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Discrete Biharmonic Green Function β

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As a discrete analogue to the results related to the biharmonic Green function β of a Riemannian manifold due to Sario et al., we discuss the existence and uniqueness of the discrete biharmonic Green function β of an infinite network. A discrete analogue to the normal derivative plays an important role in our study.

Introduction

On a regular subregion Ω of a Riemannian manifold, there exist two biharmonic Green functions, to be denoted by β and γ, with a biharmonic fundamental singularity, and with boundary data $\beta=\partial\beta/\partial n=0$ and $\gamma=\Delta\gamma=0$ (cf. [2]). A discrete analogue to the biharmonic Green function γ was studied in [5]. In this paper, we shall discuss the existence and uniqueness of a discrete analogue to the biharmonic Green function β. Discrete analogues to the biharmonic fundamental singularity $\Delta^2\beta_a=\varepsilon_a$ and the boudary data $\beta_a=0$ are easily formulated. The discrete analogue to the boundary data $\partial\beta_a/\partial n=0$ seems to be not easy as in [1] or [3]. We replace this boundary data by the condition that a weak normal derivative of β_a vanishes on the boundary in §2 and §3. For a finite subnetwork N' of a locally finite infinite network N, the existence and uniqueness of the biharmonic Green function $\beta_a^{N'}$ of N' with pole at a satisfying explicit boundary data will be shown in §2. The biharmonic Green function β_a of N with pole at a will be studied in §3 related to the ideal boundary of N. For an exhaustion $\{N_n\}$ of N, the convergence of the sequence of the biharmonic Green functions $\beta_a^{N_n}$ of N_n will be discussed in §4.

§1. Preliminaries

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive function on Y. Assume that the quartet $N=\{X, Y, K, r\}$ is an infinite network i. e., the graph $\{X, Y, K\}$ is connected, locally finite and has no self-loop. For notation and terminology, we mainly follow [4] and [6].

For a finite subnetwork $N'=\langle X', Y'\rangle$ of N, denote by $nb(N')$ the subnetwork $\langle nb(X'), nb(Y')\rangle$ of N defined by $nb(X')=\cup\{X(x); x\in X'\}$ and $nb(Y')=\{y\in Y;$ $e(y)\subset nb(X')\}$, where $e(y)=\{x\in X; K(x, y)\neq 0\}$ (the set of end nodes of y) and $X(x)=$

$\cup \{e(y);\ K(x, y) \neq 0\}$ (the set of neighboring nodes of x). Let us put $b(X') = nb\,(X') - X'$ and $b\,(Y') = nb\,(Y') - Y'$ and regard the pair $\{b(X'), b(Y')\}$ as the boundary of N'.

Let $L(X)$ be the set of all real functions on X. For $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ of u is defined by

$$\Delta u(x) = -\sum_{y \in Y} K(x, y) r(y)^{-1} [\sum_{z \in X} K(z, y) u(z)].$$

A function $u \in L(X)$ is called harmonic or biharmonic on a set A according as $\Delta u(x) = 0$ or $\Delta^2 u(x) = \Delta(\Delta u)(x) = 0$ on A respectively.

For a finite subnetwork $N' = \langle X', Y' \rangle$ of N, the harmonic Green function $g'_a = g_a^{N'}$ of N' with pole at $a \in X'$ is defined by

(1.1) $\Delta g'_a(x) = -\varepsilon_a(x)$ on X',

(1.2) $g'_a(x) = 0$ on $X - X'$,

where $\varepsilon_a(x) = 0$ if $x \neq a$ and $\varepsilon_a(a) = 1$.

The existence and uniqueness of g'_a was studied in [4]. Let $\{N_n\}$ be an exhaustion of N and let $g_a^{(n)}$ be the harmonic Green function of N_n with pole at a. Then we see that $g_a^{(n)} \leq g_a^{(n+1)}$ and the limit g_a of $\{g_a^{(n)}\}$ exists and does not depend on the choice of an exhaustion of N. We have either $g_a \in L(X)$ or $g_a = \infty$. In case $g_a = \infty$, we say that N has no Green function and denote by O_G the set of all infinite networks which have no Green function, In case $g_a \in L(X)$, we call it the harmonic Green function of N with pole at a. We have $\Delta g_a(x) = -\varepsilon_a(x)$ on X and $g_a(x) = g_x(a)$ for every $a, x \in X$.

For $\mu \in L^+(X)$, the (harmonic) Green potential $G\mu$ of μ is defined by

$$G\mu(x) = \sum_{z \in X} g_z(x) \mu(z).$$

We have either $G\mu \in L(X)$ or $G\mu = \infty$. Let us put $M(G) = \{\mu \in L^+(X);\ G\mu \in L(X)\}$.

§2. Biharmonic Green function $\beta_a^{N'}$

Let m be a strictly positive function on X. We call it a weight function. For $u, v \in L(X)$, the inner product $((u, v))$ of u and v and the norm $\|u\|$ of u are defined by

$$((u, v)) = \sum_{x \in X} m(x) u(x) v(x) \quad \text{and} \quad \|u\| = [((u, u))]^{1/2}$$

if the sum is well-defined. Denote by $L_2(X; m)$ the set of all $u \in L(X)$ with finite norm. Note that $L_2(X; m)$ is a Hilbert space with respect to the inner product $((u, v))$.

We give some examples of the weight functions.

EXAMPLE 2.1. (1) $m(x) = 1$ on X.
(2) $m(x) = \sum_{y \in Y} |K(x, y)| r(y)$ on X. In this case, $\|u\|^2 = \sum_{y \in Y} r(y) \sum_{x \in X} |K(x, y)| \cdot u(x)^2$. If $\|1\| < \infty$, then $N \notin O_G$.
(3) $m(x) = \sum_{y \in Y} |K(x, y)| r(y)^{-1}$ on X. In this case, $\|u\|^2 = \sum_{y \in Y} r(y)^{-1} \sum_{x \in X} |K(x, y)| \cdot$

$u(x)^2$. If $\|1\|<\infty$, then $N\in O_G$.

Let $N'=\langle X', Y'\rangle$ be a finite subnetwork of N and denote by $H(N')$ the set of all $u\in L(X)$ which is harmonic on X' and satisfies the boundary condition: $u(x)=0$ on $X-nb(X')$. Then $H(N')$ is a closed subspace of $L_2(X; m)$.

In order to construct a discrete analogue to the biharmonic Green function β, we introduce a discrete analogue to the weak normal derivative in [2]. Let $u\in L(X)$. For any $h\in H(N')$, $((h, \Delta u))$ is a continuous linear functional on $H(N')$. Thus there exists a unique $\partial u\in H(N')$ such that $((h, \Delta u))=((h, \partial u))$ for all $h\in H(N')$ by Riesz's theorem. We call ∂u the weak normal derivative of u on $b(X')$ (with respect to the weight function m).

LEMMA 2.1. *The weak normal derivative ∂u of u on $b(X')$ vanishes on $b(X')$ if and only if $((h, \Delta u))=0$ for all $h\in H(N')$.*

PROOF. Let $\partial u(x)=0$ on $b(X')$. Since ∂u is harmonic on X', we see by the maximum principle (cf. [5; Lemma 1.1]) that $\partial u(x)=0$ on $nb(X')$, so that $((h, \Delta u))=((h, \partial u))=0$ for all $h\in H(N')$. On the other hand, assume that $((h, \Delta u))=0$, for all $h\in H(N')$. Since $\partial u\in H(N')$, we have $((\partial u, \partial u))=0$, so that $\partial u(x)=0$ on X.

Denote by $W(N'; m)$ the orthogonal complement of $H(N')$ in $L_2(X; m)$, i.e.,

$$W(N'; m)=\{v\in L_2(X; m);\ ((h, v))=0 \text{ for all } h\in H(N')\}.$$

Now we define the biharmonic Green function $\beta'_a=\beta_a^{N'}$ of N' with pole at $a\in X'$ by the following conditions:

(2.1) $\Delta^2\beta'_a(x)=\varepsilon_a(x)$ on X'

(2.2) $\Delta\beta'_a\in W(N'; m)$, i.e., $\partial\beta'_a(x)=0$ on $b(X')$,

(2.3) $\beta'_a(x)=0$ on $X-nb(X')$.

The uniqueness of β'_a follows from the following lemma.

LEMMA 2.2. *Assume that $u\in L(X)$ satisfies the conditions:*

(2.4) $\Delta^2u(x)=0$ *on* X',

(2.5) $\Delta u\in W(N'; m)$,

(2.6) $u(x)=0$ *on* $X-nb(X')$.

Then $u(x)=0$ on X.

PROOF. Define $v\in L(X)$ by $v(x)=\Delta u(x)$ for $x\in nb(X')$ and $v(x)=0$ for $x\in X-nb(X')$. Then $\Delta v(x)=0$ on X' by (2.4), so that $v\in H(N')$. We have $\|v\|^2=((v, \Delta u))=0$ by (2.5), and hence $v(x)=0$ on X. Thus u is harmonic on $nb(X')$. It follows from (2.6) and the maximum principle that $u(x)=0$ on X.

In order to prove the existence of β'_a, we consider the following extremum problem:

(2.7) Find $c(N'; a)=\inf\{\|h-g'_a\|^2; h\in H(N')\}$,

where g'_a is the harmonic Green function of N' with pole at a.

We have by the standard projection theorem

LEMMA 2.3. *Problem* (2.7) *has a unique solution* h'_a, *i.e.*, $h'_a\in H(N')$ *such that* $c(N'; a)=\|h'_a-g'_a\|^2$. *Put* $k'_a=h'_a-g'_a$. *Then* k'_a *is the projection of* $-g'_a$ *onto* $W(N'; m)$ *and*

(2.8) $((h, k'_a))=0$ *for every* $h\in H(N')$.

REMARK 2.1. We have $c(N'; a)>0$. In fact, if $c(N'; a)=0$, then $k'_a=0$ and $g'_a\in H(N')$. This is a contradiction.

REMARK 2.2. It should be noted that k'_a is not of constant sign. In fact, assume that k'_a is non-negative (non-positive resp.) on X and let $\tilde{g}'_z$ be the harmonic Green function of $nb(N')$ with pole at $z\in b(X')$. Since $\tilde{g}'_z>0$ on $nb(X')$ and $\tilde{g}'_z\in H(N')$, (2.8) implies that $k'_a(x)=0$ on $nb(X')$, i.e., $c(N'; a)=0$. This is a contradiction.

We give a simple example of β'_a.

EXAMPLE 2.2. Let J be the set of all non-negative integers. Let us take $X=\{x_n; n\in J\}$, $Y=\{y_{n+1}; n\in J\}$ and define $K(x, y)$ by $K(x_n, y_{n+1})=-1$ and $K(x_{n+1}, y_{n+1})=1$ for $n\in J$ and $K(x, y)=0$ for any other pair (x, y). For any positive function r on Y, $N=\{X, Y, K, r\}$ is a locally finite infinite network. Let $X'=\{x_0, x_1, x_2\}$ and $Y'=\{y_1, y_2\}$. Then $N'=\langle X', Y'\rangle$ is a finite subnetwork of N and $b(X')=\{x_3\}$ and $b(Y')=\{y_3\}$. We have

$$H(N')=\{h\in L(X); h(x_n)=h(x_0)\ (1\leq n\leq 3),\ h(x_n)=0\ (n\geq 4)\}.$$

Let $a=x_0$ and put $r_n=r(y_n)$. Then $g'_a(a)=r_1+r_2+r_3$, $g'_a(x_1)=r_2+r_3$, $g'_a(x_2)=r_3$ and $g'_a(x_n)=0$ $(n\geq 3)$. We see by Lemma 2.3 that $k'_a(x_n)=c-g'_a(x_n)$ $(n=0, 1, 2)$, $k'_a(x_3)=c$ and $k'_a(x_n)=0$ $(n\geq 4)$ with a constant c. It follows from (2.8) that $c=\sum_{n=0}^{2} m(x_n)g'_a(x_n)/\sum_{n=0}^{3} m(x_n)$.

LEMMA 2.4. *Let* $u\in L(X)$. *If* u *is harmonic on* X', *then* $((u, k'_a))=0$.

PROOF. Define $h\in L(X)$ by $h(x)=u(x)$ for $x\in nb(X')$ and $h(x)=0$ for $x\in X-nb(X')$. Then $h\in H(N')$. Since $k'_a(x)=0$ on $X-nb(X')$, we have $((u, k'_a))=((h, k'_a))=0$ by (2.8).

We have

THEOREM 2.1. *For every finite subnetwork $N'=\langle X', Y'\rangle$ of N and $a\in X'$, the biharmonic Green function $\beta'_a=\beta_a^{N'}$ of N' with pole at a is given by*

(2.9) $\beta'_a(x)=-\sum_{z\in nb(X')}\tilde{g}'_z(x)k'_a(z),$

where $\tilde{g}'_z$ is the harmonic Green function of $nb(N')$ with pole at $z\in nb(X')$.

PROOF. Since $\tilde{g}'_z(x)=0$ for every $x\in X-nb(X')$, condition (2.3) is fulfilled. We have

$$\Delta\beta'_a(x)=-\sum_{z\in nb(X')}[\Delta\tilde{g}'_z(x)]k'_a(z)=k'_a(x)$$

for every $x\in nb(X')$. It follows from Lemma 2.3 that $((h, \Delta\beta'_a))=((h, k'_a))=0$ for every $h\in H(N')$, which shows condition (2.2). Since $k'_a=-g'_a+h'_a$ with $h'_a\in H(N')$ and $\Delta\beta'_a(x)=k'_a(x)$ on $nb(X')$, we have for $x\in X'$

$$\Delta^2\beta'_a(x)=\Delta k'_a(x)=\Delta(-g'_a+h'_a)(x)=-\Delta g'_a(x)=\varepsilon_a(x).$$

Namely condition (2.3) is fulfilled.

REMARK 2.3. Let $\tilde{g}'_z$ be the harmonic Green function of $nb(N')$ with pole at $z\in nb(X')$. For any $v\in L(X)$, define the potential $\tilde{G}'v\in L(X)$ of v by

$$\tilde{G}'v(x)=\sum_{z\in nb(X')}\tilde{g}'_z(x)v(z).$$

Then $H(N')=\{\tilde{G}'v;\ v\in L(X),\ v(x)=0 \text{ on } X'\}$.

§3. Biharmonic Green function β_a^N

In the rest of this paper, we always assume that N has a harmonic Green function, i.e., $N\notin O_G$.

Let us put

$$HL_2(N;m)=\{h\in L_2(X;m);\ h \text{ is harmonic on } X\}.$$

Then $HL_2(N;m)$ is a closed subspace of $L_2(X;m)$. Denote by $W(N;m)$ the orthogonal complement of $HL_2(N;m)$ in $L_2(X;m)$, i.e.,

$$W(N;m)=\{u\in L_2(X;m);\ ((h, u))=0 \text{ for all } h\in HL_2(N;m)\}.$$

Let $u\in L(X)$ such that $\Delta u\in L_2(X;m)$. For any $h\in HL_2(N;m)$, $((h, \Delta u))$ is a continuous linear functional on $HL_2(N;m)$. Thus there exists a unique $\partial u\in HL_2(N;m)$ such that $((h, \Delta u))=((h, \partial u))$ for all $h\in HL_2(N;m)$ by Riesz's theorem. We call ∂u the weak normal derivative of u on the ideal boundary (with respect to the weight function m).

We say that the weak normal derivative ∂u of u on the ideal boundary vanishes on the ideal boundary if $\Delta u \in W(N; m)$.

We say that $u \in L(X)$ vanishes on the ideal boundary if it belongs to the following functional space

$$P(G) = \{G\mu_1 - G\mu_2;\ \mu_1, \mu_2 \in M(G),\ \mu_1(x)\mu_2(x) = 0 \text{ on } X\}.$$

We define the biharmonic Green function $\beta_a = \beta_a^N$ of N with pole at $a \in X$ by the following conditions:

(3.1) $\Delta^2 \beta_a(x) = \varepsilon_a(x)$ on X,

(3.2) $\Delta \beta_a(x) \in W(N; m)$,

(3.3) $\beta_a \in P(G)$.

To prove the uniqueness of β_a, it suffices to show the following lemma:

LEMMA 3.1. *Assume that u is biharmonic on X. If $u \in P(G)$ and $\Delta u \in W(N; m)$, then $u(x) = 0$ on X.*

PROOF. Let $u = G\mu_1 - G\mu_2$ with $\mu_1, \mu_2 \in M(G)$ and $\mu_1(x)\mu_2(x) = 0$ on X. Then $\Delta u(x) = \mu_2(x) - \mu_1(x)$ on X. By our assumption, $\Delta u \in HL_2(N; m)$. Since $\Delta u \in W(N; m)$, we have

$$0 = (\!(\Delta u, \Delta u)\!) = \|\Delta u\|^2 = \|\mu_1\|^2 + \|\mu_2\|^2,$$

and hence $\mu_1(x) = \mu_2(x) = 0$ on X. Thus $u(x) = 0$ on X.

Similarly to problem (2.7), we consider the following extremum problem:

(3.4) Find $c(N; a) = \inf\{\|h - g_a\|^2;\ h \in HL_2(N; m)\}$,

where g_a is the harmonic Green function of N with pole at a.

REMARK 3.1. The value $c(N; a)$ of problem (3.4) is finite if and only if the norm $\|g_a\|$ of g_a is finite.

By the standard projection theorem, we have

LEMMA 3.2. *If $c(N; a)$ is finite, then there exists a unique optimal solution h_a of problem (3.4), i.e., $h_a \in HL_2(N; m)$ such that $c(N; a) = \|h_a - g_a\|^2$. Put $k_a = h_a - g_a$. Then k_a is the projection of $-g_a$ onto $W(N; m)$.*

COROLLARY 1. *$\Delta k_a(x) = \varepsilon_a(x)$ on X.*

COROLLARY 2. *If $g_a \in W(N; m)$, then $k_a = -g_a$.*

We have

THEOREM 3.1. *Assume that*

(3.5) $c(N; a)<\infty$,

(3.6) $\sum_{z\in X} g_x(z)|k_a(z)|<\infty$ *for some* $x\in X$.

Then the biharmonic Green function β_a *of* N *with pole at* a *is given by*

(3.7) $\beta_a(x)=-\sum_{z\in X} g_x(z)k_a(z)$.

Proof. By Harnack's principle [5; Lemma 1.3] and by (3.6), $\beta_a(x)$ is well-defined for all $x\in X$. We have $\Delta\beta_a(x)=k_a(x)\in W(N; m)$ by Lemma 3.2 and $\Delta^2\beta_a(x)=\varepsilon_a(x)$ on X. We see by (3.6) that $\beta_a\in P(G)$.

Corollary. *If* $g_a\in W(N; m)$, *then* $\beta_a(x)=\sum_{z\in X} g_x(z)g_a(z)$, *which is equal to the discrete analogue to the biharmonic Green function* γ (cf. [5]).

Remark 3.2. If $m(x)\geq m_0>0$ on X, then $HL_2(N; m)=\{0\}$ by [5; Theorem 1.1]. If we further assume that $\|g_a\|<\infty$, then $g_a\in W(N; m)$.

Lemma 3.3. *Assume that the norm of* g_a *is finite. Then* $c(N; a)=\|g_a\|^2$ *if and only if* $k_a=-g_a$.

Proof. It suffices to show the "only if" part. Assume that $c(N; a)=\|g_a\|^2$. Then we have by Lemma 3.2

$$0=(\!(h_a, k_a)\!)=-(\!(h_a, g_a)\!)+\|h_a\|^2,$$

$$\|k_a\|^2=(\!(-g_a+h_a, -g_a+h_a)\!)=\|g_a\|^2-\|h_a\|^2.$$

Since $c(N; a)=\|k_a\|^2$, we have $\|h_a\|=0$, and hence $h_a(x)=0$ on X.

We show that β_a is not equal to the biharmonic Green function γ in general.

Example 3.1. Let N be the infinite network defined in Example 2.1. Assume that $\sum_{y\in Y} r(y)<\infty$ and $\sum_{x\in X} m(x)=1$. Let $a=x_0$. Then $g_a(x_n)=\sum_{j=n+1}^{\infty} r(y_j)$ and $HL_2(N; m)$ consists only of constant functions. Since $g_a(x)\leq g_a(a)$ on X, the norm of g_a is finite. We see easily that $k_a=-g_a+(\!(1, g_a)\!)$. In case $\sum_{x\in X} g_a(x)<\infty$, i.e., $N\notin O_{QP}$ (cf. [6]), $\beta_a(x)$ exists by Theorem 3.1 and

$$\beta_a(x)=\sum_{z\in X} g_x(z)g_a(z)-(\!(1, g_a)\!)\sum_{z\in X} g_x(z).$$

As for condition (3.5), we have

Theorem 3.2. *Let* $a, b\in X$. *Then* $c(N; a)<\infty$ *if and only if* $c(N; b)<\infty$.

Proof. By Harnack's principle [5; Lemma 1.3], there exists a constant $\alpha>0$ such that $\alpha^{-1}g_b(x)\leq g_a(x)\leq\alpha g_b(x)$ on X. We have $\alpha^{-1}\|g_b\|\leq\|g_a\|\leq\alpha\|g_b\|$. Our assertion follows from Remark 3.1.

As for condition (3.6), we have

THEOREM 3.3. *Assume that the norm of g_a is finite and that $\sum_{z\in X} m(z)^{-1}g_a(z)^2<\infty$. Then condition* (3.6) *is fulfilled.*

PROOF. By Remark 3.1, $\|k_a\|<\infty$. Let $x\in X$. We see by Harnack's principle that $\sum_{z\in X} m(z)^{-1}g_x(z)^2<\infty$ (cf. the proof of Theorem 3.2). We have

$$[\sum_{z\in X} g_x(z)|k_a(z)|]^2\leq[\sum_{z\in X} m(z)^{-1}g_x(z)^2][\sum_{z\in X} m(z)k_a(z)^2]<\infty.$$

§4. Convergence of $\beta_a^{(n)}$

Let $\{N_n\}(N_n=\langle X_n, Y_n\rangle)$ be an exhaustion of N. We are concerned with the convergence of the sequence $\{\beta_a^{(n)}\}$ of the biharmonic Green functions of N_n with pole at a.

We have

LEMMA 4.1. *For any finite subnetwork $N'=\langle X', Y'\rangle$ of N and $a\in X'$, $c(N'; a)\leq c(N; a)$.*

PROOF. Let $h\in HL_2(N; m)$ and put $u=-g_a+h$. Define $v\in L(X)$ by $v(x)=u(x)$ for $x\in nb(X')$ and $v(x)=0$ for $x\in X-nb(X')$. Then $h'=v+g'_a\in H(N')$, so that $c(N'; a)\leq\|v\|^2\leq\|u\|^2$.

LEMMA 4.2. *Assume that $c(N; a)$ is finite and let $k_a^{(n)}$ be the optimal solution of problem* (2.7) *replacing N' by N_n. Then $\|k_a^{(n)}-k_a\|\to 0$ as $n\to\infty$.*

PROOF. If $n<m$, then $k_a^{(n)}-k_a^{(m)}$ is harmonic on X_n and $(\!(k_a^{(n)}-k_a^{(m)}, k_a^{(n)})\!)=0$ by Lemma 2.4, so that

$$\|k_a^{(n)}-k_a^{(m)}\|^2=\|k_a^{(m)}\|^2-\|k_a^{(n)}\|^2.$$

Since $\|k_a^{(n)}\|^2\leq c(N; a)$ by Lemma 4.1, we see that $\{k_a^{(n)}\}$ is a Cauchy sequence in the Hilbert space $L_2(X; m)$. There exists $v\in L_2(X; m)$ such that $\|k_a^{(n)}-v\|\to 0$ as $n\to\infty$. Note that $k_a^{(n)}(x)$ converges to $v(x)$ for each $x\in X$. Let $g_a^{(n)}$ be the harmonic Green function of N_n with pole at a. Since $h_a^{(n)}=k_a^{(n)}+g_a^{(n)}$ is harmonic on X_n, we see that $h^*=v+g_a$ is harmonic on X. Thus $h^*\in HL_2(N; m)$. Let h be any element of $HL_2(N; m)$. Then $(\!(h, k_a^{(n)})\!)=0$ by Lemma 2.4, so that $(\!(h, v)\!)=0$, i.e., $v\in W(N; m)$. It follows from Lemma 3.2 that $v=k_a$.

We have

THEOREM 4.1. *Assume that the norm of g_a is finite and that $\sum_{z\in X} m(z)^{-1}g_a(z)^2<\infty$. Then $\{\beta_a^{(n)}(x)\}$ converges to $\beta_a(x)$ for each $x\in X$.*

PROOF. The existence of β_a follows from Theorems 3.1 and 3.3. For each $x\in X$, let us define $p_x^{(n)}$ and p_x by

$$p_x^{(n)}(z)=m(z)^{-1}\tilde{g}_z^{(n)}(x) \qquad \text{for} \quad z\in nb(X_n),$$

$$p_x^{(n)}(z)=0 \qquad \text{for} \quad z\in X-nb(X_n),$$

$$p_x(z)=m(z)^{-1}g_z(x) \qquad \text{for} \quad z\in X,$$

where $\tilde{g}_z^{(n)}$ is the harmonic Green function of $nb(N_n)$ with pole at $z\in nb(X_n)$. We see by Theorems 2.1 and 3.1 that $\beta_a^{(n)}(x)=-(\!(p_x^{(n)}, k_a^{(n)})\!)$ and $\beta_a(x)=-(\!(p_x, k_a)\!)$. We have

$$|\beta_a(x)-\beta_a^{(n)}(x)|\le|(\!(p_x-p_x^{(n)}, k_a^{(n)})\!)|+|(\!(p_x, k_a^{(n)}-k_a)\!)|$$
$$\le\|p_x-p_x^{(n)}\|\,\|k_a^{(n)}\|+\|p_x\|\,\|k_a^{(n)}-k_a\|$$

Note that $\|p_x\|^2=\sum_{z\in X}m(z)^{-1}g_x(z)^2$ is finite by our assumption and Harnack's principle. By Lemmas 4.1 and 4.2, it suffices to show that $\|p_x-p_x^{(n)}\|\to 0$ as $n\to\infty$ for each $x\in X$. For any $\varepsilon>0$, there exists n_1 such that $x\in X'=X_{n_1}$ and

$$\textstyle\sum_{z\in X-X'}m(z)^{-1}g_z(x)^2<\varepsilon/3.$$

Since $\tilde{g}_z^{(n)}(x)=\tilde{g}_x^{(n)}(z)$ for any $z\in nb(X_n)$ if $n\ge n_1$ and $\tilde{g}_z^{(n)}(x)$ converges to $g_z(x)$, there exists n_2 such that

$$|g_z(x)-\tilde{g}_z^{(n)}(x)|^2<\varepsilon/3t \quad \text{with} \quad t=\textstyle\sum_{z\in X'}m(z)^{-1}$$

for all $n\ge n_2$ and $z\in X'$. Let $n\ge\max\{n_1, n_2\}$. Since $0\le\tilde{g}_x^{(n)}(z)\le g_x(z)$ on X, we have

$$\textstyle\sum_{z\in X-X'}m(z)^{-1}[g_x(z)-\tilde{g}_x^{(n)}(z)]^2\le\sum_{z\in X-X'}m(z)^{-1}g_x(z)^2<\varepsilon/3.$$

We have

$$\|p_x-p_x^{(n)}\|^2<\textstyle\sum_{z\in nb(X_n)}m(z)^{-1}[g_z(x)-\tilde{g}_z^{(n)}(x)]^2+\varepsilon/3$$
$$<\textstyle\sum_{z\in X}m(z)^{-1}[g_x(z)-\tilde{g}_x^{(n)}(z)]^2+\varepsilon/3$$
$$<\textstyle\sum_{z\in X'}m(z)^{-1}[g_x(z)-\tilde{g}_x^{(n)}(z)]^2+2\varepsilon/3$$
$$<[\textstyle\sum_{z\in X'}m(z)^{-1}]\varepsilon/3t+2\varepsilon/3=\varepsilon.$$

Therefore $\|p_x-p_x^{(n)}\|\to 0$ as $n\to\infty$.

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