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IMPROVED OPERATOR MONOTONICITY OF AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for all T a positive operator on a complex Hilbert space H.

Let $A > 0$ and assume that there exist positive numbers $d > c > 0$ such that $d \geq B - A \geq c > 0$, then, we show that,

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d + \|A\|)\right] \geq 0.$$

As a consequence we derive that

$$f(A)A^{-1} - f(B)B^{-1} \geq \frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(d + \|A\|)}{d + \|A\|}\right) \geq 0,$$

if f is operator monotone on $[0, \infty)$ with $f(0) = 0$ and

$$f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right)$$
$$\geq \frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(d + \|A\|)}{(d + \|A\|)^2}\right] - \frac{cf'_+(0)}{\|A\|(d + \|A\|)} \geq 0$$

provided that f is operator convex on $[0, \infty)$ with $f(0) = 0$. Some examples of interest are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

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We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ with $f'_+(0)\in\mathbb{R}$ if and only if it has the representation*

$$f(t)=f(0)+f'_+(0)t+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where $c\geq 0$ and a positive measure μ on $[0,\infty)$ such that (1.2) holds.

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\geq m>0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$\begin{aligned} f(B)-f(A) &\geq f(\|A\|+m)-f(\|A\|)\\ &\geq f(\|B\|)-f(\|B\|-m)>0. \end{aligned} \tag{1.4}$$

If $B>A>0$, then

$$\begin{aligned} f(B)-f(A) &\geq f\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-f(\|A\|)\\ &\geq f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0. \end{aligned} \tag{1.5}$$

The inequality between the first and third term in (1.5) was obtained earlier by H. Zuo and G. Duan in [8].

By taking $f(t) = t^r$, $r \in (0,1]$ in (1.5) Furuta obtained the following refinement of the celebrated Löwner-Heinz inequality [5]

$$B^r - A^r \geq \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r - \|A\|^r \geq \|B\|^r - \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right)^r > 0 \tag{1.6}$$

provided $B > A > 0$.

With the same assumptions for A and B, we have the logarithmic inequality [4]

$$\ln B - \ln A \geq \ln \left(\|A\| + \frac{1}{\left\|(B-A)^{-1}\right\|} \right) - \ln(\|A\|) \geq \ln(\|B\|) - \ln \left(\|B\| - \frac{1}{\left\|(B-A)^{-1}\right\|} \right) > 0. \tag{1.7}$$

Notice that the inequalities between the first and third terms in (1.6) and (1.7) were obtained earlier by M. S. Moslehian and H. Najafi in [7].

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Motivated by the above results, in this paper we show that

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d + \|A\|)\right] \geq 0,$$

where $A > 0$ and provided that there exist positive numbers $d > c > 0$ such that $d \geq B - A \geq c > 0$. As a consequence, we derive the following alternative lower bound to the one provided by Furuta's result in (1.4),

$$f(A)A^{-1} - f(B)B^{-1} \geq \frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(d + \|A\|)}{d + \|A\|} \right) \geq 0,$$

if f is operator monotone on $[0,\infty)$ with $f(0) = 0$ and

$$f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) \geq \frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(d + \|A\|)}{(d + \|A\|)^2} \right] - \frac{cf'_+(0)}{\|A\|(d + \|A\|)} \geq 0$$

provided that f is operator convex on $[0,\infty)$ with $f(0) = 0$. Some examples of interest are also given.

2. Preliminary Facts

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{2.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{2.2}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{2.3}$$

where μ is a positive measure on $(0, \infty)$ and the integral (2.3) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{2.4}$$

Now, assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{2.5}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{2.6}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{2.7}$$

We define the *upper incomplete Gamma function* as [9]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [10]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{2.8}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (2.8) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{2.9}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (2.9) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{2.10}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{2.11}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (2.9) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{2.12}$$

If we define the generalized exponential integral [11] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [11, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-z)^k,$$

we get

$$\begin{aligned}(2.13)\qquad &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)\\ &=(n-1)!e^{t}E_{n}(t)\\ &=(n-1)!e^{t}\left[\frac{(-t)^{n-1}}{(n-1)!}E_{1}(t)+\frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^{k}\right]\\ &=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!t^{k}+(-1)^{n-1}t^{n-1}e^{t}E_{1}(t)\end{aligned}$$

for $n\geq 2$ and $t>0$.

If $T>0$, then we have

$$(2.14)\qquad \mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda^{-a}e^{-\lambda}(t+\lambda)^{-1}d\lambda=\Gamma(1-a)T^{-a}\exp(T)\Gamma(a,T)$$

for $a<1$.

In particular,

$$(2.15)\qquad \mathcal{D}\left(w_{e^{-\cdot}}\right)(T)=\int_{0}^{\infty}e^{-\lambda}(T+\lambda)^{-1}d\lambda=\exp(T)E_{1}(T)$$

and, for $n\geq 2$

$$\begin{aligned}(2.16)\qquad &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(T)\\ &=\int_{0}^{\infty}\lambda^{n-1}e^{-\lambda}(T+\lambda)^{-1}d\lambda\\ &=\sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!T^{k}+(-1)^{n-1}T^{n-1}\exp(T)E_{1}(T),\end{aligned}$$

where $T>0$.

For $n=2$, we also get

$$(2.17)\qquad \mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T)=\int_{0}^{\infty}\lambda e^{-\lambda}(T+\lambda)^{-1}d\lambda=1-T\exp(T)E_{1}(T)$$

for $T>0$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda):=\frac{1}{\lambda+a}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$(2.18)\qquad \mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t):=\int_{0}^{\infty}\frac{1}{(\lambda+t)(\lambda+a)}d\lambda=\frac{\ln t-\ln a}{t-a}$$

for all $a>0$ and $t>0$ with $t\neq a$.

From this, we get

$$\ln t=\ln a+(t-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a>0$.

If $T > 0$, then

$$\ln T = \ln a + (T-a)\,\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) = \ln a + (T-a)\int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}\,d\lambda. \tag{2.19}$$

Let $a > 0$. Assume that either $0 < T < a$ or $T > a$, then by (2.19) we get

$$(\ln T - \ln a)(T-a)^{-1} = \int_0^\infty \frac{1}{(\lambda+a)}(\lambda+T)^{-1}\,d\lambda. \tag{2.20}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)}d\lambda = \frac{\pi t}{2a\left(t^2+a^2\right)} - \frac{\ln t - \ln a}{t^2+a^2}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)}d\lambda = \frac{\pi t}{2\left(t^2+1\right)} - \frac{\ln t}{t^2+1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\frac{\pi}{2a}T\left(T^2+a^2\right)^{-1} - (\ln T - \ln a)\left(T^2+a^2\right)^{-1} = \int_0^\infty \frac{1}{\left(\lambda^2+a^2\right)}(\lambda+T)^{-1}\,d\lambda \tag{2.21}$$

and, in particular,

$$\frac{\pi}{2}T\left(T^2+1\right)^{-1} - \left(T^2+1\right)^{-1}\ln T = \int_0^\infty \frac{1}{\left(\lambda^2+1\right)}(\lambda+T)^{-1}\,d\lambda. \tag{2.22}$$

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (2.3) exists and is finite for all $t > 0$.

Lemma 1. *For all A, $B > 0$ we have the representation*

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) = \int_0^\infty \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1}(B-A)(\lambda + sB + (1-s)A)^{-1}\,ds\right) \times w(\lambda)\,d\mu(\lambda). \tag{2.23}$$

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[(\lambda+B)^{-1} - (\lambda+A)^{-1}\right]d\mu(\lambda). \tag{2.24}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.25}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.26}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.27}$$

Now, if we take in (2.27) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}(\lambda+B)^{-1}-(\lambda+A)^{-1} &\\ =\int_0^1((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B)&\\ \times((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt&\\ =\int_0^1(\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt&\end{aligned} \tag{2.28}$$

and by (2.24) we derive

$$\begin{aligned}&\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}\,dt\right)\\ &\times w(\lambda)\,d\mu(\lambda),\end{aligned}$$

which, by the change of variable $t=1-s$, gives (2.23). ∎

Remark 1. *By making use of the examples provided above, we can infer the following identities for* A, $B > 0$,

$$A^{r-1} - B^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) (\lambda + sB + (1-s)A)^{-1} ds \right) d\lambda, \tag{2.29}$$

and

$$\Gamma(1-a) \left[A^{-a} \exp(A) \Gamma(a, A) - B^{-a} \exp(B) \Gamma(a, B) \right] = \int_0^\infty \lambda^{-a} e^{-\lambda} \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) (\lambda + sB + (1-s)A)^{-1} ds \right) d\lambda, \tag{2.30}$$

for $a < 1$.

In particular,

$$\exp(A) E_1(A) - \exp(B) E_1(B) = \int_0^\infty e^{-\lambda} \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) (\lambda + sB + (1-s)A)^{-1} ds \right) d\lambda \tag{2.31}$$

and

$$B \exp(B) E_1(B) - B \exp(B) E_1(B) = \int_0^\infty \lambda e^{-\lambda} \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) (\lambda + sB + (1-s)A)^{-1} ds \right) d\lambda. \tag{2.32}$$

Let $a > 0$*. Assume that either* $0 < A$, $B < a$ *or* A, $B > a$*, then*

$$(\ln A - \ln a)(A - a)^{-1} - (\ln B - \ln a)(B - a)^{-1} = \int_0^\infty \frac{1}{(\lambda + a)} \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1} (B - A) (\lambda + sB + (1-s)A)^{-1} ds \right) d\lambda. \tag{2.33}$$

3. Main Results

Our first main result is as follows:

Theorem 3. *Let $A > 0$ and assume that there exist positive numbers $d > c > 0$ such that*

$$d \geq B - A \geq c > 0, \tag{3.1}$$

then

$$\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d + \|A\|)\right] \geq 0. \tag{3.2}$$

Proof. Since $B - A \geq c$, then by multiplying both sides with $(\lambda + sB + (1-s)A)^{-1}$, we get

$$\begin{aligned} &(\lambda + sB + (1-s)A)^{-1}(B-A)(\lambda + sB + (1-s)A)^{-1} \\ &\geq c(\lambda + sB + (1-s)A)^{-2} \end{aligned}$$

for all $s \in [0,1]$ and $\lambda > 0$.

By integration over $s \in [0,1]$ we get

$$\begin{aligned} &\int_0^1 (\lambda + sB + (1-s)A)^{-1}(B-A)(\lambda + sB + (1-s)A)^{-1}ds \\ &\geq c\int_0^1 (\lambda + sB + (1-s)A)^{-2}ds \end{aligned}$$

for all $\lambda > 0$.

If we multiply this inequality with $w(\lambda)$ and integrate, then we get

$$\begin{aligned} &\int_0^\infty w(\lambda) \\ &\times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1}(B-A)(\lambda + sB + (1-s)A)^{-1}ds\right)d\mu(\lambda) \\ &\geq c\int_0^\infty w(\lambda)\left(\int_0^1 (\lambda + sB + (1-s)A)^{-2}ds\right)d\mu(\lambda). \end{aligned} \tag{3.3}$$

Since $A \leq \|A\|$, then

$$\begin{aligned} \lambda + sB + (1-s)A &= \lambda + A + s(B-A) \leq \lambda + \|A\| + sd \\ &= \lambda + (1-s)\|A\| + s(d + \|A\|) \end{aligned}$$

for all $s \in [0,1]$ and $\lambda > 0$, which implies that

$$(\lambda + sB + (1-s)A)^{-1} \geq (\lambda + (1-s)\|A\| + s(d + \|A\|))^{-1}$$

and

$$(\lambda + sB + (1-s)A)^{-2} \geq (\lambda + (1-s)\|A\| + s(d + \|A\|))^{-2} \tag{3.4}$$

for all $s \in [0,1]$ and $\lambda > 0$.

From (3.4) we get by integration twice the inequality

$$\begin{aligned}
(3.5)\qquad & \int_0^\infty w(\lambda)\left(\int_0^1 (\lambda + sB + (1-s)A)^{-2}\, ds\right) d\mu(\lambda) \\
& \geq \int_0^\infty w(\lambda)\left(\int_0^1 (\lambda + (1-s)\|A\| + s(d+\|A\|))^{-2}\, ds\right) d\mu(\lambda) \ \ (\geq 0) \\
& = \frac{1}{d}\int_0^\infty w(\lambda)\left[\int_0^1 (\lambda + (1-s)\|A\| + s(d+\|A\|))^{-1} (d + \|A\| - \|A\|)\right. \\
& \left. \times (\lambda + (1-s)\|A\| + s(d+\|A\|))^{-1}\, ds\right] d\mu(\lambda) \\
& = \frac{1}{d}\left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d+\|A\|)\right] \geq 0 \text{ (by (2.23))}.
\end{aligned}$$

By utilizing (3.3) and (3.5) we obtain

$$\begin{aligned}
& \int_0^\infty w(\lambda) \\
& \times \left(\int_0^1 (\lambda + sB + (1-s)A)^{-1}(B-A)(\lambda + sB + (1-s)A)^{-1}\, ds\right) d\mu(\lambda) \\
& \geq \frac{c}{d}\left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(d+\|A\|)\right],
\end{aligned}$$

which by the representation (2.23) gives (3.2). ∎

Its is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned}
0 &\leq \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\
&\leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle
\end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequalities

$$(3.6)\qquad \left\|T^{-1}\right\|^{-1} \leq T \leq \|T\|.$$

Corollary 1. *Assume that $A > 0$ and $B - A > 0$. Then*

$$\begin{aligned}
(3.7)\qquad \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) &\geq \frac{1}{\|B-A\| \left\|(B-A)^{-1}\right\|} \\
&\times \left[\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(\|B-A\| + \|A\|)\right] \\
&\geq 0.
\end{aligned}$$

The proof follows by (3.2) since, by (3.6),

$$0 < \left\|(B-A)^{-1}\right\|^{-1} \le B - A \le \|B-A\| .$$

We can state the following result for operator monotone functions on $[0,\infty)$:

Proposition 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$. If A, $B>0$ satisfy condition (3.1), then*

$$\begin{aligned} &f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1}-B^{-1}\right) \\ &\ge \frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(d+\|A\|)}{d+\|A\|}\right) - \frac{cf(0)}{\|A\|(d+\|A\|)} \ge 0. \end{aligned} \tag{3.8}$$

If $f(0)=0$, then we have the simpler inequality

$$f(A)A^{-1} - f(B)B^{-1} \ge \frac{c}{d}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(d+\|A\|)}{d+\|A\|}\right) \ge 0. \tag{3.9}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is an operator monotone, then by (1.1)

$$\frac{f(t)-f(0)}{t} - b = \mathcal{D}(\ell,\mu)(t), \ t>0$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$.

By the inequality (3.2) we have

$$\begin{aligned} &[f(A)-f(0)]A^{-1} - [f(B)-f(0)]B^{-1} \\ &\ge \frac{c}{d}\left[\frac{f(\|A\|)-f(0)}{\|A\|} - \frac{f(d+\|A\|)-f(0)}{d+\|A\|}\right] \ge 0, \end{aligned}$$

which is equivalent to (3.8). ∎

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$, $A>0$ and $B-A>0$. Then*

$$\begin{aligned} &f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1}-B^{-1}\right) \\ &\ge \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}\right) \\ &- \frac{f(0)}{\|A\|\left\|(B-A)^{-1}\right\|(\|B-A\|+\|A\|)} \\ &\ge 0. \end{aligned} \tag{3.10}$$

If $f(0)=0$, then

$$\begin{aligned} &f(A)A^{-1} - f(B)B^{-1} \\ &\ge \frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}\left(\frac{f(\|A\|)}{\|A\|} - \frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}\right) \ge 0. \end{aligned} \tag{3.11}$$

In the case of operator convex functions, we have:

Proposition 2. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$. If A, $B > 0$ satisfy condition (3.1), then*

$$f(A)A^{-2} - f(B)B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right) \tag{3.12}$$
$$\geq \frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(d + \|A\|)}{(d + \|A\|)^2}\right] - \frac{cf(0)(d + 2\|A\|)}{\|A\|^2 (d + \|A\|)^2}$$
$$- \frac{cf'_+(0)}{\|A\|(d + \|A\|)}$$
$$\geq 0.$$

If $f(0) = 0$, then

$$f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) \tag{3.13}$$
$$\geq \frac{c}{d}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(d + \|A\|)}{(d + \|A\|)^2}\right] - \frac{cf'_+(0)}{\|A\|(d + \|A\|)} \geq 0.$$

Proof. If $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t) - f(0) - f'_+(0)t}{t^2} - c = \mathcal{D}(\ell, \mu)(t),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$.

By the inequality (3.2) we have

$$\left[f(A) - f(0) - f'_+(0)A\right]A^{-2} - \left[f(B) - f(0) - f'_+(0)B\right]B^{-2}$$
$$\geq \frac{c}{d}\left[\frac{f(\|A\|) - f(0) - f'_+(0)\|A\|}{\|A\|^2}\right.$$
$$\left. - \frac{f(d + \|A\|) - f(0) - f'_+(0)(d + \|A\|)}{(d + \|A\|)^2}\right]$$
$$\geq 0,$$

which is equivalent to (3.12). ∎

Corollary 3. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, $A > 0$ and $B - A > 0$. Then*

$$f(A)A^{-2} - f(B)B^{-2} - f(0)\left(A^{-2} - B^{-2}\right) - f'_+(0)\left(A^{-1} - B^{-1}\right) \tag{3.14}$$
$$\geq \frac{1}{\left\|(B - A)^{-1}\right\| \|B - A\|}\left[\frac{f(\|A\|)}{\|A\|^2} - \frac{f(\|B - A\| + \|A\|)}{(\|B - A\| + \|A\|)^2}\right]$$
$$- \frac{f(0)(\|B - A\| + 2\|A\|)}{\left\|(B - A)^{-1}\right\| \|A\|^2 (\|B - A\| + \|A\|)^2}$$
$$- \frac{f'_+(0)}{\left\|(B - A)^{-1}\right\| \|A\|(\|B - A\| + \|A\|)}$$
$$\geq 0.$$

If $f(0)=0$, *then*

$$f(A)A^{-2}-f(B)B^{-2}-f'_{+}(0)\left(A^{-1}-B^{-1}\right) \tag{3.15}$$
$$\geq \frac{1}{\left\|(B-A)^{-1}\right\|\left\|B-A\right\|}\left[\frac{f(\|A\|)}{\|A\|^{2}}-\frac{f(\|B-A\|+\|A\|)}{(\|B-A\|+\|A\|)^{2}}\right]$$
$$-\frac{f'_{+}(0)}{\left\|(B-A)^{-1}\right\|\|A\|(\|B-A\|+\|A\|)}$$
$$\geq 0.$$

4. Some Examples

In this section we give some example of the above general inequalities that hold for some particular operator monotone or operator convex functions of interest.

If we take $f(t)=t^{r}$, $r\in(0,1]$ in (3.9), then we get

$$A^{r-1}-B^{r-1}\geq \frac{c}{d}\left(\|A\|^{r-1}-(d+\|A\|)^{r-1}\right)>0, \tag{4.1}$$

provided A, $B>0$ satisfy condition (3.1).

If $A>0$ and $B-A>0$, then

$$A^{r-1}-B^{r-1} \tag{4.2}$$
$$\geq \frac{1}{\left\|(B-A)^{-1}\right\|\left\|B-A\right\|}\left[\|A\|^{r-1}-(\|B-A\|+\|A\|)^{r-1}\right]\geq 0.$$

If we take $f(t)=-\ln(t+1)$, which is operator convex on $[0,\infty)$, then by (3.13) we get

$$B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1} \tag{4.3}$$
$$\geq \frac{c}{d}\left[\frac{\ln(d+\|A\|+1)}{(d+\|A\|)^{2}}-\frac{\ln(\|A\|+1)}{\|A\|^{2}}\right]+\frac{c}{\|A\|(d+\|A\|)}\geq 0,$$

provided that A, $B\geq 0$ and satisfy condition (3.1).

If $A\geq 0$ and $B-A>0$, then

$$B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1} \tag{4.4}$$
$$\geq \frac{1}{\left\|(B-A)^{-1}\right\|\left\|B-A\right\|}\left[\frac{\ln(\|B-A\|+\|A\|+1)}{(\|B-A\|+\|A\|)^{2}}-\frac{\ln(\|A\|+1)}{\|A\|^{2}}\right]$$
$$+\frac{1}{\left\|(B-A)^{-1}\right\|\|A\|(\|B-A\|+\|A\|)}$$
$$\geq 0.$$

Assume that A, $B > 0$ and satisfy condition (3.1) for $d > c > 0$, then

$$\begin{aligned}(4.5)\qquad & A^{-a}\exp(A)\,\Gamma(a,A) - B^{-a}\exp(B)\,\Gamma(a,B)\\ &\geq \frac{c}{d}\left[\|A\|^{-a}\exp(\|A\|)\,\Gamma(a,\|A\|)\right.\\ &\left.-(d+\|A\|)^{-a}\exp(d+\|A\|)\,\Gamma(a,d+\|A\|)\right]\\ &\geq 0\end{aligned}$$

for $a < 1$.

In particular, we have

$$\begin{aligned}(4.6)\qquad & \exp(A)\,E_1(A) - \exp(B)\,E_1(B)\\ &\geq \frac{c}{d}\left[\exp(\|A\|)\,E_1(\|A\|) - \exp(d+\|A\|)\,E_1(d+\|A\|)\right] \geq 0\end{aligned}$$

and

$$\begin{aligned}(4.7)\qquad & B\exp(B)\,E_1(B) - A\exp(A)\,E_1(A)\\ &\geq \frac{c}{d}\left[(d+\|A\|)\exp(d+\|A\|)\,E_1(d+\|A\|) - \|A\|\exp(\|A\|)\,E_1(\|A\|)\right]\\ &\geq 0.\end{aligned}$$

Let $a > 0$. Assume that A, $B > a$ and there exists $d > c > 0$ such that (3.1) holds, then by (2.20) we get

$$\begin{aligned}(4.8)\qquad & (\ln A - \ln a)(A-a)^{-1} - (\ln B - \ln a)(B-a)^{-1}\\ &\geq \frac{c}{d}\left[(\ln\|A\| - \ln a)(\|A\|-a)^{-1}\right.\\ &\left.-(\ln(d+\|A\|) - \ln a)(d+\|A\|-a)^{-1}\right]\\ &\geq 0.\end{aligned}$$

The interested author may state other similar inequalities by using the examples of operator monotone functions from [2], [3] and the references therein.

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