Mem. Fac. Sci. Shimane Univ.,
19, pp. 23–36 Dec. 20, 1985

Some Remarks on Regular Semigroups Satisfying Chain Conditions

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(Received September 14, 1985)

The class of semigroups satisfying the descending chain conditions on principal (left, right) ideals has been studied by Green [2], Munn [4] and some other papers. In this paper, we shall give a survey on the structure of the class of regular semigroups satisfying chain conditions.

Firstly, several kinds of both ascending chain conditions (a.c.c. for short) and descending chain conditions (d.c.c. for short) on (left, right) ideals or on Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$ are introduced, and the regular semigroups with 0, each of which satisfies some of these chain conditions, are studied. Secondly, the concepts of "$\mathscr{L}$($\mathscr{R}$, or $\mathscr{J}$)-bounded" and "locally $\mathscr{L}$($\mathscr{R}$, or $\mathscr{J}$)-bounded" are also introduced, and the connection between these properties and the chain conditions are discussed.

§0. Preliminary

Let S be a semigroup. Let $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$ be Green's L-, R-, J-relations respectively, and for any a of S, L_a, R_a, J_a the $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{J}$-class containing a. Further, $L(a)$, $R(a)$, $J(a)$ denote the principal left ideal, the principal right ideal and the principal ideal generated by a respectively; that is, $L(a)=S^1a$, $R(a)=aS^1$, $J(a)=S^1aS^1$, where S^1 is the adjunction of an identity 1 to S.

The following results are well-known:

RESULT 0.1. *Let S be a semigroup, and I its ideal [left ideal, right ideal]. If $I \cap J_a \neq \square[I \cap L_a \neq \square, I \cap R_a \neq \square]$ then $I \supset J_a[I \supset L_a, I \supset R_a]$. Accordingly, I is a union of $\mathscr{J}$-classes [$\mathscr{L}$-classes, $\mathscr{R}$-classes] of S.*

RESULT 0.2. *Let S be a semigroup, and $C_{\mathscr{J}}=\{J_\lambda: \lambda \in \Lambda\}[C_{\mathscr{L}}=\{L_\lambda: \lambda \in \Lambda\}$, $C_{\mathscr{R}}=\{R_\lambda: \lambda \in \Lambda\}]$ a collection of $\mathscr{J}$-classes [$\mathscr{L}$-classes, $\mathscr{R}$-classes] of S. Then $I=\cup\{J_\lambda: \lambda \in \Lambda\}[I=\cup\{L_\lambda: \lambda \in \Lambda\}, I=\cup\{R_\lambda: \lambda \in \Lambda\}]$ is an ideal [a left ideal, a right ideal] of S if and only if the following* (0.1) *is satisfied*:

(0.1) *For any $\mathscr{J}$-calss J [$\mathscr{L}$-class L, $\mathscr{R}$-class R] with $J \leq J_\lambda[L \leq L_\lambda, R \leq R_\lambda]$*[1] *for some $\lambda \in \Lambda$, $J \in C_{\mathscr{J}}[L \in C_{\mathscr{L}}, R \in C_{\mathscr{R}}]$,*

In particular, for principal (left, right) ideals

1) For the ordering $\leq$, see [3].

$$J(a)=\cup\{J\colon J \text{ is a } \mathcal{J}\text{-class such that } J\leq J_a\},$$

$$L(a)=\cup\{L\colon L \text{ is an } \mathcal{L}\text{-class such that } L\leq L_a\},$$

$$R(a)=\cup\{R\colon R \text{ is an } \mathcal{R}\text{-class such that } R\leq R_a\}.$$

It is well-known that d.c.c. and a.c.c. on ideals [left ideals, right ideals] are equivalent to the minimal condition and maximal condition on them respectively. In particular, d.c.c. and a.c.c. on principal ideals [principal left ideals, principal right ideals] are equivalent to d.c.c. and a.c.c. on $\mathcal{J}$-classes [$\mathcal{L}$-classes, $\mathcal{R}$-classes] respectively. That is,

RESULT 0.3. *Let T be one of J, L and R.*
a.c.c.: The following conditions (1), (2) *are equivalent.*

(1) *If* $T(a_1)\subset T(a_2)\subset\cdots\subset T(a_i)\subset T(a_{i+1})\cdots\cdots$. *then there exists* m *such that* $T(a_m)=T(a_{m+1}=\cdots\cdots$,

(2) *If* $T_{a_1}\leq T_{a_2}\leq\cdots\leq T_{a_i}\leq T_{a_{i+1}}\leq\cdots\cdots$, *then then there exists* m *such that* $T_{a_m}=T_{a_{m+1}}=\cdots\cdots$.

d.c.c.: The following conditions (1), (2) *are equivalnet.*

(1) *If* $T(a_1)\supset T(a_2)\supset\cdots\supset T(a_i)\supset T(a_{i+1})\cdots\cdots$, *then there exists* m *such that* $T(a_m)=T(a_{m+1})=\cdots\cdots$.

(2) *If* $T_{a_1}\geq T_{a_2}\geq\cdots\geq T_{a_i}\geq T_{a_{i+1}}\geq\cdots\cdots$, *then there exists* m *such that* $T_{a_m}=T_{a_{m+1}}=\cdots\cdots$.

Let S be a semigroup. Let $\mathcal{T}$ be one of Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$. Let T be a $\mathcal{T}$-class of S. A sequence of $\mathcal{T}$-classes, C_T: $T=T_0>T_1>\cdots>T_n$, where ecah T_i is a $\mathcal{T}$-class of S, is called *a properly descending* $\mathcal{T}$*-chain* of T, or simply *a proper* $\mathcal{T}$*-chain* of T. The number n is called the *length* of this chain, and denoted by $n=1(C_T)$. Put $\operatorname{Sup}\{1(C_T)\colon C_T$ is a proper $\mathcal{T}$-chain of $T\}=r(T)$. This $r(T)$ is called the *rank* of T. If $r(T)<\infty$ for all $\mathcal{T}$-classes T of S, then S is said to be *locally* $\mathcal{T}$*-bounded.* Further, if $\operatorname{Sup}\{r(T)\colon T$ is a $\mathcal{T}$-class of $S\}=n<\infty$ then S is said to be $\mathcal{T}$*-bounded*, and n is called *the* $\mathcal{T}$*-dimension* of S: $\dim_{\mathcal{T}} S=n$.

From the definitions above, we have the following:

RESULT 0.4. *Let S be a semigroup, and* $\mathcal{T}$ *one of* $\mathcal{J}$, $\mathcal{L}$ *and* $\mathcal{R}$. *If S is locally* $\mathcal{T}$*-bounded, then S satisfies d.c.c. on* $\mathcal{T}$*-classes. If S is* $\mathcal{T}$*-bounded, then S satisfies both d.c.c. and a.c.c. on* $\mathcal{T}$*-classes.*

§1. Regular semigroups satisfying d. c. c.

Hereafter, S will denote a non-trivial[2)] regular semigroup with 0. It is easy to

2) If a semigroup S with 0 consists of a single element, that is, if $S=\{0\}$, then S is said to be trivial.

see that an [left, right] ideal of an ideal of S is an [left, right] ideal of S. Further, we have the following:

LEMMA 1.1. *Let A be an ideal of S. For any $a \in A$, the $\mathscr{J}$-class [$\mathscr{L}$-class, $\mathscr{R}$-class] of S containing a coincides with the $\mathscr{J}$-class [$\mathscr{L}$-class, $\mathscr{R}$-class] of A containing a.*

PROOF. Let J_a be the $\mathscr{J}$-class of S containing a, and J'_a the $\mathscr{J}$-class of A containing a. It is obvious that $J'_a \subset J_a$. Conversely, let $x \in J_a$. Then, there exist u, v, y and z such that $uxv=a$ and $yaz=x$. Hence, $x \in A$, $(uxx^*)x(x^*xv)=a$, $(yaa^*)a(a^*az)=x$, uxx^*, x^*xv, yaa^*, $a^*az \in A$ for $a^* \in V(a)$ and $x^* \in V(x)$, where $V(b)$ is the set of all inverses of b. Hence, $x \in J'_a$. That is, $J_a = J'_a$. For $\mathscr{L}$-classes and $\mathscr{R}$-rlasses, we can easily prove the assertion in the same manner.

Let $\mathscr{T}$ be one of Green's relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$. Let T be a $\mathscr{T}$-class of S. If there exists no $\mathscr{T}$-class T_1 satisfying $0<T_1<T$, then T is said to be 0-*minimal*. Let I be a non-trivial (that is, non-zero) [left, right] ideal of S. If there is no [left, right] ideal I_1 satisfying $0 \subsetneqq I_1 \subsetneqq I$, I is said to be 0-*minimal*.

The following results are easily obtained by §0 and [1]:

LEMMA 1.2.

(1) *If S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes], then S contains at least one 0-minimal $\mathscr{L}$-class L [0-minimal $\mathscr{R}$-class R, 0-minimal $\mathscr{J}$-class J], and L^0 (the adjunction of a zero element 0 to L) [R^0, J^0] is a 0-minimal left ideal [0-minimal right ideal, 0-minimal ideal] of S. Accordingly, $a \in L$ [$a \in R$, $a \in J$] implies $L^0 = Sa$ [$R^0 = aS$, $J^0 = SaS$].*

(2) *For L^0[R^0, J^0] in (1), $(L^0)^2 = L^0$ [$(R^0)^2 = R^0$, $(J^0)^2 = J^0$].*

(3) *Conversely, if I is a 0-minimal left ideal [0-minimal right ideal, 0-minimal ideal], then there exists a 0-minimal $\mathscr{L}$-class L [0-minimal $\mathscr{R}$-class R, 0-minimal $\mathscr{J}$-class J] such that $I = L^0$[$I = R^0$, $I = J^0$].*

LEMMA 1.3. *Assume that S contains at least one 0-minival $\mathscr{L}$-class [0-minimal $\mathscr{R}$-class, 0-minimal $\mathscr{J}$-class]. Then, the union ${}_l\Sigma_1(S)$ of all 0-minimal $\mathscr{L}$-classes and 0 [the union ${}_r\Sigma_1(S)$ of all 0-minimal $\mathscr{R}$-classes and 0, the union ${}_j\Sigma_1(S)$ of all 0-minimal $\mathscr{J}$-classes and 0] is an ideal of S.*

The ideal ${}_l\Sigma_1(S)$[${}_r\Sigma_1(S)$, ${}_j\Sigma_1(S)$] is called *the* (*first*) *left socle* [*right socle*, *$\mathscr{J}$-socle*] of S.

LEMMA 1.4. *If S contains at least one 0-minimal $\mathscr{L}$-class, then ${}_l\Sigma_1(S)$ is a primitive regular semigroup with 0. Further, S contains both a 0-minimal $\mathscr{R}$-class of S and a 0-minimal $\mathscr{J}$-class of S, and ${}_l\Sigma_1(S) = {}_r\Sigma_1(S) \subset {}_j\Sigma_1(S)$.*

PROOF. Hereafter, $E(S)$ denotes the set of idempotents of S. Let e, f be elements of ${}_l\Sigma_1(S) \cap E(S)$ such that $ef=fe=f \neq 0$. Since $e \in {}_l\Sigma_1(S)$, there exists a 0-minimal

$\mathscr{L}$-class L_e. Since L_e is a left ideal, $f=fe\in L_e^0$. Hence, $L_f=L_e$, and accordingly $Sf=Se$. That is, $ef=e$. Therefore, $e=f$. This implies that ${}_l\sum_1(S)$ is a primitive regular semigroup with 0. Accordingly, ${}_l\sum_1(S)$ is a 0-direct union of completely 0-simple semigroups $\{C_\gamma\colon \gamma\in\Gamma\}$; that is, ${}_l\sum_1(S)=\sum\{C_\gamma\colon \gamma\in\Gamma\}$ ($\sum$ means 0-direct union). Each C_γ is an ideal of ${}_l\sum_1(S)$, and accordingly an ideal of S since S is semisimple. Hence, by Lemma 1.1 each $\mathscr{L}$-class, $\mathscr{R}$-class and $\mathscr{J}$-class are also an $\mathscr{L}$-class, $\mathscr{R}$-class and $\mathscr{J}$-class of S respectively. It is obvious that each $\mathscr{R}$-class of C_γ is 0-minimal and $C_\gamma\backslash 0$[3] is a 0-minimal $\mathscr{J}$-class. Hence, ${}_l\sum_1(S)\subset {}_r\sum_1(S)$, ${}_l\sum_1(S)\subset {}_j\sum_1(S)$. Similarly, we can prove that ${}_r\sum_1(S)\subset {}_l\sum_1(S)$. Thus, we have ${}_l\sum_1(S)={}_r\sum_1(S)$.

From the lemma above, if S has a 0-minimal $\mathscr{L}$-class or a 0-minimal $\mathscr{R}$-class then ${}_l\sum_1(S)={}_r\sum_1(S)$. Therefore, hereafter we call it *the (first) socle* of S and denote it by $\sum_1(S)$.

THEOREM 1.5. *If S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes, then S contains a 0-minimal $\mathscr{L}$-class, a 0-minimal $\mathscr{R}$-class and a 0-minimal $\mathscr{J}$-class, and $\sum_1(S)={}_j\sum_1(S)$.*

PROOF. The first part is obvious. It follows from Lemma 1.4 that ${}_l\sum_1(S)={}_r\sum_1(S)=\sum_1(S)\subset {}_j\sum_1(S)$. Assume that S satisfies d.c.c. on $\mathscr{L}$-classes. If ${}_j\sum_1(S)\backslash {}_l\sum_1(S)\ni a$, then J_a is a 0-minimal $\mathscr{J}$-class. Since S satisfies d.c.c. on $\mathscr{L}$-classes, there exists a minimal element L_b in the set of $\mathscr{L}$-classes contained in J_a. If $0\neq L_x\leq L_b$, then $Sx\subset Sb$, and hence $L_x\subset Sx\subset Sb\subset J_a^0$. By the minimality of L_b, $L_x=L_b$. Therefore, L_b is a 0-minimal $\mathscr{L}$-class of S, and accordingly $L_b^0\subset {}_l\sum_1(S)$. Hence, $J_a=J_b\subset SbS\subset {}_l\sum_1(S)$. Therefore, $a\in {}_l\sum_1(S)$. This contradicts our assumption. Thus, ${}_l\sum_1(S)={}_j\sum_1(S)$.

LEMMA 1.6. (1) *If S has a 0-minimal $\mathscr{J}$-class, then ${}_j\sum_1(S)$ is a 0-direct union of 0-simple regular semigroups.*

(2) *If S satisfies d.c.c. on $\mathscr{J}$-class, then S has a -0minimal $\mathscr{J}$-class.*

PROOF. The assertion (2) is obvious. Let $\{J_\gamma\colon \gamma\in\Gamma\}$ be the set of all 0-minimal $\mathscr{J}$-classes of S. Then, it follows from Lemma 1.1 that J_γ^0 is a 0-minimal ideal. Now, ${}_j\sum_1(S)=\cup\{J_\gamma^0\colon \gamma\in\Gamma\}$. Since $J_\alpha^0 J_\beta^0\subset J_\alpha^0\cap J_\beta^0$ and $J_\alpha^0\cap J_\beta^0$ is an ideal, for $\alpha\neq\beta$ we have $J_\alpha^0\cap J_\beta^0=0$, and accordingly $J_\alpha^0 J_\beta^0=0$. Therefore, ${}_j\sum_1(S)=\sum\{J_\gamma^0\colon \gamma\in\Gamma\}$. It is obvious that each J_γ^0 is a 0-simple regular semigroup (see also [1]).

The following is obvious from Munn [4]:

LEMMA 1.7. *If S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes, then S also satisfies d.c.c. on $\mathscr{J}$-class (see also, p. 36 of IIof* [1]).

LEMMA 1.8. *Let f be a homomorphism of S onto a regular semigroup A. If*

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3) If M, N are sets and $M\supset N$, then $M\backslash N$ means the set $\{x\in M\colon x\notin N\}$.

S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes], then A also satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes].

PROOF. We deal with only $\mathscr{L}$-classes. In the other cases, we can proceed a proof in an analogous way. For any $x \in S$, put $xf=\bar{x}$. Consider an infinite chain of $\mathscr{L}$-classes of A:

$$L_{\bar{a}_1} \geq L_{\bar{a}_2} \geq L_{\bar{a}_3} \geq \cdots\cdots .$$

Since $L_{\bar{a}_i} \geq L_{\bar{a}_{i+1}}$, there exists $\bar{x} \in A$ such that $\bar{x}\bar{a}_i=\bar{a}_{i+1}$. Let $xa_i=b_{i+1}$. Then, $\bar{b}_{i+1}=\bar{a}_{i+1}$, $\bar{x}\bar{a}_i=\bar{b}_{i+1}$, and $L_{a_i} \geq L_{b_{i+1}}$. Similarly, it follows from $L_{\bar{b}_{i+1}} \geq L_{\bar{a}_{i+2}}$ that there exists $b_{i+2} \in S$ such that $\bar{b}_{i+2}=\bar{a}_{i+2}$, $L_{\bar{b}_{i+1}} \geq L_{\bar{b}_{1+2}}$ and $L_{b_{i+1}} \geq L_{b_{i+2}}$. Hence,

$$L_{\bar{a}1} \geq L_{\bar{b}_2} \geq L_{\bar{b}_3} \geq \cdots\cdots, \text{ and } \bar{a}_i=\bar{b}_i \text{ for all } i>1. \quad \text{Further, } L_{a_1} \geq L_{b_2} \geq L_{b_3} \geq \cdots\cdots .$$

Since S satisfies d.c.c. on $\mathscr{L}$-classes, there exists m such that $L_{b_m}=L_{b_{m+1}}=\cdots$. Hence $L_{\bar{b}_m}=L_{\bar{b}_{m+1}}=\cdots$. That is, $L_{\bar{a}_m}=L_{\bar{a}_{m+1}}=\cdots$.

As a special case of the result above, we have the following:

COROLLARY. *Let I be an ideal of S. If S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes], then the Rees factor semigroup S/I moudlo I satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes].*

REMARK. An analogous assertion as was shown in Corollary above is also satisfied for a.c.c.

Assume that S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes]. As was shown above, the socle $\sum_1(S)$ is an ideal. Of course, $\sum_1(S)={}_r\sum_1(S)={}_l\sum_1(S)={}_j\sum_1(S)$. If $S \neq \sum_1(S)$, $S/\sum_1(S) \neq 0$, and $S/\sum_1(S)$ is a non-trivial regular semigroup with 0 and satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes]. Hence, we can consider the socle $\sum_1(S/\sum_1(S))$. Let $\sum_1(S/\sum_1(S))=\sum_2^*(S)$, and put $\sum_1(S) \cup \{\sum_2^*(S)\backslash 0\}=\sum_2(S)$. Then, $\sum_2(S)$ is an ideal of S and $\sum_2(S)/\sum_1(S)$ is a primitive regular semigroup with 0. Of course, $\sum_2(S)/\sum_1(S) \cong \sum_2^*(S)$. This $\sum_2(S)$ is called *the second socle* of S. Inductively, we can define the nth socle of S as follows: Assume that the $(n-1)$th socle $\sum_{n-1}(S)$ of S was defined and it is an ideal of S. Suppose that $\sum_{n-1}(S) \neq S$. Then, $S/\sum_{n-1}(S)$ is a non-trivial regular semigroup with 0 and satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes]. Hence, there exists the socle $\sum_1(S/\sum_{n-1}(S))=\sum_n^*(S)$. Put $\sum_{n-1}(S) \cup \{\sum_n^*(S)\backslash 0\}=\sum_n(S)$. Then, $\sum_n(S)$ is an ideal of S. This $\sum_n(S)$ is called *the nth socle* of S. Of course, $\sum_n(S)/\sum_{n-1}(S)$ is a primitive regular semigroup and isomorphic to $\sum_n^*(S)$.

There are two cases as follows:

Case 1. There exists m such that $S=\sum_m(S)$.

Case 2. For any i, $S \neq \sum_i(S)$.

$$\text{Let } \Sigma(S)=\begin{cases} \Sigma_m(S) \text{ in Case 1,} \\ \Sigma_1(S) \cup [\cup\{\Sigma_i^*(S)\backslash 0: i=2, 3, 4,\dots\}] \text{ in Case 2.}\end{cases}$$

Then, $\Sigma(S)$ is of course an ideal of S. Similarly, we define the nth $\mathscr{J}$-socle of S as follows: Assume that S satisfies d.c.c. on $\mathscr{J}$-classes. By Lemma 1.6. there exists the (first) $\mathscr{J}$-socle ${}_j\Sigma_1(S)$, and ${}_j\Sigma_1(S)$ is an ideal which is a 0-direct union of 0-simple regular semigroups. If ${}_j\Sigma_1(S)\neq S$, then $S/{}_j\Sigma_1(S)$ is a non-trivial regular semigroup with 0 and satisfies d.c.c. on $\mathscr{J}$-classes. Therefore, there exists ${}_j\Sigma_1(S/{}_j\Sigma_1(S))= {}_j\Sigma_2^*(S)$. This $\Sigma_2^*(S)$ is a 0-direct union of 0-simple regular semigroups. Put ${}_j\Sigma_1(S)\cup\{{}_j\Sigma_2^*(S)\backslash 0\}={}_j\Sigma_2(S)$. Then, ${}_j\Sigma_2(S)$ is an ideal of S, and ${}_j\Sigma_2(S)/{}_j\Sigma_1(S)\cong {}_j\Sigma_2^*(S)$. This ${}_j\Sigma_2(S)$ is called *the second $\mathscr{J}$-socle* of S. Assume that we can define the $(n-1)$th $\mathscr{J}$-cole ${}_j\Sigma_{n-1}(S)$ which is an ideal of S. Suppose that ${}_j\Sigma_{n-1}(S)\neq S$. Then, $S/{}_j\Sigma_{n-1}(S)$ is a non-trivial regular semigroup with 0 and satisfies d.c.c. on $\mathscr{J}$-classes. Therefore, there exists ${}_j\Sigma_1(S/{}_j\Sigma_{n-1}(S))={}_j\Sigma_n^*(S)$. This ${}_j\Sigma_n^*(S)$ is a 0-direct union of 0-simple regular semigroups. Put ${}_j\Sigma_{n-1}(S)\cup\{{}_j\Sigma_n^*(S)\backslash 0\}={}_j\Sigma_n(S)$. Then, ${}_j\Sigma_n(S)$ is an ideal of S and ${}_j\Sigma_n(S)/{}_j\Sigma_{n-1}(S)\cong {}_j\Sigma_n^*(S)$. This ${}_j\Sigma_n(S)$ is called *the nth $\mathscr{J}$-socle* of S.
There are two cases as follows:

Case 1. There exists m such that ${}_j\Sigma_m(S)=S$.

Case 2. For any i, ${}_j\Sigma_i(S)\neq S$.

$$\text{Let } {}_j\Sigma(S)=\begin{cases} {}_j\Sigma_m(S) \text{ in Case 1,} \\ {}_j\Sigma_1(S) \cup [\cup\{{}_j\Sigma_i^*(S)\backslash 0: i=2, 3,\dots\}] \text{ in Case 2.}\end{cases}$$

Then, ${}_j\Sigma(S)$ is an ideal of S. Further, it follows from Theorem 1.5 and the definitions of $\Sigma_i(S)$ and ${}_j\Sigma_i(S)$ that if S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes] then $\Sigma_i(S)={}_j\Sigma_i(S)$. Hence, if $\Sigma_i(S)\neq S$ for all i, then ${}_j\Sigma_i(S)\neq S$ for all i and $\Sigma(S)= {}_j\Sigma(S)$.

REMARK. Assume that S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes]. Let $\Sigma_0(S)= {}_j\Sigma_0(S)=\{0\}$. Then, $\Sigma_0(S)\subsetneqq {}_1\Sigma(S)\subsetneqq \Sigma_2(S)\subsetneqq\cdots[{}_j\Sigma_0(S)\subsetneqq {}_j\Sigma_1(S)\subsetneqq {}_j\Sigma_2(S)\subsetneqq\cdots]$ and $\Sigma_{i+1}(S)/\Sigma_i(S)[{}_j\Sigma_{i+1}(S)/{}_j\Sigma_i(S)]$ is a primitive regular semigroup with 0 [a 0-direct union of 0-simple regular semigroups] for all possible $i\geq 0$.

LEMMA 1.9.

(1) *If S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes], then the rank of every $\mathscr{L}$-class [$\mathscr{R}$-class] contained in the ith socle $\Sigma_i(S)$ has at most i.*

(2) *If S satisfies d.c.c. on $\mathscr{J}$-classes, then the rank of every $\mathscr{J}$-class contained in the ith $\mathscr{J}$-socle ${}_j\Sigma_i(S)$ is at most i.*

PROOF. (1) Let L_a be an $\mathscr{L}$-class contained in $\Sigma_i(S)$. Suppose that $r(L_a)\geq i+1$. Then, there exists a chain of $\mathscr{L}$-classes such that

$$L_a > L_{i+1} > L_i > \cdots > L_2 > L_1,$$

where each L_j is an $\mathscr{L}$-class.

Now, $L_3 \not\subset \sum_1(S)$, $L_4 \not\subset \sum_2(S), \ldots, L_{i+1} \not\subset \sum_{i-1}(S)$, $L_a \not\subset \sum_i(S)$. This contradicts our assumption. Hence, $r(L_a) \leq i$.

The part (2) can be proved in the same manner.

LEMMA 1.10. (1) *Let S satisfy d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes. If an $\mathscr{L}$-class L satisfies $r(L) \leq k$ then $L \subset \sum_k(S)$. Similarly, if an $\mathscr{R}$-class R satisfies $r(R) \leq k$ then $R \subset \sum_k(S)$.*

(2) *Let S satisfy d.c.c. on $\mathscr{J}$-classes. If a $\mathscr{J}$-class J satisfies $r(J) \leq k$ then $J \subset {}_j\sum_k(S)$.*

PROOF. Assume that an $\mathscr{L}$-class L satisfies $r(L) = i \leq k$. Then, there exists a chain of $\mathscr{L}$-classes such that

$$0 = L_1 < L_2 < \cdots < L_i < L_{u+1} = L,$$

where each L_j is an $\mathscr{L}$-class of S. Further, for any j with $1 \leq j \leq i$ there is no $\mathscr{L}$-class L' such that $L_j < L' < L_{j+1}$. Now, $L_2 \subset \sum_1(S)$. Next, we show that, for $j < i$, $L_j \subset \sum_{j-1}(S)$ implies $L_{j+1} \subset \sum_j(S)$. It is obvious that $L_{j+1} \not\subset \sum_{j-1}(S)$ implies $L_{j+1} \subset S/\sum_{j-1}(S)$. Hence, $L_{j+1} \subset \sum_{j-1}(S) \subset \sum_j(S)$ or $L_{j+1} \subset S/\sum_{j-1}(S)$. In the latter case, L_{j+1} is a 0-minimal in $S/\sum_{j-1}(S)$. If not: There exists an $\mathscr{L}$-class $L_a \subset S/\sum_{j-1}(S)$ such that $L_a < L_{j+1}$. Since $L_a \not\subset \sum_{j-1}(S)$, there exists an $\mathscr{L}$-class $L'_j \subset \sum^*_{j-1}(S)$ such that $L'_j < L_a$. Similarly, $L'_j \not\subset \sum_{j-2}(S)$ implies that there exists an $\mathscr{L}$-class $L'_{j-1} \subset \sum^*_{j-2}(S)$ such that $L'_{j-1} < L'_j$. Continuing this process successively, we have a chain of $\mathscr{L}$-classes

$$0 = L_1 < L'_2 < \cdots < L'_{j-1} < L'_j < L_a < L_{j+1} < \cdots < L_{i+1} = L.$$

Hence, $r(L) > i$. This contradicts our assumption. Therefore, L_{j+1} is 0-minimal $\mathscr{L}$-class in $S/\sum_{j-1}(S)$, and accordingly $L_{j+1} \subset \sum_j(S)$. Hence, $L \subset \sum_i(S)$. The part (2) can be proved in the same manner.

THEOREM 1.11.

(1) *If S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes, then*

  (i) *for any $\mathscr{L}$-class L of S contained in $\sum^*_i(S)$, $r(L) = i$, and*

  (ii) $\sum(S) = \sum_1(S) \cup [\cup \{\sum^*_i(S)\backslash 0 \colon i = 2, 3, \ldots$ *(finite or infinite)*$\}]$
  $= \cup \{L \colon L$ *is an $\mathscr{L}$-class of S such that* $r(L) < \infty\}$
  $= \cup \{R \colon R$ *is an $\mathscr{R}$-class of S such that* $r(R) < \infty\}$.

(2) *If S satisfies d.c.c. on $\mathscr{J}$-classes, then*

  (i) *for any $\mathscr{J}$-class J of S contained in $\sum^*_i(S)$, $r(J) = i$, and*

  (ii) ${}_j\sum(S) = {}_j\sum_1(S) \cup [\cup \{{}_j\sum^*_i(S)\backslash 0 \colon i = 2, 3, \ldots$ *(finite or infinite)*$\}]$
  $= \cup \{J \colon J$ *is a $\mathscr{J}$-class such that* $r(J) < \infty\}$.

PROOF. Obvious from Lemmas 1.9 and .1.0.

Let S satisfy d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes. Then, it is easy to see that S is locally $\mathscr{L}$-bounded [locally $\mathscr{R}$-bounded] if and only if $S=\sum(S)$. Hence, in this case, "locally $\mathscr{L}$-bounded "and" locally $\mathscr{R}$-bounded" are equivalent to each other. Hence, in this case we simply say that S is *locally bounded*. Similarly, it is also easy to see that for S satisfying d.c.c. on $\mathscr{J}$-classes S is locally $\mathscr{J}$-bounded if and only if $S={}_j\sum(S)$.

Both $\sum(S)$ and ${}_j\sum(S)$ above are ideals of S, and accordingly a regular subsemigroups with 0. Further, $\sum(S)$ satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes] and is locally bounded, while ${}_j\sum(S)$ satisfies d.c.c. on $\mathscr{J}$-classes and is locally $\mathscr{J}$-bounded. These $\sum(S)$ and ${}_j\sum(S)$ are called *the locally bounded part* of S and *the locally $\mathscr{J}$-bounded part* of S respectively. Conversely, it is also easy to see that if S is locally $\mathscr{L}$-bounded [locally $\mathscr{J}$-bounded] then S satisfies d.c.c. on both $\mathscr{L}$-classes and $\mathscr{R}$-classes [$\mathscr{J}$-classes]. Further, if S is locally $\mathscr{L}$-bounded then S is also locally $\mathscr{R}$-bounded, and accordingly S is locally bounded.

THEOREM 1.12. *Assume that S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes [$\mathscr{J}$-classes]. Then, $\sum(S)[{}_j\sum(S)]$ is the greatest locally bounded [the greatest locally $\mathscr{J}$-bounded] ideal of S.*

PROOF. Every $\mathscr{L}$-class of S contained in $\sum(S)$ coincides with an $\mathscr{L}$-class of $\sum(S)$. Now, let L_a be an $\mathscr{L}$-class of $\sum(S)$. Then, L_a is an $\mathscr{L}$-class of S. Hence, there exists $\sum_i(S)$ such that $\sum_i(S)\subset L_a$. From Lemma 1.9, $r(L_a)\leq i$. Therefore, $\sum(S)$ is locally bounded. Next, let I be a locally bounded ideal of S. Let L_b be an $\mathscr{L}$-class of I. Then L_b is also an $\mathscr{L}$-class of S. Since $r(L_b)<\infty$ in I, $r(L_b)<\infty$ in S, too. Hence, it follows from Theorem 1.11 that $L_b\subset\sum(S)$. Hence, $I\subset\sum(S)$. For ${}_j\sum(S)$, we can prove the assertion in the same manner.

If S is $\mathscr{L}$-bounded [$\mathscr{R}$-bounded], then S satisfies d.c.c. on both $\mathscr{L}$-classes and $\mathscr{R}$-calsses. Hence, $S=\sum_m(S)$ for some m. Therefore, S is $\mathscr{R}$-bounded [$\mathscr{L}$-bounded]. Thus, "$\mathscr{L}$-bounded" is equivalent to "$\mathscr{R}$-bounded". Hereafter, in this case we simply say that S is *bounded*. Further, in this case S satisfies d.c.c. on $\mathscr{J}$-classes. Hence, we can consider ${}_j\sum(S)$, and $\sum(S)=\sum_m(S)$ (for some m)$={}_j\sum_m(S)=\sum_j(S)$. Accordingly, if S is bounded then S is also $\mathscr{J}$-bounded.

§ 2. Locally bounded regular semigroups with 0.

Firstly, we consider the following three conditions:

(2.1) For any $a\in S$, $L(a)$ contains only a finite number of $\mathscr{L}$-classes whose ranks are different.

(2.2) For any $a\in S$, $R(a)$ contains only a finite number of $\mathscr{R}$-classes whose ranks are different.

(2.3) For any $a\in S$, $J(a)$ contains only a finite number of $\mathscr{J}$-classes whose ranks are different.

Now,

THEOREM 2.1. *If S satisfies d.c.c. on $\mathcal{L}$-classes or $\mathcal{R}$-classes, then the following* (1)–(4) *are equivalent*:

(1) *S is locally bounded.*
(2) *S satisfies* (2.1).
(3) *S satisfies* (2.2).
(4) *S satisfies* (2.3).

PROOF. (1)$\Rightarrow$(2): Assume that $S\backslash \cup\{\sum_i(S)\colon i=1, 2,\dots$ (finite or infinite)$\}\neq\square$. Let L_a be one of minimal $\mathcal{L}$-classes contained in $S\backslash \cup\{\sum_i(S)\colon i=1, 2,\dots\}$. Put $L(a)\backslash L_a=L[a]$. Since $L[a]\ni 0$, $L[a]$ is non-empty. Since $L[a]$ contains only a finite number of $\mathcal{L}$-classes whose ranks are different, there exists a $\mathcal{L}$-class L_b which has the greatest rank in $L[a]$. Since $L_b\subset L[a]$, $L_b<L_a$. Hence, $L_b\subset \cup\{\sum_i(S)\colon i=1, 2,\dots\}$. Therefore, there exists $\sum_i(S)$ such that $L_b\subset\sum_i(S)$ and $L_b\not\subset\sum_{i-1}(S)$. Since L_b has the greatest rank in $L[a]$, every $\mathcal{L}$-class contained in $L[a]$ is contained also in $\sum_i(S)$. Hence, $L[a]\subset\sum_i(S)$. Now, L_a is a 0-minimal $\mathcal{L}$-class of $S/\sum_i(S)$. Therefore, $L_a\subset\sum_{i+1}(S)$. This contradicts our assumption. Thus, $S=\sum(S)$.
(2)$\Rightarrow$(1): Assume that S is locally bounded. Then, $S=\sum(S)$. Let a be any element of S. If $a=0$, $L(a)=\{0\}$. Hence, $L(a)$ contains only one $\mathcal{L}$-class 0. If $a\neq 0$, then there exists $\sum_k(S)$ such that $\sum_k(S)\ni a$ and $\sum_{k-1}(S)\not\ni a$. Since $L(a)\subset\mathcal{L}_k(S)$, the rank of an $\mathcal{L}$-class contained in $L(a)$ is less than $k+1$. Therefore, $L(a)$ contains only a finite number of $\mathcal{L}$-classes whose ranks are different.

(1)$\Leftrightarrow$(3) can be proved in the same manner.

(1)$\Leftrightarrow$(4): Since S satisfies d.c.c. on $\mathcal{L}$-classes or $\mathcal{R}$-classes, it follows from Lemma 1.7 that S satisfies d.c.c. on $\mathcal{J}$-classes. Therefore, we can easily prove by an analogous way to the proof of (1)$\Leftrightarrow$(2) that the condition (4) is equivalent to the condition "S is locally $\mathcal{J}$-bounded". Now, suppose that S is locally bounded. Then, $S=\sum(S)={}_j\sum(S)$. Therefore, S is locally $\mathcal{J}$-bounded, and accordingly S satisfies (4). Conversely, suppose that S satisfies (4). Since S satisfies d.c.c. on $\mathcal{L}$-classes or $\mathcal{R}$-classes, $\sum(S)={}_j\sum(S)$. On the other hand, the condition (4) implies that S is locally $\mathcal{J}$-bounded. Hence, $S={}_j\sum(S)=\sum(S)$. That is, S is locally bounded.

From the proof of Theorem 2.1, we have:

THEOREM 2.2. *If S satisfies d.c.c. on $\mathcal{J}$-classes, then S is locally $\mathcal{J}$-bounded if and only if S satisfies* (3.3).

LEMMA 2.3. *If S satisfies d.c.c. on left ideals* [*right ideals*] *and satisfies the following conditon* (2.4), *S is locally bounded*:
(2.4) *For any $a\in S$, a.c.c. is satisfied with respect to $\mathcal{L}$-classes of S contained in $L(a)$* [*$\mathcal{R}$-classes of S contained in $R(a)$*].

PROOF. Assume that S satisfies d.c.c. on left ideals and the condition (2.4).

It is clear that S satisfies d.c.c. on $\mathscr{L}$-classes. Suppose that $S\backslash\sum(S)\neq\square$. There exists a minimal $\mathscr{L}$-class contained in $A=S\backslash\sum(S)$. Let L_a be such an $\mathscr{L}$-class. Then, there exists a maximal $\mathscr{L}$-class of S contained in $L[a]=L(a)\backslash L_a$. Let $\Omega=\{L_\lambda: \lambda\in\Lambda\}$ be the maximal $\mathscr{L}$-classes of S contained in $L[a]$. If the cardinality of Λ is infinite, then there exist L_{λ_i} $(i=1, 2,\ldots)$, $\lambda_i\in\Lambda$. Now, for any $i\geq 1$, put $A_i=\cup\{L$: L is an $\mathscr{L}$-class such that $L\leq L_{\lambda_i}\}$. Then, it follows from Result 0.2 that A_i is a left ideal and $A_i\subset L[a]$. Now, $\cup\{A_i: i=1, 2,\ldots\}\supsetneqq\cup\{A_i: i=2, 3,\ldots\}\supsetneqq\cup\{A_i: i=3, 4,\ldots\}\supsetneqq\cdots$. This contradicts the assumption that S satisfies d.c.c. on left ideals. Hence, Ω is a finite set; that is, $|\Omega|<\infty$. Let $\Omega=\{L_{\lambda_i}: i=1, 2,\ldots, n\}$. Since $L_{\lambda_i}\subset L[a]\subset L(a)$, $L_{\lambda_i}<L_a$. Hence $L_{\lambda_i}\subset\sum(S)$. Therefore, there exists $\sum_{j_i}(S)$ such that $\sum_{j_i}(S)\supset L_{\lambda_i}$ and $\sum_{j_i-1}(S)\not\supset L_{\lambda_i}$. Let $m=\operatorname{Max}\{j_i: i=1, 2,\ldots, n\}$. Then, $L_{\lambda_i}\subset\sum_m(S)$ for all $i=1, 2,\ldots, n$. If there exists an $\mathscr{L}$-class L such that $S\backslash\sum_m(S)\subset L$ and $L<L_a$, then $L\subset L[a]$. Hence, there exists L_{λ_k} such that $L_{\lambda_k}\subset L$, and accordingly $L\subset\sum_m(S)$. This is a contradiction. Therefore, L_a is a 0-minimal $\mathscr{L}$-class of $S/\sum_m(S)$. Thus, we have $L_a\subset\sum_{m+1}(S)$. This contradicts the fact that $L_a\subset S\backslash\sum(S)$. Hence, $S=\sum(S)$, that is, S is locally bounded.

Similarly, we have the following:

LEMMA 2.4. *If S satisfies d.c.c. on ideals and if S satisfies the following* (2.5), *then S is locally $\mathscr{J}$-bounded*:

(2.5) *For any a of S, a.c.c. is satisfied with respect to $\mathscr{J}$-classes of S contained in $J(a)$.*

LEMMA 2.5. (1) *If S satisfies d.c.c. on left ideals [right ideals], then each $\sum_i^*(S)$ is a* 0-*direct union of completely* 0-*simple semigroups $\{S_{i,k}: k\in I$ (finite set)$\}$. Further, every $S_{i,k}$ consists of only a finite number of $\mathscr{L}$-classes [$\mathscr{R}$-classes].*

(2) *If S satisfies d.c.c. on ideals, then each ${}_j\sum_i^*(S)$ is a* 0-*direct union of only a finite number of* 0-*simple regular semigroups.*

PROOF. Obvious.

COROLLARY. (1) *Assume that S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes. If S satisfies a.c.c. on left ideals [right ideals], then each $\sum_i^*(S)$ contains only a finite number of $\mathscr{L}$-classes [$\mathscr{R}$-classes].*

(2) *Assume that S satisfies d.c.c. on $\mathscr{J}$-classes. If S satisfies a.c.c. on ideals, then each ${}_j\sum_i^*(S)$ is a* 0-*direct union of only a finite number of* 0-*simple regular semigroups.*

PROOF. Obvious.

§3. Bounded regular semigroups with 0.

If S is $\mathscr{L}$-bounded [$\mathscr{R}$-bounded], then S satisfies d.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes]. Hence, $\sum_m(S)=S$ for some m. Accordingly, S is also $\mathscr{R}$-bounded [$\mathscr{L}$-bounded].

Therefore, two concepts "$\mathscr{L}$-bounded" and "$\mathscr{R}$-bounded" are equivalent to each other; that is, in this case S is *bounded*. It is obvious that if S is bounded [$\mathscr{J}$-bounded] then S satisfies both a.c.c. and d.c.c. on both $\mathscr{L}$-classes and $\mathscr{R}$-classes [$\mathscr{J}$-classes]. Further, it is also obvious that if S satisfies d.c.c. on $\mathscr{L}$-classes or $\mathscr{R}$-classes [$\mathscr{J}$-classes] and satisfies a.c.c. on ideals, then S is bounded [$\mathscr{J}$-bounded].

THEOREM 3.1. *If S satisfies d.c.c. on left ideals [right ideals] and satisfies a.c.c. on $\mathscr{L}$-classes [$\mathscr{R}$-classes], then S is bounded.*

PROOF. For any $a \in S$, a.c.c. is satisfied with respect to $\mathscr{L}$-classes contained in $L(a)$. Hence, it follows from Lemma 2.3 that S is locally bounded; that is, $S=\sum(S)$. Since S satisfies a.c.c. on $\mathscr{L}$-classes, there is at least one maximal $\mathscr{L}$-class in S. Let $\Omega=\{L_\lambda: \lambda \in \Lambda\}$ be the set of all maximal $\mathscr{L}$-classes L_λ. If $|\Lambda|=\infty$ (where $|\ |$ means cardinality), then there exist L_{λ_i} $(i=1, 2,\dots)$, $\lambda_i \in \Lambda$. Take a_{λ_i} from each L_{λ_i}. Then, $\cup\{L(a_{\lambda_i}): i=1, 2,\dots\} \supsetneqq \cup\{L(a_{\lambda_i}): i=2, 3,\dots\} \supsetneqq \cup\{L(a_{\lambda_i}): i=3, 4,\dots\} \supsetneqq \cdots\cdots$.
Hence, we obtain an infinite chain of different left ideals. This contradicts our assumption. Therefore, $\Lambda=m<\infty$. Then, $\Omega=\{L_{\lambda_1}, L_{\lambda_2},\dots, L_{\lambda_m}\}$. Since $S=\sum(S)$, for any i with $1 \leq i \leq m$, there exists $\sum_{\lambda_i}(S)$ such that $L_{\lambda_i} \subset \sum_{\lambda_i}(S)$. Let $k=\mathrm{Max}\{\lambda_i: i=1, 2,\dots, m\}$. Then, $L_{\lambda_i} \subset \sum_k(S)$ fo rall i. Accordingly, $S=\sum_k(S)$. That is, S is bounded.

Similarly, we can easily obtain the following:

THEOREM 3.2. *If S satisfies d.c.c. on ideals and a.c.c. on $\mathscr{J}$-classes, then S is $\mathscr{J}$-bounded.*

REMARK. If S is bounded, S satisfies d.c.c. on $\mathscr{L}$-classes. Hence, $\sum_i(S)={}_J\sum_i(S)$ for all possible i. Hence, $S=\sum_m(S)={}_J\sum_m(S)$ for some m. Thus, S is also $\mathscr{J}$-bounded.

§4. Construction.

If S is a non-trivial bounded regular semigroup with 0 and if $\dim_{\mathscr{L}} S=n$ (hence, also $\dim_{\mathscr{R}} S=n$), then

$$0=\sum_0(S) \subset \sum_1(S) \subset \cdots \subset \sum_i(S) \subset \sum_{i+1}(S) \subset \cdots \subset \sum_n(S)=S.$$

In this case, each $\sum_{i+1}^*(S)=\sum_{i+1}(S)/\sum_i(S)$ is a 0-direct union of completely 0-simiple semigroups. Hence, $\sum_{i+1}(S)$ is an ideal extension of $\sum_i(S)$ by the primitive regular semigroup $\sum_{i+1}^*(S)$. Now, $\dim_{\mathscr{L}} \sum_{i+1}(S)=i+1$ and $\dim_{\mathscr{L}} \sum_i(S)=i$. Hence, we can say that every bounded regular semigroup B with $\dim_{\mathscr{L}} B=i+1$ can be obtained by an ideal extension of a bounded regular semigroup A with $\dim_{\mathscr{L}} A=i$ by a non-trival primitive regular semigroup C with 0. However, every ideal extension D of A by C does not necessarily satisfy $\dim_{\mathscr{L}} D=i+1$. Now, we have the following:

LEMMA 4.1. *For any* $a\in(\sum_{i+1}^{*}(S)\backslash 0)$, *there exists* $x\in(\sum_{i+1}^{*}(S)\backslash 0)\cup(\sum_{i}^{*}(S)\backslash 0)$ *(where* $i\geq 2$*) such that* $xa\in(\sum_{i+1}^{*}(S)\backslash 0)$.

PROOF. Since $L_a\subset\sum_{i+1}(S)$ and $L_a\not\subset\sum_i(S)$, there exists an $\mathscr{L}$-class L_x such that $L_x\subset S\backslash\sum_{i-1}(S)$ and $L_x<L_a$. Therefore, $L_x\subset\sum_i(S)\backslash\sum_{i-1}(S)$ and $L_x<L_a$. There exists $y\in S$ such that $x=ya$. Since $x\in\sum_i^*(S)$, $ya\in\sum_i^*(S)$ and $y\notin\sum_{i-1}(S)$. If $y\in\sum_{i+1}(S)$ then we can take y as x in the lemma. Suppose that $y\notin\sum_{i+1}(S)$. Since $ya=ya(ya)^*ya$, where $(ya)^*\in V(ya)$ (the set of inverses of ya), if we put $z=ya(ya)^*y$ then $za\in\sum_i^*(S)\backslash 0$ and $z\in\sum_{i+1}(S)\backslash\sum_{i-1}(S)$. Therefore, we can take z as x in the lemma. Accordingly, in any cases this lemma holds.

LEMMA 4.2. *Let A be a bounded regular semigroup with* 0, *and C a non-trivial primitive regular semigroup with* 0. *Assume that* $\dim_{\mathscr{L}}A=i\geq 1$. *Then, if an ideal extesion B of A by C satisfies the following condition, then B is a bounded regular semigroup and* $\dim_{\mathscr{L}}B=i+1$. *Further,* $\sum_i(B)=A$.

(4.1) *For any* $a\in C\backslash 0$, *there exists* $x\in(C\backslash 0)\cup(\sum_i^*(A)\backslash 0)$ *such that* $xa\in\sum_i^*(A)\backslash 0$.

PROOF. It is obvious that B is a bounded regular semigroup with 0. We need only to show that each non-zero $\mathscr{L}$-class of C is not a 0-minimal $\mathscr{L}$-class of $B/\sum_{i-1}(A)$. Let L_a be a non-zero $\mathscr{L}$-class of C. There exists $x\in(C\backslash 0)\cup(\sum_i^*(A)\backslash 0)$ such that $xa\in\sum_i^*(A)\backslash 0$. Then, $xa\in\sum_i^*(A)$ and $L_{xa}<L_a$ in B. Since $B/\sum_{i-1}(A)\supset L_{xa}$ $(\neq 0)$ and $L_{xa}<L_a$, L_a is not 0-minimal in $B/\sum_{i-1}(A)$.

Now, let us consider to construct all bounded regular semigroups with 0 by using induction concerning $\mathscr{L}$-dimension.

If $\dim_{\mathscr{L}}S=1$ then S is a primitive regular semigroup with 0 (hence, it is a 0-direct union of completely 0-simple semigroups). Therefore, in this case the construction of such semigroups is obvious. Assume that we can construct all bounded regular semigroups S of $\mathscr{L}$-dimension i (where $i\geq 1$). Let us consider the construction of all bounded regular semigroups of $\mathscr{L}$-dimension $i+1$. Let A be a bounded regular semigroup with 0 such that $\dim_{\mathscr{L}}A=i$, C a non-trivial primitive regular semigroup with 0. Let B be an ideal extension of A by C which satisfies (4.1). Then, by Lemmas 4.1 and 4.2 it follows that $\dim_{\mathscr{L}}B=i+1$. Further, every bounded regular semigroup of $\mathscr{L}$-dimension $i+1$ can be obtained in this way. Therefore, we need only to determine ideal extensions of A by C which satisfies (4.1).

Since a bounded regular semigroup with 0 is of course weakly reductive, by slightly modifying Theorem 4.21 of [1] we obtain the following:

Let A be a bounded regular semigroup with 0. Let $\dim_{\mathscr{L}}A=i\geq 1$. Let C be a non-trivial primitive regular semigroup with 0. Denote the semigroups of left and right translations of A by $\bar{A}$, $\bar{\bar{A}}$ respectively.

THEOREM 4.3. *Let ϕ be a ramification of C into A (see* [1]*), and let* $A\to\lambda_A$ *and* $A\to\rho_A$ *be mappings of* $C^*=C\backslash 0$ *into* $\bar{A}$ *and* $\bar{\bar{A}}$ *respectively such that*

$$(\mathrm{I})\quad \lambda_X\lambda_Y=\begin{cases}\lambda_{XY} & \textit{if}\quad X, Y\in C^* \quad \textit{and}\quad XY\neq 0,\\ \lambda_{(X,Y)\phi} & \textit{if}\quad X, Y\in C^* \quad \textit{and}\quad XY=0,\end{cases}$$

where, for $t\in A$, λ_t *is the inner left translation of* A *induced by* t.

$$(\mathrm{II})\quad \rho_X\rho_Y=\begin{cases}\rho_{XY} & \textit{if}\quad X, Y\in C^* \quad \textit{and}\quad XY\neq 0,\\ \rho_{(X,Y)\phi} & \textit{if}\quad X, Y\in C^* \quad \textit{and}\quad XY=0,\end{cases}$$

where, for $t\in A$, ρ_t *is the inner right translation of* A *induced by* t.

(III) *For any* $X\in C^*$, λ_X *and* ρ_X *are linked.*

(IV) *For every* $Y\in C^*$, *there exists* $Z\in C^*$ *such that* $ZY=0$ *and* $(Z, Y)\phi\in\sum_i^*(A)\backslash 0$; *or there exists* $z\in\sum_i^*(A)\backslash 0$ *such that* $z\rho_Y\in\sum_i^*(A)\backslash 0$.

Then $B=C^*\cup A$ *is a bounded regular semigroup of* $\mathscr{L}$*-dimension* $i+1$ *by the multiplication* $\circ$ *defined as follows:*

$$(\mathrm{N1})\quad X\circ Y=\begin{cases}XY & \textit{if}\quad X, Y\in C^* \quad \textit{and}\, XY=0,\\ (X, Y)\phi & \textit{if}\quad X, Y\in C^* \quad \textit{and}\quad XY=0.\end{cases}$$

(N2) $Y\circ s=s\lambda_Y$ *if* $s\in A$ *and* $Y\in C^*$.

(N3) $s\circ Y=s\rho_Y$ *if* $s\in A$ *and* $Y\in C^*$.

(N4) $s\circ t=st$ *if* $s, t\in A$.

Further, every bounded regular semigroup of $\mathscr{L}$*-dimension* $i+1$ *can be constructed in this way.*

PROOF. Obvious from the results above and Theorem 4.21 of [1].

REMARKS. (1) For any given A, C, the existence of at least one triple of mappings $\{X\rightarrow\lambda_X, X\rightarrow\rho_X, \phi\}$ satisfying (I)–(IV) above is easily verified. However, omit its proof.

(2) For a bounded regular semigroup S, $\dim_{\mathscr{L}} S=\dim_{\mathscr{R}} S$. Further, $\dim_{\mathscr{L}}\sum_i(S)=\dim_{\mathscr{R}}\sum_i(S)$ for all $i\leq\dim_{\mathscr{L}} S$. Hence, the theorem above is still holds, even if (IV) is changed as follows:

(IV)* For every $Y\in C^*$, there exists $Z\in C^*$ such that $YZ=0$ and $(Y, Z)\phi\in\sum_i^*(A)\backslash 0$; or there exists $z\in\sum_i^*(A)\backslash 0$ such that $z\lambda_Y\in\sum_i^*(A)\backslash 0$.

(3) For the construction of $\mathscr{J}$-bounded regular semigroups, we can proceed an analogous discussion by changing "bounded" "a primitive regular semigroup with 0" to "$\mathscr{J}$-bounded", "a 0-direct union of 0-simple regular semigroups" respectively. However, we omit here the discussion.

References

[1] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, Vol. I and II, Math. Surveys 7, Amer. Math. Soc., Providence, Rhode Island, 1961 and 1967.

[2] J. A. Green, On the structure of semigroups, Ann. of Math. (2) **54** (1951), 163–172.

[3] J. M. Howie, An introduction to semigroup theory, Academic Press, London, 1975.

[4] W. D. Munn, Semigroups satisfying minimal conditions, Proc. Glasgow Math. Assoc. 3 (1957), 145–152.