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GEOMETRIC PROPERTIES OF GEODESIC SPHERES IN A COMPLEX PROJECTIVE SPACE

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Abstract. We survey geometric properties of geodesic spheres in a complex projective space. These spheres can be regarded as the simplest examples in the class of all real hypersurfaces isometrically immersed into this projective space. Moreover, geodesic spheres with sufficiently big radii in this space are well-known examples of Berger spheres.

1. Two viewpoints on geodesic spheres

We first investigate geodesic spheres in a complex projective space from the viewpoint of submanifold theory. We denote by $\mathbb{C}P^n(c)$ a complex $n(\geqq 2)$-dimensional complex projective space of constant holomorphic sectional curvature $c(> 0)$, and by M a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure (J, g) of $\mathbb{C}P^n(c)$.

Standard examples of connected real hypersurfaces in $\mathbb{C}P^n(c)$ are *homogeneous* real hypersurfaces, i.e., real hypersurfaces given as orbits under the subgroups of the full isometry group $\mathrm{I}(\mathbb{C}P^n(c))$ of the ambient space $\mathbb{C}P^n(c)$, which is the unitary group $\mathrm{U}(n+1)$. R. Takagi ([28]) classifies homogeneous real hypersurfaces of $\mathbb{C}P^n(c)$ and he shows that a homogeneous real hypersurface is locally congruent to one of the six model spaces of types (A_1), (A_2), (B), (C), (D) and (E). Furthermore, we find that the number of distinct principal curvatures of homogeneous real hypersurfaces is 2, 3, 3, 5, 5, 5, respectively (see [29]). Every homogeneous real hypersurface of type (A_1) is congruent to a geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$. Note that $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ is congruent to a tube of radius $(\pi/\sqrt{c}\, - r)$ over a totally geodesic complex hypersurface $\mathbb{C}P^{n-1}(c)$.

It is known that there exist no totally umbilic real hypersurfaces in $\mathbb{C}P^n(c)$ and T. Cecil and P. Ryan show that every geodesic sphere is the only example of

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real hypersrfaces in $\mathbb{C}P^n(c)\,(n \geqq 3)$ with at most two distinct principal curvatures at each point (cf. [7]). This implies that a geodesic sphere is the simplest real hypersurface of $\mathbb{C}P^n(c)$.

We next study those geodesic spheres from the viewpoint of length spectrum. Klingenberg ([11]) proved the following: Let M be an even dimensional compact simply connected Riemannian manifold having the sectional curvature K with $0 < K \leqq L$ on M, where L is a constant. Then the length ℓ of every closed geodesic on M satisfies $\ell \geqq 2\pi/\sqrt{L}$.

In this context Berger gave examples of metrics on S^3 for which this inequality does *not* hold. This 3-sphere is called a *Berger sphere* with a Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. Motivated by this fact, Chavel constructed similar metrics on higher odd-dimensional spheres.

Weinstein ([32]) gave a description of Berger and Chavel examples as geodesic spheres $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$. Then it is known that our geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ admits a closed geodesic, say γ on $G(r)$ whose length is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$. In the following, we explain this fact in details. We see that the sectional curvature K of every geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ satisfies sharp inequalities $0 < (c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2)(= L)$ at its each point (cf. [19]). The shape operator A of $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ (with a unit normal vector field $\mathcal{N}$ on $G(r)$) in $\mathbb{C}P^n(c)$ is written as: $A\xi = \sqrt{c}\,\cot(\sqrt{c}\,r)\xi$ and $AX = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)X$ for each $X(\perp \xi)$, where ξ is the characteristic vector field on $G(r)$ defined by $\xi = -J\mathcal{N}$. The closed geodesic $\gamma = \gamma(s)$ is nothing but an integral curve γ_ξ of the characteristic vector field ξ. Since the curve γ_ξ lies on a holomorphic line $\mathbb{C}P^1(c)(= S^2(c))$ as a circle of curvature $k = \sqrt{c}\,\cot(\sqrt{c}\,r)$, it is closed with length

$$\ell = \frac{2\pi}{\sqrt{k^2 + c}} = \frac{2\pi}{\sqrt{c\cot^2(\sqrt{c}\,r) + c}} = \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r).$$

Moreover, we see easily that $\nabla_\xi \xi = \phi A\xi = 0$, so that the curve $\gamma_\xi = \gamma_\xi(s)$ is a geodesic on $G(r)$, where ∇ is the Riemannian connection on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ induced from the Riemannian connection $\widetilde{\nabla}$ in the ambient space $\mathbb{C}P^n(c)$. We set

$$\frac{2\pi}{\sqrt{c + \frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right)}} > \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r).$$

Then, solving the above inequality, we have $\tan^2(\sqrt{c}\,r/2) > 2$ and vice versa.

In this paper, geodesic spheres $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ are called *Berger spheres* (cf. [32]).

2. Comments on type A hypersurfaces

First of all we review fundamental terminologies on the theory of real hypersurfaces (see [21]).

Let M be a connected real hypersurface immersed into $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. The Riemannian connections $\widetilde{\nabla}$ of $\mathbb{C}P^n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the standard metric of the ambient space $\mathbb{C}P^n(c)$ and A is the shape operator of M in $\mathbb{C}P^n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\mathbb{C}P^n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\mathbb{C}P^n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the *principal distribution* associated to the principal curvature λ.

It is well-known that M has an almost contact metric structure induced from the Kähler structure (J, g) of the ambient space $\mathbb{C}P^n(c)$. That is, we have a quadruple (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi\xi = 0$ and $\eta(\phi X) = 0$. In the following, we call ϕ, ξ and η *the structure tensor*, *the characteristic vector* and *the contact form* on M, respectively.

It follows from (2.1), (2.2), $\widetilde{\nabla} J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Indeed, for the second equality, we get

$$\begin{aligned}\nabla_X \xi &= -\nabla_X(J\mathcal{N}) = -\widetilde{\nabla}_X(J\mathcal{N}) + g(AX, J\mathcal{N})\mathcal{N} \\ &= -J\widetilde{\nabla}_X \mathcal{N} + g(AX, J\mathcal{N})\mathcal{N} = JAX - g(JAX, \mathcal{N})\mathcal{N} = \phi AX.\end{aligned}$$

For the first, we see

$$\begin{aligned}(\nabla_X \phi)Y &= \nabla_X(\phi Y) - \phi\nabla_X Y = \nabla_X(JY - \eta(Y)\mathcal{N}) - \phi\nabla_X Y \\ &= \widetilde{\nabla}_X(JY - \eta(Y)\mathcal{N}) - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= J(\nabla_X Y + g(AX, Y)\mathcal{N}) - X(\eta(Y))\xi + \eta(Y)AX \\ &\qquad - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= \phi\nabla_X Y + g(\nabla_X Y, \xi)\mathcal{N} - g(AX, Y)\xi - g(\nabla_X Y, \xi)\mathcal{N} \\ &\quad - g(Y, \phi AX)\mathcal{N} + \eta(Y)AX - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\ &= \eta(Y)AX - g(AX, Y)\xi.\end{aligned}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z,W) = (c/4)\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ &+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W). \end{aligned} \tag{2.5}$$

The following is called the equation of Codazzi:

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi).$$

Let K be the sectional curvature of M. That is, K is defined by $K(X,Y) = g(R(X,Y)Y,X)$, where X and Y are orthonormal vectors on M. Then it follows from (2.5) that

$$K(X,Y) = (c/4)\big(1 + 3g(\phi X,Y)^2\big) + g(AX,X)g(AY,Y) - g(AX,Y)^2. \tag{2.6}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. Type (A) hypersurfaces are the simplest examples of Hopf hypersurfaces.

In the following, we study type (A) hypersurfaces. Here, type (A) hypersurfaces mean either homogeneous real hypersurfaces of type (A_1) or type (A_2). They are

(A_1): a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$,

(A_2): a tube $T^\ell(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic complex submanifold $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$, $n \geqq 3$.

The shape operator A of type (A_1) hypersurface is expressed in Section 1. The tangent bundle TM of type (A_2) hypersurface is decomposed as: $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\lambda_1} \oplus V_{\lambda_2}$, where $A\xi = \sqrt{c}\,\cot(\sqrt{c}\,r)\xi$, $\dim V_{\lambda_1} = 2n-2\ell-2 (\geqq 2)$, $\dim V_{\lambda_2} = 2\ell (\geqq 2)$, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and $\lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$.

Type (A) hypersurfaces have many common geometric properties. For examples, we know

Theorem 1 ([15, 20, 25])**.** *For a connected real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$, the following local statements hold*:

(1) *The length of the derivative of the shape operator A of each real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$ satisfies $\|\nabla A\|^2 \geqq (c^2/4)(n-1)$. In particular, $\|\nabla A\|^2 = (c^2/4)(n-1)$ holds on M if and only if M is of type* (A).

(2) *A real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$ is of type* (A) *if and only if the equality $\phi A = A\phi$ holds on M, where ϕ and A are the structure tensor and the shape operator of M, respectively.*

(3) *A real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$ is of type* (A) *if and only if the characteristic vector field ξ is Killing, that is, $L_\xi g = 0$ holds on M, where L is the Lie derivative on M.*

(4) *A real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$ is of type* (A) *if and only if every geodesic $\gamma = \gamma(s)$ on M is mapped to a curve having constant the first curvature $\kappa_1 = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$ along γ, where $\widetilde{\nabla}$ is the Riemannian connection on $\mathbb{C}P^n(c)$. In this case, κ_1 depends on the choice of γ.*

In some sense Theorem 1 shows that it is *not so* easy to distinguish between hypersurfaces of type (A_1) and type (A_2). Inspired by this result, we pay particular attention to geometric properties which discriminate hypersurfaces of type (A_1) from those of type (A_2). It follows from (2.6) that

Proposition 1 ([19]). *For a homogeneous real hypersurface M in $\mathbb{C}P^n(c)$ we have the following*:
(1) *M is of type* (A_1) *if and only if M has positive sectional curvature K at its each point. In this case, M has sharp inequalities*

$$0 < (c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2).$$

(2) *M is of type* (A_2) *if and only if the minimal sectional curvature of M is null. Here, M has sharp inequalities*

$$0 \leqq K \leqq c + (c/4)\max\{\cot^2(\sqrt{c}\,r/2),\ \tan^2(\sqrt{c}\,r/2)\}.$$

(3) *There exist no homogeneous real hypersurfaces all of whose sectional curvatures are nonpositive.*

We next observe the extrinsic shape of geodesics on type (A) hypersurfaces in $\mathbb{C}P^n(c)$, $n \geqq 2$. In this paper for a submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion, a curve γ on M^n is called an *extrinsic geodesic* if the curve γ, considered as a curve in the ambient space, is a geodesic in $\widetilde{M}^{n+p}$. Needless to say, such a curve γ is also a geodesic on the submanifold M^n.

We compute the number of extrinsic geodesics on type (A) hypersurfaces in a complex projective space.

Proposition 2 ([26]). *For a type* (A) *hypersurface we find the following*:
(1) *A geodesic sphere $G(r)\,(0 < r < \pi/(2\sqrt{c}\,))$ has no extrinsic geodesics*;
(2) *A geodesic sphere $G(r)\,(\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}\,)$ has just one congruent class of extrinsic geodesics up to isometries on this sphere*;
(3) *Every type* (A_2) *hypersurface M has uncountably infinite congruent classes of extrinsic geodesics up to isometries on this sphere.*

We here recall the congruence theorem for geodesics γ on a type (A) hypersurface M. For a geodesic $\gamma = \gamma(s)$ on M we denote by $\rho_\gamma(s) := g(\dot\gamma(s), \xi_{\gamma(s)})$ the *structure torsion* of the curve γ. Note that the structure torsion ρ_γ is constant along γ. Indeed, from (2.4), Theorem 1(2), the symmetry of A and the skew-symmetry of ϕ we find

$$\begin{aligned} \dot\gamma\rho_\gamma &= \dot\gamma\big(g(\dot\gamma(s), \xi_{\gamma(s)})\big) = g(\dot\gamma, \nabla_{\dot\gamma}\xi) = g(\dot\gamma, \phi A\dot\gamma) \\ &= g(\dot\gamma, A\phi\dot\gamma) = -g(\phi A\dot\gamma, \dot\gamma) = 0, \end{aligned} \tag{2.7}$$

so that ρ_γ is a constant with $-1 \leqq \rho_\gamma \leqq 1$. Next, for a geodesic $\gamma = \gamma(s)$ on a type (A) hypersurface M we denote by $\kappa_\gamma(s) := g(A\dot\gamma(s), \dot\gamma(s))$ the *normal curvature* of the curve γ. The normal curvature κ_γ is also constant. In fact, from Theorem 1(4) we see $\dot\gamma\kappa_\gamma = \dot\gamma(g(A\dot\gamma, \dot\gamma)) = g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma) = 0$. By using these two invariants ρ_γ and κ_γ, the congruence theorem for geodesics can be described as follows:

Lemma 1 ([26]). (1) *Two geodesics γ_1, γ_2 on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ are congruent by isometries of this sphere if and only if $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

(2) *Two geodesics* γ_1, γ_2 *on a type* (A_2) *hypersurface* M *are congruent by isometries of* M *if and only if* $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ *and* $\kappa_{\gamma_1} = \kappa_{\gamma_2}$.

Using the notion of the structure torsion $\rho_\gamma(s) := g(\dot\gamma(s), \xi_{\gamma(s)})$ for a geodesic $\gamma = \gamma(s)$ on a real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$, we give a geometric meaning of the equality $\phi A = A\phi$ in Theorem 1(2).

Proposition 3. *For a connected real hypersurface* M *isometrically immersed into* $\mathbb{C}P^n(c)$, $n \geqq 2$ *the following two conditions are mutually equivalent*:

(1) *The equality* $\phi A = A\phi$ *holds on* M, *where* ϕ *and* A *are the structure tensor and the shape operator of* M, *respectively*;
(2) *The structure torsion* ρ_γ *of every geodesic* $\gamma = \gamma(s)$ *on* M *is constant along the curve* γ.

Proof. (1) $\Longrightarrow$ (2): See (2.7).
(2) $\Longrightarrow$ (1): We see that

$$0 = \dot\gamma\rho_\gamma = \dot\gamma(g(\dot\gamma, \xi)) = g(\dot\gamma, \phi A\dot\gamma) = (1/2)g((\phi A - A\phi)\dot\gamma, \dot\gamma)$$

holds for each geodesic γ on M, which implies that $g((\phi A - A\phi)X, X) = 0$ for every vector X of M. This, together with a fact that $\phi A - A\phi$ is symmetric, yields that $g((\phi A - A\phi)X, Y) = 0$ for every $X, Y \in TM$. Hence we get Condition (1). □

At the end of this section we consider the exterior derivative $d\eta$ of the contact form η on a real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$, which is given by

$$d\eta(X, Y) := (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])\} \quad \text{for all } X, Y \in TM. \tag{2.8}$$

By (2.4) and (2.8) we have

$$d\eta(X, Y) = \frac{1}{2}g((\phi A + A\phi)X, Y).$$

This implies that $d\eta = 0$, i.e., the contact form η is closed if and only if the equality $\phi A + A\phi = 0$ holds on M. However there does *not* exist such a real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$ (see [21]).

So it is natural to recall the following:

Lemma 2 ([21])**.** *Suppose that a connected real hypersurface* M *in* $\mathbb{C}P^n(c)$, $n \geqq 2$ *satisfies the derivative of the contact form* η *on* M : $d\eta(X, Y) = kg(\phi X, Y)$ *for all* $X, Y \in TM$ *with nonzero constant* k. *Then* M *is locally congruent to either a geodesic sphere* $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ *or a homogeneous real hypersurface of type* (B), *that is, it is realized as a tube of radius* $r\,(0 < r < \pi/(2\sqrt{c}\,))$ *around a totally real totally geodesic* n*-dimensional real projective space* $\mathbb{R}P^n(c/4)$ *of constant sectional curvature* $c/4$.

3. Characterization of geodesic spheres

We first recall the notion of circles in Riemannian geometry (cf. [24]). A smooth real curve $\gamma = \gamma(s)$ parametrized by its arclength s in a Riemannian manifold M endowed with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies $\nabla_{\dot\gamma}\dot\gamma = kY$, $\nabla_{\dot\gamma}Y = -k\dot\gamma$ with some nonnegative constant k and a unit vector field Y along γ. We may regard a geodesic as a circle of null curvature. It is well-known that a smooth real curve $\gamma = \gamma(s)$ parametrized by its arclength is a circle if and only if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -k^2\dot\gamma$ with some nonnegative constant k and that the circle $\gamma = \gamma(s)$ can be defined for $-\infty < s < \infty$ when M is complete.

Next, standing on the ambient space $\mathbb{C}P^n(c)$, $n \geqq 2$, we observe the extrinsic shape of some geodesics on geodesic spheres $G(r)$, $(0 < r < \pi/\sqrt{c}\,)$. For every point p of $G(r)$ and each unit vector $v(\in T_pG(r))$ orthogonal to the characteristic vector ξ_p we take the geodesic $\gamma = \gamma(s)$ on $G(r)$ with initial condition that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then it follows from the constancy of the structure torsion $\rho_\gamma := g(\dot\gamma, \xi)$ for the curve γ that $\dot\gamma(s)$ is orthogonal to $\xi_{\gamma(s)}$ for each $s \in (-\infty, \infty)$, so that the shape operator A of $G(r)$ satisfies $A\dot\gamma(s) = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)\dot\gamma(s)$. This, together with (2.1) and (2.2), yields that $\widetilde\nabla_{\dot\gamma}\dot\gamma = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)\mathcal{N}$ and $\widetilde\nabla_{\dot\gamma}\mathcal{N} = -(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)\dot\gamma$. Hence we get the following:

Proposition 4 ([18])**.** *Every geodesic $\gamma = \gamma(s)$ on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ satisfying that the initial vector $\dot\gamma(0)$ is perpendicular to the characteristic vector $\xi_{\gamma(0)}$ is mapped to a circle of the same positive curvature which is independent of the choice of γ.*

Motivated by Proposition 4, we consider the following two conditions on the extrinsic shape of geodesics on real hypersurfaces M:

(ES1) At each point p of M, there exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2} \in T_pM$ orthogonal to ξ_p satisfying the following:

(i) geodesics $\gamma_i = \gamma_i(s)\,(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$,

(ii) geodesics $\gamma_{ij} = \gamma_{ij}(s)\,(1 \leqq i < j \leqq 2n-2)$ on M with $\gamma_{ij}(0) = p$ and $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$.

(ES2) At each point p of M, there exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2} \in T_pM$ orthogonal to ξ_p satisfying that geodesics $\gamma_i = \gamma_i(s)\,(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of the same positive curvature in $\mathbb{C}P^n(c)$.

Weakening Condition (ES2), we also consider the following condition:

(ES2') At each point p of M, there exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2} \in T_pM$ orthogonal to ξ_p satisfying that geodesics $\gamma_i = \gamma_i(s)\,(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of positive curvature in $\mathbb{C}P^n(c)$. The difference between Conditions (ES2) and (ES2') is that the curvatures of circles are the same or not. Note that Condition (ES2') is related to a characterization of *all* homogeneous real hypersurfaces of types (A_1), (A_2), (B), (C), (D) and (E) in $\mathbb{C}P^n(c)$.

Theorem 2 ([2, 6, 10])**.** *For a connected real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$, M is locally congruent to a homogeneous real hypersurface in this ambient space if and only if M satisfies Condition* (ES2′)*.*

Combining Conditions (ES1), (ES2), (ES2') with Propositions 1, 2 and Lemma 2, we establish the following:

Theorem 3 ([16, 18, 26])**.** *For a connected real hypersurface M in $\mathbb{C}P^n(c)$, $n \geqq 2$, the following four conditions are mutually equivalent*:

1. *M is locally congruent to a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$*;
2. *M satisfies Condition* (ES2) *and has at most one congruent class of extrinsic geodesics with respect to the full isometry group* $\mathrm{I}(M)$ *of M*;
3. *M satisfies Condition* (ES2$'$) *and has positive sectional curvature K on M*;
4. *M has nonnegative sectional curvature K on M and the exterior derivative $d\eta$ of the contact form η on M satisfies $d\eta(X,Y) = kg(\phi X, Y)$ for all $X, Y \in TM$ with some nonzero constant k.*

As immediate consequences of Theorem 3 we get the following which are characterizations of Berger spheres:

Proposition 5 ([12])**.** *Let M be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere, namely a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$, with respect to the full isometry group* $\mathrm{U}(n+1)$ *of the ambient space $\mathbb{C}P^n(c)$ if and only if at each point p of M there exists an orthonormal basis $v_1, v_2, \ldots, v_{2n-2}, \xi_p$ of T_pM such that all geodesics $\gamma_i = \gamma_i(s)\,(1 \leqq i \leqq 2n-2)$ with initial condition that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ are mapped to circles of the same positive curvature $k(p)$ with $k(p) < \sqrt{c}/(2\sqrt{2}\,)$ in the ambient space $\mathbb{C}P^n(c)$, where ξ_p is the characteristic vector of M at $p \in M$. In this case, the function $k = k(p)$ on M is automatically constant with $k = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$.*

Proposition 6 ([12])**.** *Let M be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then M is locally congruent to a Berger sphere if and only if M satisfies the following two conditions.*

1. *There exists a positive constant k with $k < \sqrt{c}/(2\sqrt{2}\,)$ such that the exterior derivative $d\eta$ of the contact form on M satisfies either $d\eta(X,Y) = kg(\phi X, Y)$ for all $X, Y \in TM$ or $d\eta(X,Y) = -kg(\phi X, Y)$ for all $X, Y \in TM$, where g and ϕ are the Riemannian metric and the structure tensor on M, respectively.*
2. *There exists a point x of M satisfying that every sectional curvature of M at x is positive.*

In the statement of Proposition 6, if we remove Condition (2), this proposition does not hold. The Berger sphere and a certain homogeneous real hypersurface of type (B) satisfy Proposition 6(1). The following lemma is worth mentioning.

Lemma 3. *Let $G(r)$ be a geodesic sphere of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ and L be the maximal sectional curvature of this sphere. Then the following three conditions are mutually equivalent*:

1. *The radius r satisfies an inequality $\tan^2(\sqrt{c}\,r/2) > 2$*;
2. *The sectional curvature K of $G(r)$ satisfies sharp inequalities $\delta L \leqq K \leqq L$ for some $\delta \in (0, 1/9)$ at its each point*;
3. *The length of every integral curve of the characteristic vector field ξ on $G(r)$ is shorter than $2\pi/\sqrt{L}$.*

Needless to say, for each geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ every integral curve of the characteristic vector field ξ on $G(r)$ is a geodesic.

We here recall the work of Li, Vrancken and and Wang ([14]), which gives a characterization of three dimensional Berger spheres as Lagrangian submanifolds of $\mathbb{C}P^3$. They show the following (for details, see Theorem 1.2 in [14]): Let ι be a Lagrangian isometric immersion from an open part of one of the homogeneous 3-manifolds into a complex space form $M_3(c)(= \mathbb{C}P^3(c)$, $\mathbb{C}H^3(c)$ or $\mathbb{C}^3)$. Then $c > 0$ and ι is minimal and M^3 is locally congruent to the Berger sphere.

4. Length spectrum of geodesic spheres

Using the structure torsion ρ_γ for a geodesic γ on $G(r)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$, we see that the curve γ is closed or not. Moreover, when the curve γ is closed, we find its length which is denoted by length(γ).

Theorem 4 ([4])**.** *Let γ be a geodesic on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$.*

1. *If the structure torsion of γ is ± 1, then γ is closed and its length is $(2\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r)$.*
2. *If γ has null structure torsion, then γ is also closed and its length is $(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$.*
3. *When the structure torsion of γ is of the form $\sin\theta$ $(0 < |\theta| < \pi/2)$, it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin(\sqrt{c}\,r/2)\sqrt{p^2\tan^2(\sqrt{c}\,r/2) + q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2(\sqrt{c}\,r/2)$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} (4\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\,r/2) + q^2\cos^2(\sqrt{c}\,r/2)} \\ \qquad\qquad \textit{if } pq \textit{ is even,} \\ (2\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\,r/2) + q^2\cos^2(\sqrt{c}\,r/2)} \\ \qquad\qquad \textit{if } pq \textit{ is odd.} \end{cases}$$

When we study the length spectrum of geodesics on a Riemannian manifold N, in order to avoid the influence of the action of the isometry group of N, we consider the moduli space of geodesics under the action of isometries. The moduli space Geod(N) of geodesics on N is the quotient space of the set of all geodesics on N under the congruency relation. We call a smooth curve σ *open* if it is not closed. For convenience we set length$(\sigma) = \infty$ for an open curve σ. We define the *length spectrum* $\mathcal{L}_N : \text{Geod}(N) \to \mathbb{R} \cup \{\infty\}$ of N by $\mathcal{L}_N([\gamma]) = \text{length}(\gamma)$, where $[\gamma]$ denotes the congruent class containing a geodesic γ. We also call the image $\text{Lspec}(N) = \mathcal{L}_N(\text{Geod}(N)) \cap \mathbb{R}$ the length spectrum of N. For example, the length spectrum of a standard unit sphere is $\text{Lspec}(S^m) = \{2\pi\}$.

As a direct consequence of Theorem 4, for a geodesic sphere $G(r)$ of radius r in $\mathbb{C}P^n(4)$, we can see that

$$\begin{aligned}\mathrm{Lspec}(G(r)) &= \{\pi \sin 2r\} \cup \{2\pi \sin r\}\\ &\cup \left\{ 2\pi\sqrt{p^2\sin^2 r + q^2\cos^2 r} \,\middle|\, \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p\tan^2 r \end{array} \right\}\\ &\cup \left\{ \pi\sqrt{p^2\sin^2 r + q^2\cos^2 r} \,\middle|\, \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p\tan^2 r \end{array} \right\}.\end{aligned}$$

Therefore we obtain the following.

Theorem 5 ([4]). *On a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, there exist countably infinite congruent classes of closed geodesics. Moreover the length spectrum* $\mathrm{Lspec}(G(r))$ *of $G(r)$ is a discrete unbounded subset in the real line $\mathbb{R}$.*

For a length spectrum $\lambda \in \mathrm{Lspec}(N)$ we call the cardinality $m_N(\lambda)$ of the set $\mathcal{L}_N^{-1}(\lambda)$ the *multiplicity* of λ. When the multiplicity of a length spectrum is 1 we say it is *simple*. Clearly for a geodesic sphere $G(r)$ in a complex projective space, we see by the expression of $\mathrm{Lspec}(G(r))$ that $m_{G(r)}(\lambda) < \infty$ at each λ. We here study the first, the second and the third length spectrum, that is, the minimum, the second minimum and the third minimum of the length spectrum.

Proposition 7 ([4]). *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

(1) *The first length spectrum of $G(r)$ is $(2\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r)$, which is the length of geodesics with structure torsion ± 1. It is simple.*
(2) *The second length spectrum of $G(r)$ is also simple. When $0 < r \leqq \pi/(2\sqrt{c}\,)$, it is $(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$, which is the length of geodesics with null structure torsion. When $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$, it is $2\pi/\sqrt{c}$, which is the length of geodesics with structure torsion $\pm\cot(\sqrt{c}\,r/2)$.*
(3) *The third length spectrum is also simple. When $\pi/(2\sqrt{c}\,) < r < \pi/\sqrt{c}$, it is $(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$, which is the length of geodesics with null structure torsion. When $\sqrt{2m-1} \leqq \cot(\sqrt{c}\,r/2) < \sqrt{2m+1}$ $(m = 1, 2, \ldots)$, in particular, $0 < r \leqq \pi/(2\sqrt{c}\,)$, it is $(2\pi/\sqrt{c}\,)\sqrt{4m(m+1)\sin^2(\sqrt{c}\,r/2)+1}$, which is the length of geodesics with structure torsion $\pm 1/\big(\sin(\sqrt{c}\,r/2)\sqrt{(2m+1)^2\tan^2(\sqrt{c}\,r/2)+1}\,\big)$.*

Every length spectrum except the first one we find the following lemma of Klingenberg's type holds, which is well-known for some geometers.

Corollary 1 ([4]). *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. Except geodesics with structure torsion ± 1, every geodesic γ on $G(r)$ satisfies* $\mathrm{length}(\gamma) > 4\pi/\sqrt{c\big(4+\cot^2(\sqrt{c}\,r/2)\big)}$.

Length spectrum is of course not necessarily simple. For example when M is a geodesic sphere of radius $\pi/4$ in $\mathbb{C}P^n(4)$, we have

$$\begin{aligned}\mathrm{Lspec}(M) = \{&\pi, \sqrt{2}\,\pi, \sqrt{5}\,\pi, \sqrt{10}\,\pi, \sqrt{13}\,\pi, \sqrt{17}\,\pi, 5\pi, \sqrt{26}\,\pi, \sqrt{29}\,\pi \\ &\sqrt{34}\,\pi, \sqrt{37}\,\pi, \sqrt{41}\,\pi, \sqrt{50}\,\pi, \sqrt{53}\,\pi, \sqrt{58}\,\pi, \sqrt{61}\,\pi, \sqrt{65}\,\pi, \sqrt{73}\,\pi, \dots\}\end{aligned}$$

and the multiplicity of $\sqrt{65}\,\pi$ is two; it is the common length of geodesics of structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$. Every spectrum which is smaller than $\sqrt{65}\,\pi$ is simple.

Theorem 6 ([4])**.** *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

(1) *If $\tan^2(\sqrt{c}\,r/2)$ is irrational, then every length spectrum of $G(r)$ is simple.*
(2) *If $\tan^2(\sqrt{c}\,r/2)$ is rational, then the multiplicity of each length spectrum of $G(r)$ is finite. But it is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_{G(r)}(\lambda) = \infty$. *In this case, the growth order of $m_{G(r)}$ is not so rapid. It satisfies $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{G(r)}(\lambda) = 0$ for arbitrary positive δ.*

This theorem guarantees that on a geodesic sphere of radius r with irrational $\tan^2(\sqrt{c}\,r/2)$ in a complex projective space, two closed geodesics are congruent if and only if they have the same length. On the other hand, if $\tan^2(\sqrt{c}\,r/2)$ is rational, this theorem shows that we can not classify congruent classes of geodesics only by their length.

Finally we make mention of the growth of the number of congruent classes of geodesics with respect to their length spectrum for a geodesic sphere in a complex projective space. For a Riemannian manifold N we denote by $n_N(\lambda)$ the cardinality of the set $\{[\gamma] \in \mathrm{Geod}(N) \mid \mathcal{L}_N([\gamma]) \leqq \lambda\}$.

Theorem 7 ([4])**.** *For a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ we have*

$$\lim_{\lambda\to\infty} \frac{n_{G(r)}(\lambda)}{\lambda^2} = \frac{3c\sqrt{c}\,r}{8\pi^4 \sin(\sqrt{c}\,r)}.$$

The following problem is still open.

Problem 1. Find a geometric meaning of the equality

$$\lim_{r\to 0} \lim_{\lambda\to\infty} \frac{n_{G(r)}\lambda}{\lambda^2} = \frac{3c}{8\pi^4}.$$

Remark. On the other hand, it is easy to find a geometric meaning of the equality

$$\lim_{r\to\pi/\sqrt{c}} \lim_{\lambda\to\infty} \frac{n_{G(r)}\lambda}{\lambda^2} = \infty$$

because of a fact that $\lim\limits_{r\to\pi/\sqrt{c}} G(r) = \mathbb{C}P^{n-1}(c)$.

5. Appendixes

Let M denote a connected real hypersurface isometrically immersed into $\mathbb{C}P^n(c)$, $n \geqq 2$. Then M has two almost contact metric structures (ϕ, ξ, η, g) associated to a local unit normal vector $\mathcal{N}$ on M and $(\phi, -\xi, -\eta, g)$ associated to $-\mathcal{N}$ (see Section 2). So it is natural that we call a real hypersurface M *Sasakian* if M satisfies either $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$ for all $X, Y \in TM$ or $(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$ for all $X, Y \in TM$ (see Theorem 6.3 in [5]). A Sasakian manifold is called a *Sasakian space form* of constant ϕ-sectional curvature c if the sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ satisfies $K(u, \phi u) = c$ for each unit vector u orthogonal to ξ.

In pp. 114-115 of [5], we find a standard construction of Sasakian space forms. But, in general all Sasakian space forms $M(c)\,(c \neq 1)$ can be realized as totally η-umbilic real hypersurfaces in nonflat complex space forms $(= \mathbb{C}P^n(c), \mathbb{C}H^n(c))$ (see [4]). The following is the unique existence theorem of Sasakian space forms.

Lemma 4 ([31])**.** *For any two simply connected complete Sasakian manifolds of constant ϕ-sectional curvature c, there exists an isomorphism between them which preserves their almost contact metric structures.*

We obtain easily the following:

Proposition 8 ([1])**.** *For a connected real hypersurface M isometrically immersed into $\mathbb{C}P^n(c)$, $n \geqq 2$, the following are mutually equivalent*:
(1) *M is a Sasakian manifold*;
(2) *M is a Sasakian space form of constant ϕ-sectional curvature d, where d is automatically expressed as $d = c + 1$*;
(3) *The shape operator A of M is written either $AX = -X + (c/4)\eta(X)\xi$ for all $X, Y \in TM$ or $AX = X - (c/4)\eta(X)\xi$ for all $X, Y \in TM$*;
(4) *M is locally congruent a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ in $\mathbb{C}P^n(c)$.*

By virtue of the works ([3, 4]) and Lemma 4 we find

Theorem 8. *Let $M(c)$ be a complete simply connected Sasakian space form of constant ϕ-sectional curvature $c(> 1)$. Suppose that c is irrational. Then for any closed geodesics $\gamma_1 = \gamma_1(s)$ and $\gamma_2 = \gamma_2(s)$ on $M(c)$ they are congruent by some isometry on this space if and only if they have the same common length.*

We next recall a fact that an isometric immersion f of a Kähler manifold with Kähler structure J into a Euclidean sphere has parallel second fundamental form σ if and only if σ is J-invariant , i.e., $\sigma(JX, JY) = \sigma(X, Y)$ holds for each vector X, Y on M (cf. [9, 13, 22]). Motivated by this fact, for a real hypersurface M isometrically immersed into $\mathbb{C}P^n(c)$, $n \geqq 2$ we consider the following condition concerning ϕ-invariance of the shape operator A of M.

The shape operator A of M is called *ϕ-invariant* if A satisfies

$$g(A\phi X, \phi Y) = g(AX, Y), \text{ i.e., } \sigma(\phi X, \phi Y) = \sigma(X, Y)$$

for all vectors X and Y on M. We classify real hypersurfaces M having ϕ-invariant shape operator A in $\mathbb{C}P^n(c)$, $n \geqq 2$ and give a geometric meaning such hypersurfces M by observing the extrinsic shape of some geodesics and integral curves of ξ on M.

Theorem 9 ([17]). *Let M be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion. Then the following conditions* (1), (2) *and* (3) *are mutually equivalent*:

(1) *M is locally congruent to a homogeneous real hypersurface of type* (A) *of radius $\pi/(2\sqrt{c}\,)$. That is, M is locally congruent to either a geodesic sphere $G(\pi/(2\sqrt{c}\,))$ or a tube $T^\ell(\pi/(2\sqrt{c}\,))$ around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$ in the ambient space $\mathbb{C}P^n(c)$.*
(2) *The shape operator A of M is ϕ-invariant.*
(3) *M satisfies both of Condition* (ES2) *in Section 3 and a condition that there exists at least one integral curve of the characteristic vector field ξ of M which is mapped to a geodesic in $\mathbb{C}P^n(c)$.*

At the end of this paper, as an application of the theory of real hypersurfaces in $\mathbb{C}P^n(c)$ we shall discuss some homogeneous submanifolds in Euclidean sphere. We shall explain an idea to construct a certain class of (Riemannian) submanifolds in a sphere. We denote by (M, ι_M) a real hypersurface M of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion $\iota_M : M \to \mathbb{C}P^n(c)$. In the following, we regard real hypersurfaces M in $\mathbb{C}P^n(c)$ as submanifolds of the sphere $S^{n(n+2)-1}((n+1)c/(2n))$ through $f_1 \circ \iota_M$, where f_1 is the first standard (parallel equivariant) minimal embedding of $\mathbb{C}P^n(c)$ into $S^{n(n+2)-1}((n+1)c/(2n))$.

We here give the definition and fundamental geometric properties of f_1. The embedding f_1 is defined by eigenfunctions of the first eigenvalue of the Laplacian Δ on $\mathbb{C}P^n(c)$ (for details, see [8, 30]). In submanifold theory, this minimal embedding f_1 is well-known as the only example of a minimal full immersion with parallel second fundamental form of a complex projective space endowed with Fubini-Study metric into a Euclidean sphere. The inner product of the first normal space of f_1 is given by

$$\begin{aligned}\langle\sigma_1(X,Y),\sigma_1(Z,W)\rangle &= -(c/(2n))\langle X,Y\rangle\langle Z,W\rangle + (c/4)(\langle X,W\rangle\langle Y,Z\rangle \\ &\quad + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle).\end{aligned} \tag{5.1}$$

Here, σ_1 is the second fundamental form of the embedding f_1. Equation (5.1) yields the following properties of f_1:

(i) f_1 is minimal;
(ii) $\sigma_1(JX, JY) = \sigma_1(X, Y)$ for all X, $Y \in T\mathbb{C}P^n(c)$ (namely, σ_1 is J-invariant);
(iii) $\|\sigma_1(X,X)\| = \sqrt{(n-1)c/(2n)}$ for each unit vector X on $\mathbb{C}P^n(c)$ (that is, f_1 is $\sqrt{(n-1)c/(2n)}$-isotropic (cf. [27])).

Thus we obtain a family of submanifolds $\{(M^{2n-1}, f_1 \circ \iota_M)\}$ in the sphere. This class contains some homogeneous submanifolds of $S^{n(n+2)-1}((n+1)c/(2n))$, that is, they are expressed as orbits of some subgroups of the isometry group $\mathrm{SO}(n(n+2))$ of the ambient sphere. Indeed, if we take a homogeneous real hypersurface M of

$\mathbb{C}P^n(c)$, the immersion $f_1 \circ \iota_M$ gives a homogeneous submanifold M of the sphere, so that M has constant mean curvature, i.e., the length of the mean curvature vector of M is constant. However, the second fundamental form of the immersion $f_1 \circ \iota_M$ is *not* parallel for each real hypersurface M of $\mathbb{C}P^n(c)$ (cf. [9]). So, it is natural to pose the following problem:

Problem 2. Classify submanifold $(M, f_1 \circ \iota_M)$ of $S^{n(n+2)-1}((n+1)c/(2n))$ satisfying that the immersion $f_1 \circ \iota_M$ has parallel mean curvature vector with repect to the normal connection.

The following is an answer to Problem 2:

Lemma 5 ([23])**.** *Let M be a connected real hypersurface of $\mathbb{C}P^n(c)$, $n \geqq 2$ through an isometric immersion ι_M and $f_1 : \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$ be the first standard minimal embedding. Then M is locally congruent to the geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ in $\mathbb{C}P^n(c)$ if and only if the immersion $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$ has parallel mean curvature vector with respect to the normal connection. Moreover, this submanifold $(M, f_1 \circ \iota_M)$ is locally homogeneous in this ambient sphere.*

In order to establish our Theorem 10 we need the following two lemmas:

Lemma 6 ([23])**.** *The geodesic sphere $G(r)$ with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ is a Sasakian manifold with respect to the almost contact metric structure induced from the Kähler structure (J, g) on $\mathbb{C}P^n(c)$, $n \geqq 2$ if and only if $c = 8n+4$. Furthermore, this geodesic sphere is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.*

Lemma 7 ([23])**.** *We consider the following isometric embedding $\tilde{f}$ of the geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ into an $N(\geqq n(n+2)-1)$-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c}(\leqq (n+1)c/(2n))$.*

(1) *When $N > n(n+2)-1$, $\tilde{f}$ is given by*

$$\tilde{f} = \iota \circ (f_1 \circ \iota_{G(r)}) : G(r) \xrightarrow{f_1 \circ \iota_{G(r)}} S^{n(n+2)-1}((n+1)c/(2n)) \xrightarrow{\iota} S^N(\tilde{c}),$$

where ι is a totally umbilic embedding, so that $(n+1)c/(2n) \geqq \tilde{c}$.

(2) *When $N = n(n+2)-1$, $\tilde{f}$ is nothing but $f_1 \circ \iota_{G(r)}$, so that $(n+1)c/(2n) = \tilde{c}$.*

Then our geodesic sphere is homogeneous in $S^N(\tilde{c})$ and has nonzero parallel mean curvature vector with respect to the normal connection in this sphere.

Therefore, in view of Lemmas 5, 6 and 7 we establish the following:

Theorem 10 ([23])**.** (1) *For each of $c > 0$, $n(\geqq 2)$, $N > n(n+2)-1$ and $\tilde{c} \leqq (n+1)c/(2n)$, there exists a $(2n-1)$-dimensional submanifold M in an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c}$, which has the following properties*:

(A) *M is a homogeneous submanifold which has nonzero parallel mean curvature vector with respect to the normal connection in $S^N(\tilde{c})$;*

(B) *M is a Berger sphere. Here, this means that M is an odd dimensional compact simply connected Riemannian manifold having the property that all sectional curvatures of M lie in the interval $[\delta K, K]$ with some constant $\delta \in (0, 1/9)$ but it has a closed geodesic whose length is shorter than $2\pi/\sqrt{K}$, where K is a positive constant.*

(C) *When $c = 8n + 4$, M is a Sasakian space form of constant ϕ-sectional curvature $8n+5$.*

(2) *For each of $c > 0$, $n(\geqq 2)$, when $N = n(n+2) - 1$, there exists also a $(2n-1)$-dimensional submanifold M in an N-dimensional sphere $S^N(\tilde{c})$ of constant sectional curvature $\tilde{c} = (n+1)c/(2n)$, which has the above properties* (A), (B), (C).

We finally pose the following open problem:

Problem 3. Let f_1 be a mimimal parallel full immersion of a complex n-dimensional compact Hermitian symmetric space $\widetilde{M}_n$ into a Euclidean sphere $S^{2n+p}(\tilde{c})$. If there exists a real hypersurface $(M^{2n-1}, \iota_{M^{2n-1}})$ of $\widetilde{M}_n$ satisfying that the corresponding submanifold $(M^{2n-1}, f_1 \circ \iota_{M^{2n-1}})$ has parallel mean curvature vector with respect to the normal connection in the ambient sphere $S^{2n+p}(\tilde{c})$, is our Hermitian symmetric space $\widetilde{M}_n$ holomorphically isometric to a complex projective space $\mathbb{C}P^n(c)$, $n \geqq 2$ of constant holomorphic sectional curvature $c = 2n\tilde{c}/(n+1)$ and $p = n^2 - 1$?

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