Math.* **83**(1979), 187–197.

7. I. T. Kiguradze, Oscillation properties of solutions of certain ordinary differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **144**(1962), 33–36.

8. L. H. Erbe, Qingkai Kong, and B. G. Zhang, Oscillation theorem for functional differential equations. *Marcel Dekker, New York,* 1995.

9. K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics. *Kluwer Academic Publishers, Dordrecht,* 1992.

10. I. Győri and G. Ladas, Oscillation theory of delay differential equations with applications. *Clarendon Press, Oxford,* 1991.

11. G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, Necessary and sufficient conditions for oscillations of higher order delay differential equations. *Trans. Amer. Math. Soc.* **285**(1984), 81–90.

12. G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, Oscillation theory of differential equations with deviating arguments. *Marcel Dekker, New York,* 1987.

13. V. Lakshmikantham, L. Z. Wen, and B. G. Zhang, Theory of differential equations with unbounded delay. *Kluwer Academic Publishers, Dordrecht,* 1994.

14. W. E. Mahfoud, Oscillation theorems for a second order delay differential equation. *J. Math. Anal. Appl.* **63**(1978), 339–346.

(Recieved 5.04.2000)

Authors' address:
Department of Mathematics and Computer Science
Shimane University
Matsue 690-8504
Japan
E-mail: jsugie@math.shimane-u.ac.jp