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TENSORIAL AND HADAMARD PRODUCT INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA A CARTWRIGHT-FIELD RESULT

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if the functions f and g are continous and positive on the interval I and such that there exist the positive numbers $m<M$ with

$$0<m\leq\frac{f(t)}{g(t)}\leq M \text{ for all } t\in I,$$

then, for the selfadjoint operators A, B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A)\subset I$, we have the tensorial inequalities

$$\begin{aligned}0&\leq\frac{1}{M}\nu(1-\nu)\\&\times\left[\frac{\left(f^{2}(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^{2}(B)g^{-1}(B)\right)}{2}-f(A)\otimes f(B)\right]\\&\leq(1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B)\\&-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)g^{1-\nu}(B)\right)\\&\leq\frac{1}{m}\nu(1-\nu)\\&\times\left[\frac{\left(f^{2}(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^{2}(B)g^{-1}(B)\right)}{2}-f(A)\otimes f(B)\right]\end{aligned}$$

for all $\nu\in[0,1]$. Some similar inequalities for Hadamard product are also given.

1. Introduction

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\max\{a,b\}}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\min\{a,b\}}\tag{1.1}$$

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for any $a,\ b>0$ and $\nu\in[0,1]$.

This result was obtained in 1978 by Cartwright and Field [4] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Since $\max\{a,b\}\min\{a,b\}=ab$ for $a,\ b>0$, then by (1.1) we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab}&\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\le\frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab},\end{aligned}$$

namely

$$\begin{aligned}0&\le\frac{1}{2}\nu(1-\nu)\min\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right)\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\&\le\frac{1}{2}\nu(1-\nu)\max\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right),\end{aligned}\tag{1.2}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_k)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k)\tag{1.3}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f(A_1,...,A_k)=f_1(A_1)\otimes...\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k)=f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is known that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st)\ge(\le)f(s)f(t)\ \text{ for all } s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [7, p. 173]

$$f(A\otimes B)\ge(\le)f(A)\otimes f(B)\ \text{ for all } A,\ B\ge0.\tag{1.4}$$

This follows by observing that, if

$$A=\int_{[0,\infty)} tdE\left(t\right) \text{ and } B=\int_{[0,\infty)} sdF\left(s\right)$$

are the spectral resolutions of A and B, then

$$f\left(A\otimes B\right)=\int_{[0,\infty)}\int_{[0,\infty)} f\left(st\right)dE\left(t\right)\otimes dF\left(s\right) \tag{1.5}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators A, $B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [9] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\leq\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\leq\frac{1}{2}\left(A\otimes B+B\otimes A\right)\end{aligned} \tag{1.6}$$

for A, $B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)\,e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B=\mathcal{U}^*\left(A\otimes B\right)\mathcal{U} \tag{1.7}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative operator concave (sub-multiplicative operator convex)* on $[0,\infty)$, then also [7, p. 173]

$$f\left(A\circ B\right)\geq(\leq)\,f\left(A\right)\circ f\left(B\right) \text{ for all } A,\ B\geq 0. \tag{1.8}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1 \text{ for } A>0. \tag{1.9}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}\text{ for }A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2}\text{ for }A,\ B\ge 0.$$

It has been shown in [8] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we obtain some lower and upper bounds for the quantities

$$(1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B)-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)g^{1-\nu}(B)\right)$$

and

$$(1-\nu)f(A)\circ g(B)+\nu g(A)\circ f(B)-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\circ\left(f^{\nu}(B)g^{1-\nu}(B)\right)$$

with $\nu\in[0,1]$, under the assumptions that the functions f and g are continuous and positive on the interval I and such that there exists the positive numbers $m<M$ such that

$$0<m\le\frac{f(t)}{g(t)}\le M\text{ for all }t\in I,$$

while the selfadjoint operators A, B are with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A)\subset I$.

2. Main Results

We have the following main result:

Theorem 1. *Assume that the functions f and g are continuous and positive on the interval I and such that there exist the positive numbers $m<M$ such that*

$$0<m\le\frac{f(t)}{g(t)}\le M\textit{ for all }t\in I,$$

then for the selfadjoint operators A, B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A)\subset I$, we have the tensorial inequalities

$$\begin{aligned}(2.1)\quad 0&\le\frac{1}{M}\nu(1-\nu)\\&\times\left[\frac{\left(f^2(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^2(B)g^{-1}(B)\right)}{2}-f(A)\otimes f(B)\right]\\&\le(1-\nu)f(A)\otimes g(B)+\nu g(A)\otimes f(B)\\&-\left(f^{1-\nu}(A)g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)g^{1-\nu}(B)\right)\\&\le\frac{1}{m}\nu(1-\nu)\\&\times\left[\frac{\left(f^2(A)g^{-1}(A)\right)\otimes g(B)+g(A)\otimes\left(f^2(B)g^{-1}(B)\right)}{2}-f(A)\otimes f(B)\right]\end{aligned}$$

for $\nu\in[0,1]$.

Proof. Now if $a,\ b\in[m,M]\subset(0,\infty)$, then we have from (1.1) the following inequalities

$$0\leq\frac{1}{2M}\nu(1-\nu)\left(a^{2}-2ab+b^{2}\right)\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2m}\nu(1-\nu)\left(a^{2}-2ab+b^{2}\right)\tag{2.2}$$

for $\nu\in[0,1]$.

Since

$$a=\frac{f(t)}{g(t)},\ b=\frac{f(s)}{g(s)}\in[m,M]\text{ for all }t,s\in I,$$

then by (2.2) we get

$$\begin{aligned}0&\leq\frac{1}{2M}\nu(1-\nu)\left(\left(\frac{f(t)}{g(t)}\right)^{2}-2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)}+\left(\frac{f(s)}{g(s)}\right)^{2}\right)\\&\leq(1-\nu)\frac{f(t)}{g(t)}+\nu\frac{f(s)}{g(s)}-\left(\frac{f(t)}{g(t)}\right)^{1-\nu}\left(\frac{f(s)}{g(s)}\right)^{\nu}\\&\leq\frac{1}{2m}\nu(1-\nu)\left(\left(\frac{f(t)}{g(t)}\right)^{2}-2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)}+\left(\frac{f(s)}{g(s)}\right)^{2}\right)\end{aligned}\tag{2.3}$$

for all $t,\ s\in I$ and $\nu\in[0,1]$.

If we multiply the inequalities (2.3) by $g(t)g(s)\geq0$, then we get

$$\begin{aligned}0&\leq\frac{1}{2M}\nu(1-\nu)\left(\frac{f^{2}(t)}{g(t)}g(s)-2f(t)f(s)+\frac{f^{2}(s)}{g(s)}g(t)\right)\\&\leq(1-\nu)f(t)g(s)+\nu g(t)f(s)-f^{1-\nu}(t)g^{\nu}(t)f^{\nu}(s)g^{1-\nu}(s)\\&\leq\frac{1}{2m}\nu(1-\nu)\left(\frac{f^{2}(t)}{g(t)}g(s)-2f(t)f(s)+\frac{f^{2}(s)}{g(s)}g(t)\right)\end{aligned}\tag{2.4}$$

for all $t,\ s\in I$ and $\nu\in[0,1]$.

If

$$A=\int_{I}tdE(t)\text{ and }B=\int_{I}sdF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_I \int_I$ over $dE(t) \otimes dF(s)$ in (2.4) we get

$$
\begin{aligned}
(2.5)\quad 0 &\le \frac{1}{2M}\nu(1-\nu) \\
&\times \int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s) \\
&\le \int_I \int_I \left[(1-\nu) f(t) g(s) + \nu g(t) f(s) - f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) \right] \\
&\times dE(t) \otimes dF(s) \\
&\le \frac{1}{2m}\nu(1-\nu) \\
&\times \int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s)
\end{aligned}
$$

for all $\nu \in [0,1]$.

Now, by (1.3) we get

$$
\begin{aligned}
&\int_I \int_I \left(\frac{f^2(t)}{g(t)} g(s) - 2f(t) f(s) + \frac{f^2(s)}{g(s)} g(t) \right) dE(t) \otimes dF(s) \\
&= \int_I \int_I \frac{f^2(t)}{g(t)} g(s)\, dE(t) \otimes dF(s) + \int_I \int_I g(t) \frac{f^2(s)}{g(s)} dE(t) \otimes dF(s) \\
&- 2 \int_I \int_I f(t) f(s)\, dE(t) \otimes dF(s) \\
&= \left(f^2(A) g^{-1}(A) \right) \otimes g(B) + g(A) \otimes \left(f^2(B) g^{-1}(B) \right) \\
&- 2f(A) \otimes f(B),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_I \int_I \left[(1-\nu) f(t) g(s) + \nu g(t) f(s) - f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s) \right] \\
&\times dE(t) \otimes dF(s) \\
&= (1-\nu) \int_I \int_I f(t) g(s)\, dE(t) \otimes dF(s) + \nu \int_I \int_I g(t) f(s)\, dE(t) \otimes dF(s) \\
&- \int_I \int_I f^{1-\nu}(t) g^{\nu}(t) f^{\nu}(s) g^{1-\nu}(s)\, dE(t) \otimes dF(s) \\
&= (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\
&- \left(f^{1-\nu}(A) g^{\nu}(A) \right) \otimes \left(f^{\nu}(B) g^{1-\nu}(B) \right).
\end{aligned}
$$

Then by (2.5) we get (2.1). ∎

Remark 1. *We observe that for* $\nu = 1/2$ *we obtain the following inequalities*

$$\begin{aligned} (2.6) \qquad 0 &\le \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B) + g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\right. \\ &\left. -f(A)\otimes f(B)\right] \\ &\le \frac{f(A)\otimes g(B) + g(A)\otimes f(B)}{2} \\ &- \left(f^{1/2}(A)\,g^{1/2}(A)\right)\otimes\left(f^{1/2}(B)\,g^{1/2}(B)\right) \\ &\le \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B) + g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\right. \\ &\left. -f(A)\otimes f(B)\right]. \end{aligned}$$

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} (2.7) \qquad 0 &\le \frac{1}{M}\nu(1-\nu) \\ &\times\left[\frac{(f^2(A)\,g^{-1}(A))\circ g(B) + g(A)\circ(f^2(B)\,g^{-1}(B))}{2} - f(A)\circ f(B)\right] \\ &\le (1-\nu)\,f(A)\circ g(B) + \nu g(A)\circ f(B) \\ &- \left(f^{1-\nu}(A)\,g^{\nu}(A)\right)\circ\left(f^{\nu}(B)\,g^{1-\nu}(B)\right) \\ &\le \frac{1}{m}\nu(1-\nu) \\ &\times\left[\frac{(f^2(A)\,g^{-1}(A))\circ g(B) + g(A)\circ(f^2(B)\,g^{-1}(B))}{2} - f(A)\circ f(B)\right] \end{aligned}$$

for all $\nu \in [0,1]$.

Proof. For $X, Y \in B(H)$, we have the representation

$$X\circ Y = \mathcal{U}^*(X\otimes Y)\,\mathcal{U}$$

where $\mathcal{U} : H \to H\otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ at the left and $\mathcal{U}$ at the right in the inequality (2.1), then we get

$$\begin{aligned} 0 &\le \frac{1}{M}\nu(1-\nu) \\ &\times\mathcal{U}^*\left[\frac{(f^2(A)\,g^{-1}(A))\otimes g(B) + g(A)\otimes(f^2(B)\,g^{-1}(B))}{2} - f(A)\otimes f(B)\right]\mathcal{U} \\ &\le \mathcal{U}^*\left[(1-\nu)\,f(A)\otimes g(B) + \nu g(A)\otimes f(B)\right. \\ &\left. - \left(f^{1-\nu}(A)\,g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)\,g^{1-\nu}(B)\right)\right]\mathcal{U} \\ &\le \frac{1}{m}\nu(1-\nu) \\ &\times\mathcal{U}^*\left[\frac{(f^2(A)\,g^{-1}(A))\otimes g(B) + g(A)\otimes(f^2(B)\,g^{-1}(B))}{2} - f(A)\otimes f(B)\right]\mathcal{U}, \end{aligned}$$

namely

$$0 \le \frac{1}{M}\nu(1-\nu)$$
$$\times \left[\frac{\mathcal{U}^*\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B)\right]\mathcal{U}+\mathcal{U}^*\left[g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\mathcal{U}}{2}\right.$$
$$\left.-\mathcal{U}^*\left(f(A)\otimes f(B)\right)\mathcal{U}\right]$$
$$\le (1-\nu)\,\mathcal{U}^*\left[f(A)\otimes g(B)\right]\mathcal{U}+\nu\mathcal{U}^*\left(g(A)\otimes f(B)\right)\mathcal{U}$$
$$-\mathcal{U}^*\left[\left(f^{1-\nu}(A)\,g^{\nu}(A)\right)\otimes\left(f^{\nu}(B)\,g^{1-\nu}(B)\right)\right]\mathcal{U}$$
$$\le \frac{1}{m}\nu(1-\nu)$$
$$\times \left[\frac{\mathcal{U}^*\left[\left(f^2(A)\,g^{-1}(A)\right)\otimes g(B)\right]\mathcal{U}+\mathcal{U}^*\left[g(A)\otimes\left(f^2(B)\,g^{-1}(B)\right)\right]\mathcal{U}}{2}\right.$$
$$\left.-\mathcal{U}^*\left(f(A)\otimes f(B)\right)\mathcal{U}\right],$$

which is equivalent to (2.7). ∎

Remark 2. *We observe that for $\nu = 1/2$ we obtain the following inequalities*

$$\begin{aligned}(2.8)\qquad 0 &\le \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(B)+g(A)\circ\left(f^2(B)\,g^{-1}(B)\right)\right]\right.\\ &\left.-f(A)\circ f(B)\right]\\ &\le \frac{f(A)\circ g(B)+g(A)\circ f(B)}{2}\\ &-\left(f^{1/2}(A)\,g^{1/2}(A)\right)\circ\left(f^{1/2}(B)\,g^{1/2}(B)\right)\\ &\le \frac{1}{4M}\left[\frac{1}{2}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(B)+g(A)\circ\left(f^2(B)\,g^{-1}(B)\right)\right]\right.\\ &\left.-f(A)\circ f(B)\right].\end{aligned}$$

Now, if we take $B = A$ in Corollary 1, then we get

$$\begin{aligned}(2.9)\qquad 0 &\le \frac{1}{M}\nu(1-\nu)\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right]\\ &\le f(A)\circ g(A)-\left(f^{1-\nu}(A)\,g^{\nu}(A)\right)\circ\left(f^{\nu}(A)\,g^{1-\nu}(A)\right)\\ &\le \frac{1}{m}\nu(1-\nu)\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right]\end{aligned}$$

for all $\nu \in [0,1]$.

In particular, for $\nu = 1/2$ we get

$$\begin{aligned}(2.10)\qquad 0 &\le \frac{1}{4M}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right]\\ &\le f(A)\circ g(A)-\left(f^{1/2}(A)\,g^{1/2}(A)\right)\circ\left(f^{1/2}(A)\,g^{1/2}(A)\right)\\ &\le \frac{1}{4m}\left[\left(f^2(A)\,g^{-1}(A)\right)\circ g(A)-f(A)\circ f(A)\right].\end{aligned}$$

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