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LIPSCHITZ TYPE INEQUALITIES FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we show that

$$\begin{aligned} &\|\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A)\| \\ &\leq \|B - A\| \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2) - \mathcal{M}(w,\mu)(m_1)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \mathcal{M}'(w, \mu)(m) \text{ if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

where $\mathcal{M}'(w, \mu)(t)$ is the derivative of $\mathcal{M}(w, \mu)$ as a function of t. If the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$, then

$$\|f(B) - f(A)\| \leq \|B - A\| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ f'(m) \text{ if } m_1 = m_2 = m. \end{cases}$$

In particular we have the power inequalities

$$\|B^r - A^r\| \leq \|B - A\| \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ r m^{r-1} \text{ if } m_1 = m_2 = m, \end{cases}$$

and the logarithmic inequalities

$$\|\ln B - \ln A\| \leq \|B - A\| \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{m} \text{ if } m_1 = m_2 = m. \end{cases}$$

Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

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1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right), \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following representation of operator monotone functions [15], see for instance [5, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.5}$$

where $b \geq 0$ *and a positive measure* μ *on* $[0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.6}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

Theorem 2. *A function* $f : [0, \infty) \to \mathbb{R}$ *is operator convex in* $[0, \infty)$ *with* $f'_+(0) \in \mathbb{R}$ *if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t+\lambda} d\mu(\lambda), \tag{1.7}$$

where $c \geq 0$ *and a positive measure* μ *on* $[0, \infty)$ *such that (1.6) holds.*

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.8}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.8) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.9}$$

Now, assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.11}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.12}$$

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \quad t > 0. \tag{1.13}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t) &= \int_0^\infty w(\lambda) t (t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(t + \lambda - \lambda)(t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[1 - \lambda (t + \lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{1.14}$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w, \mu)(t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(\ell w, \mu)(t), \tag{1.15}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then, after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t + \lambda} d\lambda = E_1(at) \exp(at), \quad t \geq 0$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a},$$

where the *exponential integral* is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w, \mu)(t) = tE_1(at) \exp(at), \quad t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a}, \mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t + \lambda} d\lambda = \frac{1}{a} - tE_1(at) \exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda) d\lambda - \mathcal{D}(\ell e_{-a}, \mu)(t), \quad t > 0$$

and the equality (1.15) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda = \infty$ and the equality (1.15) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1 - \lambda (T+\lambda)^{-1}\right] d\mu(\lambda). \tag{1.16}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

where $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$ and μ is the usual Lebesgue measure.

Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we show that

$$\begin{aligned}&\|\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)\| \\ &\leq \|B-A\| \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ \mathcal{M}'(w,\mu)(m) \text{ if } m_1 = m_2 = m, \end{cases}\end{aligned}$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)$ as a function of t. If the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$, then

$$\|f(B) - f(A)\| \leq \|B-A\| \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ f'(m) \text{ if } m_1 = m_2 = m. \end{cases}$$

In particular we have the power inequalities

$$\|B^r - A^r\| \leq \|B-A\| \begin{cases} \frac{m_2^r-m_1^r}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ rm^{r-1} \text{ if } m_1 = m_2 = m, \end{cases}$$

and the logarithmic inequalities

$$\|\ln B - \ln A\| \leq \|B-A\| \begin{cases} \frac{\ln m_2-\ln m_1}{m_2-m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{m} \text{ if } m_1 = m_2 = m. \end{cases}$$

Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

2. Main Results

We have the following equality that is of interest in itself:

Lemma 1. *For all A, $B > 0$ we have the representation*

$$\begin{aligned}&\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\ &= \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\,dt\right) \\ &\times \lambda w(\lambda)\,d\mu(\lambda).\end{aligned} \tag{2.1}$$

Proof. From (1.16) we have for all $A,\ B\geq 0$ that

$$\begin{aligned}(2.2)\quad &\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)\left[1-\lambda(B+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\ &=\int_0^\infty \lambda w(\lambda)\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]:\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}(2.6)\quad &(\lambda+A)^{-1}-(\lambda+B)^{-1}\\ &=\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A)\\ &\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}dt\\ &=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt.\end{aligned}$$

By employing (2.2) and (2.6), we derive (2.1). ∎

Corollary 1. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and it has the representation (1.5), then for all* $A,\ B>0$ *we have the* *equality*

$$f(B)-f(A)-b(B-A) = \int_0^\infty \left(\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \times \lambda^2 d\mu(\lambda). \tag{2.7}$$

Proof. From (1.5) we have for $T>0$ that

$$f(T)-f(0)-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(\ell,\mu)(A)=f(B)-f(A)-b(B-A)$$

and by (2.1) we get (2.7). ∎

Corollary 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and it has the representation (1.3), then for all A, $B>0$ we have the identity*

$$f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A) = \int_0^\infty \left(\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \times \lambda^2 d\mu(\lambda). \tag{2.8}$$

Proof. From (1.7) we have for $T>0$ that

$$(f(T)-f(0))T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(\ell,\mu)(A)=(f(B)-f(0))B^{-1}-(f(A)-f(0))A^{-1}-c(B-A)$$

and by (2.1) we get (2.8). ∎

Remark 1. *From the representation (2.1) we observe that if $B\geq A>0$, then $\mathcal{M}(w,\mu)(B)\geq\mathcal{M}(w,\mu)(A)$ which means that $\mathcal{M}(w,\mu)$ is operator monotone on $(0,\infty)$, see also* [6].

We have the following Lipschitz type inequality:

Theorem 3. *Assume that $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\| \leq \|B-A\| \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2, \\ \mathcal{M}'(w,\mu)(m) & \text{if } m_1=m_2=m, \end{cases} \tag{2.9}$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)$ as a function of t.

Proof. From the identity (2.6) we get by taking the norm that

$$\begin{aligned}
(2.10)\quad & \left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\right\| \\
& \leq \int_0^\infty \left\|\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right\| \\
& \times \lambda w(\lambda)\,d\mu(\lambda) \\
& \leq \int_0^\infty \left(\int_0^1 \left\|(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\right\|dt\right) \\
& \times \lambda w(\lambda)\,d\mu(\lambda) \\
& \leq \|B-A\|\int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left\|(\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right)d\mu(\lambda)
\end{aligned}$$

for all $A,\ B>0$.

Assume that $m_2>m_1$. Then

$$(1-t)A+tB+\lambda \geq (1-t)m_1+tm_2+\lambda,$$

which implies that

$$((1-t)A+tB+\lambda)^{-1} \leq ((1-t)m_1+tm_2+\lambda)^{-1},$$

and

$$\left\|((1-t)A+tB+\lambda)^{-1}\right\|^2 \leq ((1-t)m_1+tm_2+\lambda)^{-2} \tag{2.11}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore, by integrating (2.11) we derive

$$\begin{aligned}
& \int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left\|((1-t)A+tB+\lambda)^{-1}\right\|^2 dt\right)dw(\lambda) \\
& \leq \int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-2}dt\right)dw(\lambda) \\
& = \frac{1}{m_2-m_1}\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-1}\right. \\
& \left.\times(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}dt\right)dw(\lambda) \\
& = \frac{1}{m_2-m_1}\left[\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)\right] \text{ (by (2.1))}
\end{aligned}$$

and by (2.10) we deduce

$$\begin{aligned}
(2.12)\quad & \left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\right\| \\
& \leq \frac{\|B-A\|}{m_2-m_1}\left[\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_2)\right].
\end{aligned}$$

The case $m_2<m_1$ goes in a similar way and we also obtain (2.12).

Let $\epsilon > 0$. Then $B + \epsilon \geq m + \epsilon > m$. From (2.12) we get

$$\begin{aligned} &\left\| \mathcal{M}(w,\mu)(B+\epsilon) - \mathcal{M}(w,\mu)(A) \right\| \\ &\leq \frac{\|B+\epsilon - A\|}{m+\epsilon - m} \left[\mathcal{M}(w,\mu)(m+\epsilon) - \mathcal{M}(w,\mu)(m) \right] \end{aligned}$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of $\mathcal{M}(w,\mu)$ we deduce the second part of (2.9). ∎

Corollary 3. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and it has the representation (1.5). If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then,*

$$\begin{aligned} &\|f(B) - f(A) - b(B-A)\| \\ &\leq \|B-A\| \begin{cases} \left(\frac{f(m_2)-f(m_1)}{m_2-m_1} - b \right) & \text{if } m_1 \neq m_2, \\ \\ (f'(m) - b) & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{2.13}$$

Proof. From (1.5) we have for $T > 0$ that

$$f(T) - f(0) - bT = \mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda \geq 0$. Therefore

$$\mathcal{M}(\ell,\mu)(B) - \mathcal{M}(\ell,\mu)(A) = f(B) - f(A) - b(B-A),$$

$$\mathcal{M}(\ell,\mu)(m_2) - \mathcal{M}(\ell,\mu)(m_1) = f(m_2) - f(m_1) - b(m_2 - m_1)$$

and

$$\mathcal{M}'(\ell,\mu)(m) = f'(m) - b.$$

By (2.9) we obtain

$$\begin{aligned} &\|f(B) - f(A) - b(B-A)\| \\ &\leq \|B-A\| \begin{cases} \left(\frac{f(m_2)-f(m_1)}{m_2-m_1} - b \right) & \text{if } m_1 \neq m_2, \\ \\ (f'(m) - b) & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

which is equivalent to (2.13). ∎

By the properties of the norm, we have

$$\begin{aligned} &\|f(B) - f(A)\| - b\|B-A\| \\ &\leq \|f(B) - f(A) - b(B-A)\| \\ &\leq \|B-A\| \begin{cases} \left(\frac{f(m_2)-f(m_1)}{m_2-m_1} - b \right) & \text{if } m_1 \neq m_2, \\ \\ (f'(m) - b) & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

which implies the following inequalities in which the nonnegative parameter b is not involved

$$\|f(B)-f(A)\| \le \|B-A\| \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} \text{ if } m_1 \ne m_2, \\ \\ f'(m) \text{ if } m_1 = m_2 = m, \end{cases} \tag{2.14}$$

where the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$.

By employing this inequality for power and logarithmic functions we can state the following results of interest:

Proposition 1. *If $A \ge m_1 > 0$, $B \ge m_2 > 0$, then for $r \in (0,1]$ we have the power inequalities*

$$\|B^r - A^r\| \le \|B-A\| \begin{cases} \frac{m_2^r-m_1^r}{m_2-m_1} \text{ if } m_1 \ne m_2, \\ \\ rm^{r-1} \text{ if } m_1 = m_2 = m, \end{cases} \tag{2.15}$$

and the logarithmic inequalities

$$\|\ln B - \ln A\| \le \|B-A\| \begin{cases} \frac{\ln m_2-\ln m_1}{m_2-m_1} \text{ if } m_1 \ne m_2, \\ \\ \frac{1}{m} \text{ if } m_1 = m_2 = m. \end{cases} \tag{2.16}$$

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ that has the representation (1.7). If $A \ge m_1 > 0$, $B \ge m_2 > 0$, then*

$$\begin{aligned} &\left\|f(B)B^{-1} - f(A)A^{-1} - f(0)\left(B^{-1}-A^{-1}\right) - c(B-A)\right\| \\ &\le \|B-A\| \begin{cases} \left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1} - c\right) \text{ if } m_1 \ne m_2, \\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2} - c\right) \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{2.17}$$

If $f(0)=0$, then we have the simpler inequalities

$$\begin{aligned} &\left\|f(B)B^{-1} - f(A)A^{-1} - c(B-A)\right\| \\ &\le \|B-A\| \begin{cases} \left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1} - c\right) \text{ if } m_1 \ne m_2, \\ \\ \left(\frac{f'(m)m-f(m)}{m^2} - c\right) \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{2.18}$$

Proof. From (1.7) we have for $T>0$ that

$$(f(T)-f(0))T^{-1} - f'_+(0) - cT = \mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\begin{aligned} &\mathcal{M}(\ell,\mu)(B) - \mathcal{M}(\ell,\mu)(A) \\ &= f(B)B^{-1} - f(A)A^{-1} - f(0)\left(B^{-1}-A^{-1}\right) - c(B-A), \end{aligned}$$

$$\begin{aligned}&\mathcal{M}(\ell,\mu)(m_2)-\mathcal{M}(\ell,\mu)(m_1)\\&=f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)-c(m_2-m_1)\end{aligned}$$

and

$$\mathcal{M}(\ell,\mu)(m)=\frac{f'(m)m-f(m)+f(0)}{m^2}-c.$$

Then by (2.9) we get

$$\begin{aligned}&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|\\&\leq\|B-A\|\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c\text{ if }m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right)\text{ if }m_1=m_2=m,\end{cases}\end{aligned}$$

and the inequality (2.17) is obtained. ∎

By the properties of the norm, we have

$$\begin{aligned}&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\|-c\|B-A\|\\&\leq\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c\right)\text{ if }m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right)\text{ if }m_1=m_2=m,\end{cases}\end{aligned}$$

which implies the following inequalities in which the nonnegative parameter c is not involved

$$\begin{aligned}&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\|\\&\leq\|B-A\|\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}\text{ if }m_1\neq m_2,\\ \\ \frac{f'(m)m-f(m)+f(0)}{m^2}\text{ if }m_1=m_2=m.\end{cases}\end{aligned}\tag{2.19}$$

By applying this inequality to the operator convex function $f(t)=-\ln(t+1)$, then we can state the following result:

Proposition 2. *If $A\geq m_1>0$, $B\geq m_2>0$, then we have the logarithmic inequalities*

$$\begin{aligned}&\left\|B^{-1}\ln(B+1)-A^{-1}\ln(A+1)\right\|\\&\leq\|B-A\|\begin{cases}\frac{m_1^{-1}\ln(m_1+1)-m_2^{-1}\ln(m_2+1)}{m_2-m_1}\textit{ if }m_1\neq m_2,\\ \\ \frac{\ln(m+1)-m(m+1)^{-1}}{m^2}\textit{ if }m_1=m_2=m.\end{cases}\end{aligned}\tag{2.20}$$

3. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 3. *For all A, $B \geq m > 0$ we have the midpoint inequality*

$$\left\| \int_0^1 \mathcal{M}(w,\mu)\left((1-t)A+tB\right)dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\left\|B-A\right\|. \tag{3.1}$$

Proof. Since A, $B \geq m$, hence $\frac{A+B}{2} \geq m > 0$ and $(1-t)A+tB \geq m > 0$ for all $t \in [0,1]$ and by (2.9)

$$\begin{aligned} &\left\| \mathcal{M}(w,\mu)\left((1-t)A+tB\right) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \mathcal{M}'(w,\mu)(m)\left\|(1-t)A+tB-\frac{A+B}{2}\right\| \\ &= \mathcal{M}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\left\|B-A\right\| \end{aligned} \tag{3.2}$$

for all $t \in [0,1]$.

Taking the integral in (3.2), we get

$$\begin{aligned} &\left\| \int_0^1 \mathcal{M}(w,\mu)\left((1-t)A+tB\right)dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \int_0^1 \left\| \mathcal{M}(w,\mu)\left((1-t)A+tB\right) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| dt \\ &\leq \mathcal{M}'(w,\mu)(m)\left\|B-A\right\| \int_0^1 \left|t-\frac{1}{2}\right| dt = \frac{1}{4}\mathcal{M}'(w,\mu)(m)\left\|B-A\right\| \end{aligned}$$

and the inequality (3.1) is proved. ∎

We have the following trapezoid type inequalities:

Proposition 4. *For all A, $B \geq m > 0$ we have the trapezoid inequality*

$$\left\| \frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} - \int_0^1 \mathcal{M}(w,\mu)\left((1-t)A+tB\right)dt\right\| \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\left\|B-A\right\|. \tag{3.3}$$

Proof. Since A, $B \geq m$, hence $(1-s)A+s\frac{A+B}{2}$, $s\frac{A+B}{2}+(1-s)B \geq m > 0$ for all $s \in [0,1]$ and by Theorem 3 we get

$$\begin{aligned} &\left\| \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right\| \\ &\leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\left\|B-A\right\| s \end{aligned} \tag{3.4}$$

and

$$\left\| \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)\left(s\frac{A+B}{2} + (1-s)B\right)\right\| \le \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|s. \tag{3.5}$$

From (3.4) and (3.5) we derive by addition, division by 2 and triangle inequality that

$$\left\| \frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} - \frac{1}{2}\left[\mathcal{M}(w,\mu)\left((1-s)A + s\frac{A+B}{2}\right) + \mathcal{M}(w,\mu)\left(s\frac{A+B}{2} + (1-s)B\right)\right]\right\| \le \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|s$$

for all $s \in [0,1]$.

By taking the integral and using its properties, we derive

$$\left\| \frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} - \frac{1}{2}\left[\int_0^1 \mathcal{M}(w,\mu)\left((1-s)A + s\frac{A+B}{2}\right) + \mathcal{M}(w,\mu)\left(s\frac{A+B}{2} + (1-s)B\right)ds\right]\right\| \le \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|\int_0^1 sds = \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|. \tag{3.6}$$

Now, using the change of variable $t = 2s$ we have

$$\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)\left((1-t)A + t\frac{A+B}{2}\right)dt = \int_0^{1/2} \mathcal{M}(w,\mu)((1-s)A + sB)ds$$

and by the change of variable $t = 1 - v$ we have

$$\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)\left(t\frac{A+B}{2} + (1-t)A\right)dt = \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2} + vB\right)dv.$$

Moreover, if we make the change of variable $v = 2s - 1$ we also have

$$\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2} + vB\right)dv = \int_{1/2}^1 \mathcal{M}(w,\mu)((1-s)A + sB)ds.$$

Therefore

$$\frac{1}{2}\int_0^1 \left[\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds$$
$$=\int_0^{1/2}\mathcal{M}(w,\mu)((1-s)A+sB)\,dt+\int_{1/2}^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds$$
$$=\int_0^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds$$

and by (3.6) we deduce the desired result (3.3). ∎

The case of operator monotone functions is as follows:

Corollary 5. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and it has the representation (1.5). If A, $B\geq m>0$, then we have the midpoint inequality*

$$\left\|\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\right\|\leq\frac{1}{4}[f'(m)-b]\|B-A\|\leq\frac{1}{4}f'(m)\|B-A\| \tag{3.7}$$

and the trapezoid inequality

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|\leq\frac{1}{4}[f'(m)-b]\|B-A\|\leq\frac{1}{4}f'(m)\|B-A\|. \tag{3.8}$$

Proof. From (1.5) we have for $T>0$ that

$$f(T)-f(0)-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq0$.

Therefore

$$\int_0^1\mathcal{M}(\ell,\mu)((1-t)A+tB)\,dt=\int_0^1 f((1-t)A+tB)\,dt-f(0)-b\left(\frac{A+B}{2}\right),$$

$$\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right)=f\left(\frac{A+B}{2}\right)-f(0)-b\left(\frac{A+B}{2}\right)$$

and

$$\mathcal{M}'(\ell,\mu)(m)=f'(m)-b.$$

From (3.1) we derive (3.7).

Since

$$\mathcal{M}(\ell,\mu)(A)=f(A)-f(0)-bA,\text{ and }\mathcal{M}(\ell,\mu)(B)=f(B)-f(0)-bB,$$

then by (3.3) we derive (3.8). ∎

Remark 2. *If A, $B \geq m > 0$, then we have the midpoint inequality and the trapezoid inequality for power function with exponent $r \in (0,1]$*

$$\left\| \int_0^1 ((1-t)A + tB)^r \, dt - \left(\frac{A+B}{2} \right)^r \right\| \leq \frac{1}{4} r m^{r-1} \|B - A\| \tag{3.9}$$

and

$$\left\| \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r \, dt \right\| \leq \frac{1}{4} r m^{r-1} \|B - A\| . \tag{3.10}$$

The following inequalities for logarithm also hold

$$\left\| \int_0^1 \ln((1-t)A + tB) \, dt - \ln\left(\frac{A+B}{2} \right) \right\| \leq \frac{1}{4m} \|B - A\| \tag{3.11}$$

and

$$\left\| \frac{\ln A + \ln B}{2} - \int_0^1 \ln((1-t)A + tB) \, dt \right\| \leq \frac{1}{4m} \|B - A\| . \tag{3.12}$$

Corollary 6. *Assume that $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ that has the representation (1.7). If $A \geq m > 0$, $B \geq m > 0$, then*

$$\begin{aligned} &\left\| \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} dt - f\left(\frac{A+B}{2} \right) \left(\frac{A+B}{2} \right)^{-1} \right. \\ &\left. - f(0) \left(\int_0^1 ((1-t)A + tB)^{-1} dt - \left(\frac{A+B}{2} \right)^{-1} \right) \right\| \\ &\leq \frac{1}{4} \left(\frac{f'(m)m - f(m) + f(0)}{m^2} - c \right) \|B - A\| \\ &\leq \frac{f'(m)m - f(m) + f(0)}{4m^2} \|B - A\| \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\left\| \frac{f(A)A^{-1} + f(B)B^{-1}}{2} - \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} dt \right. \\ &\left. - f(0) \left(\frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1-t)A + tB)^{-1} dt \right) \right\| \\ &\leq \frac{1}{4} \left(\frac{f'(m)m - f(m) + f(0)}{m^2} - c \right) \|B - A\| \\ &\leq \frac{f'(m)m - f(m) + f(0)}{4m^2} \|B - A\| . \end{aligned} \tag{3.14}$$

Proof. From (1.7) we have for $T > 0$ that

$$\mathcal{M}(\ell, \mu)(T) = (f(T) - f(0))T^{-1} - f'_+(0) - cT,$$

for some positive measure μ. Therefore

$$\int_0^1 \mathcal{M}(\ell,\mu)((1-t)A+tB)\,dt$$
$$= \int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt - f(0)\int_0^1 ((1-t)A+tB)^{-1}dt$$
$$- f'_+(0) - c\left(\frac{A+B}{2}\right),$$

$$\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right) = f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1} - f(0)\left(\frac{A+B}{2}\right)^{-1}$$
$$- f'_+(0) - c\left(\frac{A+B}{2}\right),$$

and

$$\mathcal{M}(\ell,\mu)(m) = \frac{f'(m)m - f(m) + f(0)}{m^2} - c.$$

By utilizing (3.1) we get (3.13).

Since

$$\mathcal{M}(\ell,\mu)(A) = (f(A) - f(0))A^{-1} - f'_+(0) - cA$$

and

$$\mathcal{M}(\ell,\mu)(B) = (f(B) - f(0))B^{-1} - f'_+(0) - cB,$$

hence by (3.3) we get (3.14). ∎

Remark 3. *In the case when $f(0) = 0$ in Corollary 6, we have the simpler inequalities*

$$\left\|\int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt - f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}\right\| \tag{3.15}$$
$$\leq \frac{1}{4}\left(\frac{f'(m)m - f(m)}{m^2} - c\right)\|B-A\| \leq \frac{f'(m)m - f(m)}{4m^2}\|B-A\|$$

and

$$\left\|\frac{f(A)A^{-1} + f(B)B^{-1}}{2} - \int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt\right\| \tag{3.16}$$
$$\leq \frac{1}{4}\left(\frac{f'(m)m - f(m)}{m^2} - c\right)\|B-A\| \leq \frac{f'(m)m - f(m)}{4m^2}\|B-A\|.$$

If in these inequalities we take the operator convex function $f(t) = -\ln(t+1)$, *then we get*

$$\left\| \int_0^1 \ln((1-t)A + tB + 1)((1-t)A + tB)^{-1} dt - \ln\left(\frac{A+B}{2} + 1\right)\left(\frac{A+B}{2}\right)^{-1} \right\| \le \frac{\ln(m+1) - m(m+1)^{-1}}{m^2} \|B - A\| \tag{3.17}$$

and

$$\left\| \frac{A^{-1}\ln(A+1) + B^{-1}\ln(B+1)}{2} - \int_0^1 \ln((1-t)A + tB + 1)((1-t)A + tB)^{-1} dt \right\| \le \frac{\ln(m+1) - m(m+1)^{-1}}{m^2} \|B - A\|. \tag{3.18}$$

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