Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematics
57 (2024), pp. 27–38

WELL-POSEDNESS OF QUADRATIC HARTREE TYPE EQUATIONS BELOW L^2

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Communicated by Toshihiro Nakanishi

(Received: April 21, 2023)

Abstract. This paper studies the Cauchy problem for the nonlinear Schrödinger equation $i\partial_t u - \partial_x^2 u = f(u)$ in one space dimension. The nonlinear interaction $f(u)$ is a linear combination of $(V *_x u)u$, $(V *_x \bar{u})u$, $(V *_x u)\bar{u}$ and $(V *_x \bar{u})\bar{u}$, where $V(x)$ is a locally integrable function whose Fourier transform satisfies $|\hat{V}(\xi)| \lesssim \langle\xi\rangle^{-m}$ for some $m \geq 0$. The Cauchy problem is well-posed in H^s for $s > -(m/2+1/4)$; furthermore, if $f(u)$ contains only the first and the last types of nonlinear terms, then the Cauchy problem is well-posed for $s > -(m/2+3/4)$. The proof is based on bilinear estimates in $X^{s,b}$ spaces.

1. Introduction

In this paper, we consider the Cauchy problem for the nonlinear Schrödinger equation

$$i\partial_t u - \partial_x^2 u = f(u), \qquad u(\cdot, 0) = u_0, \tag{1}$$

where $u : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{C}$, and the nonlinear interaction $f(u)$ is defined by

$$f(u) = \sum_{j=1}^{4} \lambda_j f_j(u) = \lambda_1 (V *_x u)u + \lambda_2 (V *_x \bar{u})u + \lambda_3 (V *_x u)\bar{u} + \lambda_4 (V *_x \bar{u})\bar{u} \tag{2}$$

with $\lambda_j \in \mathbb{C}$, $1 \leq j \leq 4$, $V = V(x)$, and $*_x$ means the convolution with respect to x. We study the well-posedness of (1) below $L^2(\mathbb{R})$, namely in the Sobolev space $H^s(\mathbb{R})$ with $s < 0$.

There is a lot of literature on the solvability and asymptotic behaviour of the Hartree equation

$$i\partial_t u - \Delta_x u = \lambda (V *_x |u|^2)u, \qquad u(\cdot, 0) = u_0 \tag{3}$$

with $(x, t) \in \mathbb{R}^{n+1}$, $V(x) = |x|^{-\gamma}$, $0 < \gamma < n$ and $\lambda \in \mathbb{C}$. The equation (3) is scaling-invariant in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ with critical index

2020 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35A01, 35A02.

Key words and phrases. Nonlinear Schrödinger equations, Nonlocal interaction, Fourier restriction method.

$s_c = \gamma/2 - 1$, and (3) is known to be locally well-posed in $H^s(\mathbb{R}^n)$ under the assumption $s \geq (s_c)_+ \equiv \max\{0; s_c\}$, see e.g. Ginibre–Velo [6], Hayashi–Ozawa [7] and Hirata [8]; see also Cazenave [3] and references therein. On the contrary, apart from Cho–Hwang–Ozawa [4], there are few results on the well-posedness of (3) for $s_c \leq s < 0$. In [4], under the assumption $n \geq 3$, $3/2 < \gamma < 2$, they proved that for $u_0 \in \dot{H}^{s_c}_{\mathrm{rad}}(\mathbb{R}^n)$ small enough, there exists a unique solution $u \in C_{\mathrm{b}}(\mathbb{R}; \dot{H}^{s_c}_{\mathrm{rad}}(\mathbb{R}^n))$ which scatters in $\dot{H}^{s_c}_{\mathrm{rad}}(\mathbb{R}^n)$ as $t \to \pm\infty$. Here the subscript b means the space of bounded functions, and $\dot{H}^{s_c}_{\mathrm{rad}}(\mathbb{R}^n)$ denotes the homogeneous Sobolev space for radially symmetric functions. They also obtained an analogous result for nonradial case, but they need some positive regularity for spherical coordinates.

It would be very interesting to study well-posedness of (3) in Sobolev spaces of negative order. However, it seems quite difficult to estimate cubic terms in negative order Sobolev spaces. Therefore, as a first step, we will consider the equation (1) with quadratic nonlinear terms instead.

The well-posedness of (1) in $H^s(\mathbb{R})$ with $f(u)$ replaced by

$$f_{\mathrm{p}}(u) = \mu_1 u^2 + \mu_2 |u|^2 + \mu_3 \bar{u}^2, \tag{4}$$

namely the case of quadratic power nonlinearity, has been extensively studied. In this case, the critical exponent in $\dot{H}^s(\mathbb{R})$ is $s_c = -3/2$. Tsutsumi [17] and Kato [9] proved that the Cauchy problem (1)–(4) is locally well-posed in $L^2(\mathbb{R})$; see also Cazenave [3]. For negative order Sobolev spaces, Kenig–Ponce–Vega [12] proved the local well-posedness for (i) $s > -3/4$ with $\mu_2 = 0$, or (ii) $s > -1/4$, by the method of Fourier restriction norm initiated by Bourgain [2]. The result in case (i) was generalized for $s \geq -1$ by Bejenaru–Tao [1] and Kishimoto [13, 14].

We need the free Schrödinger group $U(t) = \exp(-it\partial_x^2)$, and the associated retarded potential $(U *_R f)(t) = \int_0^t U(t-t') f(t')\, dt'$. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function with $0 \leq \psi \leq 1$, $\operatorname{supp}\psi \subset [-2, 2]$ and $\psi(t) = 1$ for $t \in [-1, 1]$. For $0 < T < 1$, we set $\psi_T(t) = \psi(t/T)$. We convert (1) into the following integral equation localized in time:

$$u(t) = \psi(t) U(t) u_0 - i\psi_T(t)(U *_R f(u))(t). \tag{5}$$

At least formally, (5) is equivalent to (1) for $t \in [-T, T]$.

In order to state the main theorem in this paper, we define the space $X^{s,b}$, $s, b \in \mathbb{R}$, as the completion of $\mathscr{S}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{X^{s,b}} \equiv \|\langle\xi\rangle^s \langle\tau - \xi^2\rangle^b \hat{u}(\xi, \tau)\|_{L^2_\tau(L^2_\xi)} = \left(\iint_{\mathbb{R}^2} \langle\xi\rangle^{2s} \langle\tau - \xi^2\rangle^{2b} |\hat{u}(\xi, \tau)|^2 \, d\xi d\tau \right)^{1/2},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, and $\hat{u}(\xi, \tau)$ is the Fourier transform of $u = u(x, t)$ defined by

$$\hat{u}(\xi, \tau) = \mathscr{F}_{x,t} u(\xi, \tau) = \iint_{\mathbb{R}^2} u(x, t) e^{-i(x\xi + t\tau)} dx dt.$$

We note that $\|u\|_{X^{s,b}} = \|U(-\cdot)u\|_{H^b_t(H^s_x)}$. If $b > 1/2$, then by the Sobolev inequality, we have the continuous inclusion $X^{s,b} \subset C_{\mathrm{b}}(\mathbb{R}; H^s(\mathbb{R}))$.

Theorem 1.1. *Let $m \geq 0$. Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy the estimate*

$$|\hat{V}(\xi)| \leq C\langle\xi\rangle^{-m}. \tag{6}$$

Let $s \in \mathbb{R}$ satisfy (i) $s > -(m/2+3/4)$ *if* $(\lambda_2, \lambda_3) = (0,0)$; (ii) $s > -(m/2+1/4)$ *if* $(\lambda_2, \lambda_3) \neq (0,0)$. *Then, for any $u_0 \in H^s(\mathbb{R})$, there exist $b > 1/2$ and $0 < T < 1$ such that* (5) *with $f(u)$ defined by* (2) *has a unique solution $u \in X^{s,b}$.*

Remark 1.2. The result by Kenig–Ponce–Vega [12] corresponds to the case $V(x) = \delta(x)$, namely $m = 0$. The estimate (6) means that $V *_x$ is a smoothing operator of order m, and the theorem shows that the effect of $V *_x$ gains regularity of order $m/2$ in the well-posedness result.

This paper is organized as follows. In §2, we summarize some basic estimates which are repeatedly used throughout the paper. In §3, we prove Theorem 1.1 by the contraction mapping principle. The key ingredients of the proof are bilinear estimates of $f(u)$ in $X^{s,b}$-space. In §§4–6, we derive bilinear estimates for $f_j(u)$, $1 \leq j \leq 4$.

2. Preliminaries

Lemma 2.1. *Let $0 < c < 1/2 < b$, $\alpha \in \mathbb{R}$ and $\beta > 0$. The following estimates hold:*

$$\text{(i)} \int_{-\infty}^{\infty} \frac{dx}{\langle x\rangle^{2b}|x-\alpha|^{1/2}} \lesssim \langle\alpha\rangle^{-1/2}; \qquad \text{(ii)} \int_{-\beta}^{\beta} \frac{dx}{\langle x\rangle^{2c}|x-\alpha|^{1/2}} \lesssim \langle\beta\rangle^{1-2c}\langle\alpha\rangle^{-1/2}.$$

Proof. Since these estimates are elementary, we omit the proof. □

As stated in §1, let $U(t) = \exp(-it\partial_x^2)$ and $(U *_R f)(t) = \int_0^t U(t-t')f(t')\,dt'$. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function with $0 \leq \psi \leq 1$, $\operatorname{supp}\psi \subset [-2,2]$ and $\psi(t) = 1$ for $t \in [-1,1]$. For $0 < T < 1$, we set $\psi_T(t) = \psi(t/T)$. We have the following linear estimate:

Lemma 2.2. *Let $s \in \mathbb{R}$ and let $b, c \geq 0$ satisfy $b + c \leq 1$. Then, the estimate*

$$\|\psi_T(U *_R f)\|_{X^{s,b}} \lesssim T^{1-b-c}\|f\|_{X^{s,-c}}$$

holds for $f \in X^{s,-c}$.

Proof. See e.g. [5, Lemma 2.1], [11, Lemma 3.3] or [15, Lemma 7.10]. □

Lemma 2.3. *Let $b > 1/2$. The estimate*

$$\iint_{\mathbb{R}^2} |uu_1u_2|\,dxdt \lesssim \|u\|_{L^2_t(L^2_x)}\|u_1\|_{X^{0,b}}\|u_2\|_{X^{0,b}} \tag{7}$$

holds for any $u \in L^2_t(L^2_x)$ and $u_j \in X^{0,b}$, $j = 1, 2$.

Proof. See [12, Lemma 2.3]. For completeness we shall introduce an alternative proof in terms of Strichartz estimates (see [5, Lemmas 2.3–2.4]). By Hölder's inequality, the left-hand side of (7) is bounded by

$$\|u\|_{L^2_t(L^2_x)}\|u_1\|_{L^4_t(L^2_x)}\|u_2\|_{L^4_t(L^\infty_x)}.$$

It follows from the Minkowski and Sobolev inequalities together with the unitarity of $U(t)$ that $\|u_1\|_{L_t^4(L_x^2)} \lesssim \|U(-\cdot)u_1\|_{L_x^2(L_t^4)} \lesssim \|U(-\cdot)u_1\|_{L_x^2(H_t^{1/4})} = \|u_1\|_{X^{0,1/4}}$. On the other hand, from the representation $u(t) = \int e^{it\tau}U(t)(\mathscr{F}_t U(-\cdot)u_2)(\tau)\,d\tau$ and the Strichartz estimate [3, 10, 16], we obtain $\|u_2\|_{L_t^4(L_x^\infty)} \lesssim \int \|(\mathscr{F}_t U(-\cdot)u_2)(\tau)\|_{L_x^2} d\tau \lesssim \|u_2\|_{X^{0,b}}$. Combining these estimates, we obtain (7). □

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

Proof. Let $f(u_1, u_2) = \sum_{j=1}^4 \lambda_j f_j(u_1, u_2)$ be the quadratic form associated with the nonlinear inter interaction $f(u)$ defined by (2). We can show that $f(u_1, u_2)$ satisfies the estimate

$$\|f(u_1, u_2)\|_{X^{s,-c}} \le C\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}} \tag{8}$$

for some b, c with $0 < c < 1/2 < b < 1$ and with $b+c < 1$. In fact, if $(\lambda_2, \lambda_3) = (0, 0)$, then for $s = -\rho > -(m/2+3/4)$, we first choose $c < 1/2$ satisfying the assumptions of Propositions 4.1 and 6.1; for such a number c, we next choose b such that $1/2 < b < 1 - c$. Then, the inequalities (9) and (19) respectively hold for f_1 and f_4 by Propositions 4.1 and 6.1. If $(\lambda_2, \lambda_3) \ne (0, 0)$, we should further assume $s > -(m/2 + 1/4)$ so that we can choose numbers b, c with $0 < c < 1/2 < b < 1$ satisfying the assumptions of Propositions 5.1 and 5.2. Then, the inequalities (15) and (18) respectively hold for f_2 and f_3. These inequalities all together yield (8). The proofs of these propositions themselves are given in §§4–6.

For $R > 0$, we define the set $B_R = \{u \in X^{s,b} : \|u\|_{X^{s,b}} \le R\}$ equipped with the metric $d(u_1, u_2) = \|u_1 - u_2\|_{X^{s,b}}$. Clearly, (B_R, d) is a complete metric space. For suitable positive numbers R and T, we use a contraction method to find a fixed point of the mapping

$$\Phi(u) = \psi(t)U(t)u_0 - i\psi_T(t)(U *_R f(u, u))(t),$$

which solves (5). By definition, we immediately show

$$\|\psi(t)U(t)u_0\|_{X^{s,b}} = \|\psi(t)u_0\|_{H_t^b(H_x^s)} \le C\|u_0\|_{H^s}.$$

Therefore, by Lemma 2.2 together with (8), we obtain

$$\|\Phi(u)\|_{X^{s,b}} \le C\|u_0\|_{H^s} + CT^{1-b-c}\|u\|_{X^{s,b}}^2 \le C\|u_0\|_{H^s} + CT^{1-b-c}R^2$$

for $u \in B_R$. Similarly, we obtain $\|\Phi(u_1) - \Phi(u_2)\|_{X^{s,b}} \le CRT^{1-b-c}\|u_1 - u_2\|_{X^{s,b}}$. Choosing R and T such that $C\|u_0\|_{H^s} \le R/2$ and $CRT^{1-b-c} \le 1/2$, we see that Φ is a contraction mapping from B_R into itself. Thus, it follows from the contraction mapping principle that Φ has a unique fixed point in B_R, thereby obtaining the theorem. □

4. Bilinear estimates for $f_1(u)$

In this section we consider the nonlinear term $f_1(u) = (V *_x u)u$, or more generally the quadratic form $f_1(u_1, u_2) = (V *_x u_1)u_2$. Let $s = -\rho < 0$, and $0 < c < 1/2 < b < 1$. Under suitable assumptions, we shall derive the following estimate of $f_1(u_1, u_2)$ in $X^{s,b}$:

$$\|f_1(u_1, u_2)\|_{X^{s,-c}} \lesssim \|u_1\|_{X^{s,b}} \|u_2\|_{X^{s,b}}. \tag{9}$$

To prove (9), we use a duality argument. The Fourier transform of f_1 is

$$\hat{f}_1(\zeta) = \int \hat{V}(\xi_1)\hat{u}_1(\zeta_1)\hat{u}_2(\zeta - \zeta_1)\, d\zeta_1 = \int \frac{\hat{V}(\xi_1)\langle\xi_1\rangle^\rho\langle\xi_2\rangle^\rho\hat{v}_1(\zeta_1)\hat{v}_2(\zeta_2)}{\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\zeta_1.$$

Here, we write $\zeta = (\xi, \tau)$, $\sigma = \tau - \xi^2$, and $\zeta_j = (\xi_j, \tau_j)$, $\sigma_j = \tau_j - \xi_j^2$, $j = 1, 2$ for short, with taking the relation $\zeta = \zeta_1 + \zeta_2$ into account. We also set $\hat{v}_j(\zeta_j) = \langle\xi_j\rangle^{-\rho}\langle\sigma_j\rangle^b\hat{u}(\zeta_j)$. Multiplying $\langle\xi\rangle^{-\rho}\langle\sigma\rangle^{-c}\overline{\hat{v}(\zeta)}$ by $\hat{f}_1(\zeta)$ and integrating with respect to ζ, we obtain

$$\int \langle\xi\rangle^{-\rho}\langle\sigma\rangle^{-c}\overline{\hat{v}(\zeta)}\hat{f}_1(\zeta)\, d\zeta = \iint \frac{K(\xi_1, \xi_2)\overline{\hat{v}(\zeta)}\hat{v}_1(\zeta_1)\hat{v}_2(\zeta_2)}{\langle\sigma\rangle^c\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\zeta d\zeta_1 \equiv S,$$

where v is an arbitrary element of $L^2(\mathbb{R}^2)$ and $K(\xi_1, \xi_2) = \hat{V}(\xi_1)\langle\xi_1\rangle^\rho\langle\xi_2\rangle^\rho\langle\xi\rangle^{-\rho}$. Then, the estimate (9) is equivalent to

$$|S| \lesssim \|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}. \tag{10}$$

Proposition 4.1. *Let $m \geq 0$. Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy (6). Let $0 < c < 1/2 < b$ and $0 \leq \rho \leq \min\{m/2 + c + 1/4; m + 2c\}$. Then the estimate (9) holds.*

Proof. We shall prove (10). If $\min\{|\xi_1|, |\xi_2|\} \leq 1$, then we have $\langle\xi_1\rangle^\rho\langle\xi_2\rangle^\rho\langle\xi\rangle^{-\rho} \lesssim 1$, so that the contribution S_0 of this region to S is estimated as

$$|S_0| \lesssim \iint \frac{|\overline{\hat{v}(\zeta)}\hat{v}_1(\zeta_1)\hat{v}_2(\zeta_2)|}{\langle\sigma\rangle^c\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\zeta d\zeta_1 \lesssim \iint |\bar{w}w_1w_2|\, dxdt,$$

where $w = \mathscr{F}^{-1}\langle\sigma\rangle^{-c}|\hat{v}(\zeta)|$, and $w_j = \mathscr{F}^{-1}\langle\sigma_j\rangle^{-b}|\hat{v}(\zeta_j)|$, $j = 1, 2$. From Lemma 2.3, we obtain $|S_0| \lesssim \|w\|_{L^2_{x,t}}\|w_1\|_{X^{0,b}}\|w_2\|_{X^{0,b}} \lesssim \|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}$. Hence, we may assume $|\xi_1|, |\xi_2| \geq 1$. By the energy conservation

$$\sigma_1 + \sigma_2 - \sigma = (\tau_1 - \xi_1^2) + (\tau_2 - \xi_2^2) - (\tau - \xi^2) = 2\xi_1\xi_2,$$

we have $2|\xi_1\xi_2| \leq 3\max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$. We further split the integration region into subregions according to which of $|\sigma|, |\sigma_1|$ and $|\sigma_2|$ is the largest.

Case 1. Let $|\sigma_1|, |\sigma_2| \leq |\sigma|$. We estimate the contribution S_1 of this region to S. For this purpose we set

$$I_1(\zeta) \equiv \int_{|\sigma_1|,|\sigma_2|\leq|\sigma|} \frac{|K(\xi_1, \xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\zeta_1 = \int_{|\sigma_1|\leq|\sigma|} d\sigma_1 \int_{A_1} \frac{|K(\xi_1, \xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\xi_1$$

with $A_1 = A_1(\zeta, \sigma_1) = \{\xi_1 \in \mathbb{R} : |\sigma_2| \leq |\sigma|\}$. Then, it follows from the Schwarz inequality that

$$
\begin{aligned}
|S_1|^2 &\leq \iint_{|\sigma_1|,|\sigma_2| \leq |\sigma|} \frac{|K(\xi_1,\xi_2)|^2 |\hat{v}(\zeta)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\zeta d\zeta_1 \iint |\hat{v}_1(\zeta_1)|^2 |\hat{v}_2(\zeta_2)|^2\, d\zeta d\zeta_1 \\
&\lesssim \big\{ \sup_{\zeta} I_1(\zeta) \big\} \|v\|_{L^2}^2 \|v_1\|_{L^2}^2 \|v_2\|_{L^2}^2, \qquad (11)
\end{aligned}
$$

so that we should prove $\sup_\zeta I_1(\zeta) < \infty$. For fixed ζ and σ_1, we change the variable from ξ_1 to $\sigma_2 = \sigma - \sigma_1 + 2\xi_1\xi_2$. Then we have the relations $d\sigma_2 = 2(\xi_2 - \xi_1)d\xi_1$, and $(\xi_2 - \xi_1)^2 = \xi^2 - 4\xi_1\xi_2 = \xi^2 - 2(\sigma_1 + \sigma_2 - \sigma)$. Therefore,

$$
I_1(\zeta) \lesssim \int_{|\sigma_1| \leq |\sigma|} d\sigma_1 \int_{|\sigma_2| \leq |\sigma|} \frac{|K(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b} |\xi^2 - 2(\sigma_1+\sigma_2-\sigma)|^{1/2}}\, d\sigma_2.
$$

To estimate $|K(\xi_1,\xi_2)|^2$, we further split the integration region into the following three subregions:

(i) $1 \leq |\xi| \lesssim |\xi_1| \sim |\xi_2|$, so that $|K(\xi_1,\xi_2)|^2 \lesssim \langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{4\rho-2m} \sim \langle\xi\rangle^{-2\rho}\langle\xi_1\xi_2\rangle^{2\rho-m}$;

(ii) $1 \leq |\xi_1| \lesssim |\xi| \sim |\xi_2|$, so that $|K(\xi_1,\xi_2)|^2 \lesssim \langle\xi_1\rangle^{2(\rho-m)} \lesssim \langle\xi_1\xi_2\rangle^{(\rho-m)_+}$;

(iii) $1 \leq |\xi_2| \lesssim |\xi| \sim |\xi_1|$, so that $|K(\xi_1,\xi_2)|^2 \lesssim \langle\xi_1\rangle^{-2m}\langle\xi_2\rangle^{2\rho} \lesssim \langle\xi_1\xi_2\rangle^{(\rho-m)_+}$.

From the estimate $|\xi_1\xi_2| \lesssim |\sigma|$, we see

$$
|K(\xi_1,\xi_2)|^2 \lesssim \begin{cases} \langle\sigma\rangle^{(2\rho-m)_+}\langle\xi\rangle^{-2\rho}, & |\xi| \lesssim |\xi_1| \sim |\xi_2|, \\ \langle\sigma\rangle^{(\rho-m)_+}, & |\xi_1|, |\xi_2| \lesssim |\xi|. \end{cases} \qquad (12)
$$

On the other hand, from Lemma 2.1 (i), we see

$$
\begin{aligned}
&\int_{|\sigma_1| \leq |\sigma|} d\sigma_1 \int_{|\sigma_2| \leq |\sigma|} \frac{d\sigma_2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}|\xi^2 - 2(\sigma_1+\sigma_2-\sigma)|^{1/2}} \\
&\quad \lesssim \int_{|\sigma_1| \leq |\sigma|} \frac{d\sigma_1}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\xi^2 - 2(\sigma_1-\sigma)\rangle^{1/2}} \lesssim \frac{1}{\langle\sigma\rangle^{2c}\langle\xi^2+2\sigma\rangle^{1/2}}.
\end{aligned}
$$

Therefore, we obtain

$$
I_1(\zeta) \lesssim \max\left\{ \frac{\langle\sigma\rangle^{(2\rho-m)_+-2c}}{\langle\xi^2\rangle^{\rho}\langle\xi^2+2\sigma\rangle^{1/2}}; \frac{\langle\sigma\rangle^{(\rho-m)_+-2c}}{\langle\xi^2+2\sigma\rangle^{1/2}} \right\} \lesssim 1
$$

provided that $0 \leq \rho \leq \min\{m/2 + c + 1/4; m + 2c\}$.

Case 2. Let $|\sigma|, |\sigma_2| \leq |\sigma_1|$. To estimate the contribution S_2 of this region to S, we set

$$
I_2(\zeta_1) \equiv \int_{|\sigma| \leq |\sigma_1|} d\sigma \int_{A_2} \frac{|K(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\xi
$$

with $A_2 = A_2(\zeta_1, \sigma) = \{\xi \in \mathbb{R} : |\sigma_2| \leq |\sigma_1|\}$. If we obtain $\sup_{\zeta_1} I_2(\zeta_1) < \infty$, then we can estimate the contribution S_2 of this region to S as in (11), with ζ and ζ_1 interchanged. For fixed ζ_1 and σ, we change the variable from ξ to σ_2. Since $d\sigma_2 = 2\xi_1 d\xi$, taking $|\xi_1| \geq 1$ into account, we have

$$
I_2(\zeta_1) \lesssim \int_{|\sigma| \leq |\sigma_1|} d\sigma \int_{|\sigma_2| \leq |\sigma_1|} \frac{|K(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \frac{d\sigma_2}{\langle\xi_1\rangle}.
$$

As in Case 1, we have the estimate

$$\frac{|K(\xi_1,\xi_2)|^2}{\langle\xi_1\rangle} \lesssim \begin{cases} \langle\sigma_1\rangle^{(2\rho-m-1/2)_+}\langle\xi\rangle^{-2\rho}, & |\xi| \lesssim |\xi_1| \sim |\xi_2|, \\ \langle\sigma_1\rangle^{(\rho-m-1/2)_+}, & |\xi_1|, |\xi_2| \lesssim |\xi|. \end{cases} \tag{13}$$

On the other hand, by computation we see

$$\int_{|\sigma|\le|\sigma_1|} d\sigma \int_{|\sigma_2|\le|\sigma_1|} \frac{d\sigma_2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \lesssim \langle\sigma_1\rangle^{1-2(b+c)}. \tag{14}$$

Therefore, we see $I_2(\zeta_1) \lesssim \langle\sigma_1\rangle^{(2\rho-m-1/2)_+ +1-2(b+c)}$. Thus, we obtain $\sup_{\zeta_1} I_2(\zeta_1) < \infty$ provided that $\rho \le m/2 - 1/4 + b + c$.

Case 3. Let $|\sigma|, |\sigma_1| \le |\sigma_2|$. We can easily show that $|K(\xi_1,\xi_2)|^2/\langle\xi_2\rangle$ is bounded by the right-hand side of (13). Therefore, changing the variables ζ_1 and ζ_2, we can treat this case in the same way as Case 2. □

5. Bilinear estimates for $f_2(u)$ and $f_3(u)$

In this section we first consider the nonlinear term $f_2(u) = (V *_x \bar{u})u$, or more generally the quadratic form $f_2(u_1,u_2) = (V *_x \bar{u}_1)u_2$. We shall derive the following estimate

$$\|f_2(u_1,u_2)\|_{X^{s,-c}} \lesssim \|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}} \tag{15}$$

for suitable $s = -\rho < 0$ and $0 < c < 1/2 < b < 1$. To prove (15), we again use a duality argument. The Fourier transform of f_2 is

$$\hat{f}_2(\zeta) = \int \hat{V}(-\xi_1)\overline{\hat{u}_1(\zeta_1)}\hat{u}_2(\zeta+\zeta_1)\, d\zeta_1,$$

so that we should estimate the cubic form

$$S^* \equiv \iint \frac{K^*(\xi_1,\xi_2)\overline{\hat{v}(\zeta)\hat{v}_1(\zeta_1)}\hat{v}_2(\zeta_2)}{\langle\sigma\rangle^c\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\, d\zeta d\zeta_1.$$

Here, v is an arbitrary element of $L^2(\mathbb{R}^2)$, the symbols $\zeta, \sigma, \zeta_j, \sigma_j$, $j = 1, 2$ are similar to those in §4, but they should satisfy $\zeta = -\zeta_1 + \zeta_2$ instead of $\zeta = \zeta_1 + \zeta_2$, and the kernel $K(\xi_1,\xi_2)$ in S is replaced with $K^*(\xi_1,\xi_2) = \hat{V}(-\xi_1)\langle\xi_1\rangle^\rho\langle\xi_2\rangle^\rho\langle\xi\rangle^{-\rho}$. Then, the estimate (15) is equivalent to

$$|S^*| \lesssim \|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}. \tag{16}$$

Proposition 5.1. *Let $m \ge 0$. Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy* (6)*. Let $0 < c < 1/2 < b$ and $0 \le \rho \le (m+c)/2$. Then the estimate* (15) *holds.*

Proof. We shall prove (16). As in the proof of Proposition 4.1, we may assume $|\xi_1|, |\xi_2| \ge 1$. We split the integration region of S^* into subregions and estimate the contribution of each subregion separately. We first consider the case $|\xi| \le 1$. We set

$$I_0^*(\zeta_1) \equiv \iint_{|\xi|\le 1\le|\xi_1|,|\xi_2|} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\zeta.$$

If we obtain $\sup_{\zeta_1} I_0^*(\zeta_1) < \infty$, then we can estimate the contribution of this region to S^* as in (11). Since $|\xi| \le 1$, we have $|\xi_1| \sim |\xi_2|$, so that $|K^*(\xi_1,\xi_2)|^2 \lesssim \langle\xi_1\rangle^{4\rho-2m}$.

For fixed ζ_1, we change the variables from $\zeta = (\xi, \tau)$ to (σ, σ_2). Then we have $d\sigma d\sigma_2 = 2|\xi_1| d\xi d\tau$. By the energy conservation

$$\sigma + \sigma_1 - \sigma_2 = (\tau - \xi^2) + (\tau_1 - \xi_1^2) - (\tau_2 - \xi_2^2) = 2\xi\xi_1 \tag{17}$$

together with $|\xi| \le 1$, we have $|\sigma - \sigma_2| \le |\sigma_1| + 2|\xi_1|$. Therefore,

$$I_0^*(\zeta_1) \lesssim \int d\sigma_2 \int_{|\sigma-\sigma_2|\le|\sigma_1|+2|\xi_1|} \frac{\langle \xi_1 \rangle^{4\rho-2m-1}}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \, d\sigma.$$

Since $\langle \sigma \rangle^{-2c}$ is positive, even and decreasing for positive σ, the integral above becomes greater if we replace the interval of integration for σ with $|\sigma| \le |\sigma_1| + 2|\xi_1|$. Hence we obtain

$$\begin{aligned} I_0^*(\zeta_1) &\lesssim \int d\sigma_2 \int_{|\sigma|\le|\sigma_1|+2|\xi_1|} \frac{\langle \xi_1 \rangle^{4\rho-2m-1}}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \, d\sigma \\ &\lesssim \langle \xi_1 \rangle^{4\rho-2m-1} \langle \sigma_1 \rangle^{-2b} \langle |\sigma_1| + 2|\xi_1| \rangle^{1-2c}. \end{aligned}$$

If $\rho \le (m+c)/2$, then the right-hand side is bounded, so that $\sup_{\zeta_1} I_0^*(\zeta_1) < \infty$. Thus, in what follows we may assume $|\xi|, |\xi_1|, |\xi_2| \ge 1$. By (17), we have $2|\xi\xi_1| \le 3\max\{|\sigma_1|, |\sigma_2|, |\sigma|\}$. We split the integration region into subregions according to which of $|\sigma|, |\sigma_1|$ and $|\sigma_2|$ is the largest.

Case 1. Let $|\sigma_1|, |\sigma_2| \le |\sigma|$. We estimate the contribution S_1^* of this region to S^*. We set

$$I_1^*(\zeta) \equiv \int_{|\sigma_1|,|\sigma_2|\le|\sigma|} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \, d\zeta_1 = \int_{|\sigma_1|\le|\sigma|} d\sigma_1 \int_{B_1} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \, d\xi_1$$

with $B_1 = B_1(\zeta, \sigma_1) = \{\xi_1 \in \mathbb{R} : |\sigma_2| \le |\sigma|\}$. It suffices to show $\sup_\zeta I_1^*(\zeta) < \infty$. To this end, for fixed ζ and σ_1, we change the variable from ξ_1 to σ_2. Since $d\sigma_2 = -2\xi d\xi_1$, taking $|\xi| \ge 1$ into account, we have

$$I_1^*(\zeta) \lesssim \int_{|\sigma_1|\le|\sigma|} d\sigma_1 \int_{|\sigma_2|\le|\sigma|} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \frac{d\sigma_2}{\langle \xi \rangle}.$$

As in the proof of Proposition 4.1, Case 1, we split the integration region into subregions to estimate $|K^*(\xi_1,\xi_2)|^2$, but in this case we use $|\xi_j| \lesssim |\xi\xi_j| \lesssim |\sigma|$, $j = 1, 2$. Then we obtain

$$\frac{|K^*(\xi_1,\xi_2)|^2}{\langle \xi \rangle} \lesssim \begin{cases} \langle \sigma \rangle^{(4\rho-2m)_+} \langle \xi \rangle^{-2\rho-1}, & |\xi| \lesssim |\xi_1| \sim |\xi_2|, \\ \langle \sigma \rangle^{2(\rho-m)_+-1/2}, & |\xi_1|, |\xi_2| \lesssim |\xi| \end{cases} \lesssim \langle \sigma \rangle^{(4\rho-2m)_+}.$$

Hence, by computation we obtain $I_1^*(\zeta) \lesssim \langle \sigma \rangle^{(4\rho-2m)_+-2c}$, so that $\sup_\zeta I_1(\zeta) < \infty$ if $\rho \le (m+c)/2$.

Case 2. Let $|\sigma|, |\sigma_2| \le |\sigma_1|$. To estimate the contribution S_2^* of this region to S^*, we set

$$I_2^*(\zeta_1) \equiv \int_{|\sigma|\le|\sigma_1|} d\sigma \int_{B_2} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle \sigma \rangle^{2c} \langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \, d\xi$$

with $B_2 = B_2(\zeta_1, \sigma) = \{\xi \in \mathbb{R} : |\sigma_2| \le |\sigma_1|\}$. It suffices to show $\sup_{\zeta_1} I_2^*(\zeta_1) < \infty$. The estimate for $I_2^*(\zeta_1)$ is similar to that for $I_1^*(\zeta)$ in Case 1. Indeed, changing the variable from ξ to σ_2, we obtain

$$I_2^*(\zeta_1) \lesssim \int_{|\sigma|\le|\sigma_1|} d\sigma \int_{|\sigma_2|\le|\sigma_1|} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \frac{d\sigma_2}{\langle\xi_1\rangle}.$$

We can easily obtain the estimate $|K^*(\xi_1,\xi_2)|^2/\langle\xi_1\rangle \lesssim \langle\sigma_1\rangle^{(4\rho-2m-1)_+}$. Applying this estimate to the integral above, we obtain $I_2^*(\zeta_1) \lesssim \langle\sigma_1\rangle^{(4\rho-2m-1)_+ + 1 - 2(b+c)}$. Hence we can obtain $\sup_{\zeta_1} I_2(\zeta_1) < \infty$ provided that $\rho \le (m+b+c)/2$.

Case 3. Let $|\sigma|, |\sigma_1| \le |\sigma_2|$. We estimate the contribution S_3^* of this region to S^*. We set

$$I_3^*(\zeta_2) \equiv \int_{|\sigma|\le|\sigma_2|} d\sigma \int_{B_3} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\xi$$

with $B_3 = B_3(\zeta_2, \sigma) = \{\xi \in \mathbb{R} : |\sigma_1| \le |\sigma_2|\}$. It suffices to show $\sup_{\zeta_2} I_3^*(\zeta_2) < \infty$. For fixed ζ_2 and σ, we change the variable from ξ to σ_1. We have the relations $d\sigma_1 = 2(\xi_1 - \xi)d\xi$, and $(\xi_1-\xi)^2 = \xi_2^2 - 4\xi\xi_1 = \xi_2^2 - 2(\sigma+\sigma_1-\sigma_2)$. Therefore, it follow that

$$I_3^*(\zeta_2) \lesssim \int_{|\sigma|\le|\sigma_2|} d\sigma \int_{|\sigma_1|\le|\sigma_2|} \frac{|K^*(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}|\xi_2^2 - 2(\sigma+\sigma_1-\sigma_2)|^{1/2}}\, d\sigma_1.$$

As in the previous cases, we have the estimate $|K^*(\xi_1,\xi_2)|^2 \lesssim \langle\sigma_2\rangle^{(4\rho-2m)_+}$. Therefore, it follows from Lemma 2.1 (i), (ii) that

$$I_3^*(\zeta_2) \lesssim \langle\sigma_2\rangle^{(4\rho-2m)_+ + 1 - 2(b+c)}\langle\xi_2^2 + 2\sigma_2\rangle^{-1/2}.$$

Hence, we can obtain $\sup_\zeta I_3^*(\zeta) < \infty$ provided that $\rho \le (m+b+c)/2 - 1/4$. □

We next consider the quadratic form $f_3(u_1,u_2) = (V *_x u_1)\bar{u}_2$ associated with $f_3(u)$. We have the following:

Proposition 5.2. *Under the same assumption as Proposition 5.1, the following estimate holds:*

$$\|f_3(u_1,u_2)\|_{X^{s,-c}} \lesssim \|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}} \tag{18}$$

Proof. We consider the cubic form

$$\tilde{S}^* \equiv \iint \frac{\tilde{K}^*(\xi_1,\xi_2)\overline{\hat{v}(\zeta)}\hat{v}_1(\zeta_1)\overline{\hat{v}_2(\zeta_2)}}{\langle\sigma\rangle^{c}\langle\sigma_1\rangle^{b}\langle\sigma_2\rangle^{b}}\, d\zeta d\zeta_2.$$

Here, the symbols $\zeta, \zeta_j, \sigma, \sigma_j$, $j = 1, 2$ are defined as in S^*, but they should satisfy $\zeta = \zeta_1 - \zeta_2$ instead of $\zeta = -\zeta_1 + \zeta_2$, and the kernel $K^*(\xi_1,\xi_2)$ in S^* is replaced with $\tilde{K}^*(\xi_1,\xi_2) = \hat{V}(\xi_1)\langle\xi_1\rangle^\rho\langle\xi_2\rangle^\rho\langle\xi\rangle^{-\rho}$. For the proof, it suffices to show the estimate

$$\left|\tilde{S}^*\right| \lesssim \|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}.$$

We have the relation

$$\sigma - \sigma_1 + \sigma_2 = (\tau - \xi^2) - (\tau_1 - \xi_1^2) + (\tau_2 - \xi_2^2) = 2\xi\xi_2.$$

Since the upper-bounds of $|K^*(\xi_1,\xi_2)|^2$ used in the proof of Proposition 5.1 are symmetric with respect to ξ_1 and ξ_2, we can analogously prove Proposition 5.2 with the subscripts 1 and 2 interchanged. $\square$

6. Bilinear estimates for $f_4(u)$

In this section we consider the nonlinear term $f_4(u)=(V*_x\bar{u})\bar{u}$, or more generally the quadratic form $f_4(u_1,u_2)=(V*_x\bar{u}_1)\bar{u}_2$. We shall derive the following estimate

$$\|f_4(u_1,u_2)\|_{X^{s,-c}}\lesssim\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}} \tag{19}$$

for suitable $s=-\rho<0$ and $0<c<1/2<b<1$. To prove (19), we again use a duality argument. The Fourier transform of f_4 is

$$\hat{f}_4(\zeta)=\int\hat{V}(-\xi_1)\overline{\hat{u}_1(\zeta_1)\hat{u}_2(-\zeta-\zeta_1)}\,d\zeta_1,$$

so that we should estimate the cubic form

$$S^{**}\equiv\iint\frac{K^{**}(\xi_1,\xi_2)\overline{\hat{v}(\zeta)\hat{v}_1(\zeta_1)\hat{v}_2(\zeta_2)}}{\langle\sigma\rangle^c\langle\sigma_1\rangle^b\langle\sigma_2\rangle^b}\,d\zeta d\zeta_1.$$

Here, v is an arbitrary element of $L^2(\mathbb{R}^2)$, the symbols $\zeta,\zeta_j,\sigma,\sigma_j,\ j=1,2$ are defined as before, but they should satisfy $\zeta+\zeta_1+\zeta_2=0$, and the kernel $K^{**}(\xi_1,\xi_2)$ is equal to $K^*(\xi_1,\xi_2)$ in S^*. Then, the estimate (19) is equivalent to

$$|S^{**}|\lesssim\|v\|_{L^2}\|v_1\|_{L^2}\|v_2\|_{L^2}. \tag{20}$$

Proposition 6.1. *Let $m\geq 0$. Let $V\in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfy* (6). *Let $0<c<1/2<b$ and $0\leq\rho\leq\min\{m/2+c+1/4;m+2c\}$. Then the estimate* (19) *holds.*

Proof. We shall prove (20). As in the proof of Propositions 4.1 and 5.1, we may assume $|\xi_1|,|\xi_2|\geq 1$. By the energy conservation

$$\sigma_1+\sigma_2+\sigma=(\tau_1-\xi_1^2)+(\tau_2-\xi_2^2)+(\tau-\xi^2)=-(\xi^2+\xi_1^2+\xi_2^2),$$

we have $\xi^2+\xi_1^2+\xi_2^2\leq 3\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$. We split the integration region into subregions according to which of $|\sigma|,|\sigma_1|$ and $|\sigma_2|$ is the largest.

Case 1. Let $|\sigma_1|,|\sigma_2|\leq|\sigma|$. We estimate the contribution S_1^{**} of this region to S^{**}. We set

$$I_1^{**}(\zeta)\equiv\int_{|\sigma_1|,|\sigma_2|\leq|\sigma|}\frac{|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,d\zeta_1=\int_{|\sigma_1|\leq|\sigma|}d\sigma_1\int_{C_1}\frac{|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\,d\xi_1$$

with $C_1=C_1(\zeta,\sigma_1)=\{\xi_1\in\mathbb{R}:|\sigma_2|\leq|\sigma|\}$. It suffices to show $\sup_\zeta I_1^{**}(\zeta)<\infty$. For fixed ζ and σ_1, we change the variable from ξ_1 to σ_2. We have the relations $d\sigma_2=2(\xi_2-\xi_1)d\xi_1$, and $(\xi_2-\xi_1)^2=2(\xi_1^2+\xi_2^2)-\xi^2=-3\xi^2-2(\sigma_1+\sigma_2+\sigma)$. Therefore,

$$I_1^{**}(\zeta)\lesssim\int_{|\sigma_1|\leq|\sigma|}d\sigma_1\int_{|\sigma_2|\leq|\sigma|}\frac{|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}|3\xi^2+2(\sigma_1+\sigma_2+\sigma)|^{1/2}}\,d\sigma_2.$$

Clearly $K^{**}(\xi_1,\xi_2)$ satisfies the same estimate (12) for $K(\xi_1,\xi_2)$, we can estimate $I_1^{**}(\zeta)$ in the same way as $I_1(\zeta)$ in the proof of Proposition 4.1. Thus we have proved $\sup_\zeta I_1^{**}(\zeta)<\infty$ provided that $\rho\leq\min\{m/2+c+1/4;m+2c\}$.

Case 2. Let $|\sigma|, |\sigma_2| \le |\sigma_1|$. We estimate the contribution S_2^{**} of this region to S^{**}. We set

$$I_2^{**}(\zeta_1) \equiv \int_{|\sigma|,|\sigma_2|\le|\sigma_1|} \frac{|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\zeta = \int_{|\sigma|\le|\sigma_1|} d\sigma \int_{C_2} \frac{|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}\, d\xi$$

with $C_2 = C_2(\zeta_1, \sigma) = \{\xi \in \mathbb{R} : |\sigma_2| \le |\sigma_1|\}$. We further divide C_2 into the two sub-regions $C_{21} = \{\xi \in C_2 : 3|\xi| \le |\xi_1|\}$ and $C_{22} = C_2 \setminus C_{21}$. In both cases, for fixed ζ_1 and σ, we change the variable from ξ to σ_2. We have the relations $d\sigma_2 = 2(\xi_2 - \xi)d\xi$. Since $|\xi_2 - \xi| \ge |\xi_1|/3$ in C_{21}, we estimate the contribution $I_{21}^{**}(\zeta_1)$ of C_{21} to $I_2^{**}(\zeta_1)$ as

$$I_{21}^{**}(\zeta_1) \lesssim \int_{|\sigma|\le|\sigma_1|} d\sigma \int_{|\sigma_2|\le|\sigma_1|} \frac{\mathbf{1}_{C_{21}}(\xi_1)|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} \frac{d\sigma_2}{\langle\xi_1\rangle}.$$

Clearly we have the estimate (13) with K replaced by K^{**}. Hence, it follows from (14) that $I_{21}^{**}(\zeta_1) \lesssim \langle\sigma_1\rangle^{(2\rho-m-1/2)_+ +1-2(b+c)}$.

On the other hand, since $(\xi_2 - \xi)^2 = 2(\xi^2 + \xi_2^2) - \xi_1^2 = -3\xi_1^2 - 2(\sigma_1 + \sigma_2 + \sigma)$, the contribution $I_{22}^{**}(\zeta_1)$ of C_{22} to $I_2^{**}(\zeta_1)$ satisfies

$$I_{22}^{**}(\zeta_1) \lesssim \int_{|\sigma|\le|\sigma_1|} d\sigma \int_{|\sigma_2|\le|\sigma_1|} \frac{\mathbf{1}_{C_{22}}(\xi_1)|K^{**}(\xi_1,\xi_2)|^2}{\langle\sigma\rangle^{2c}\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}|3\xi_1^2 + 2(\sigma_1+\sigma_2+\sigma)|^{1/2}}\, d\sigma_2.$$

Clearly we have the estimate (12) with K and σ replaced with K^{**} and σ_1 respectively. Since $|\xi_1|, |\xi_2| \lesssim |\xi|$ on C_{22}, it follows from Lemma 2.1 (i), (ii) that $I_{22}^{**}(\zeta) \lesssim \langle\sigma_1\rangle^{(\rho-m)_+ +1-2(b+c)}$. Thus we have proved $\sup_\zeta I_2^{**}(\zeta_1) < \infty$ provided that

$$0 \le \rho \le \min\{m/2 - 1/4 + b + c; m - 1 + 2(b+c)\}.$$

Case 3. Let $|\sigma|, |\sigma_1| \le |\sigma_2|$. By symmetry, changing the variables ζ_1 and ζ_2, we can treat this case in the same way as Case 2. □

Acknowledgment. The authors thank the referee for his/her careful reading and helpful comments. The work of the second author was partially supported by JSPS, Grant-in-Aid for Scientific Research (C) 18K03370.

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