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INTEGRALS WITH POWERS OF THE ARCTAN FUNCTION VIA EULER SUMS

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Abstract. We undertake an investigation into families of integrals containing powers of the inverse tangent and log functions. It will be shown that Euler sums play an important part in the evaluation of these integrals. In a special case of the parameters, our analysis generalizes an arctan integral studied by Ramanujan [11]. In another special case, we prove that the corresponding log tangent integral can be represented as a linear combination of the product of zeta functions and the Dirichlet beta function. We also deduce formulas for log-sine and log-cosine integrals.

1. Introduction

In this paper, we study the representations of integrals of the type

$$I(a,q)=\int_0^1 \frac{x^a \ln(x)}{1+x^2}\arctan^q(x)\,dx=\int_0^{\frac{\pi}{4}} \theta^q (\tan\theta)^a \ln(\tan\theta)\,d\theta, \tag{1}$$

$$J(q)=\int_0^\infty \frac{\ln(x)}{1+x^2}\arctan^q(x)\,dx=\int_0^{\frac{\pi}{2}} \theta^q \ln(\tan\theta)\,d\theta, \tag{2}$$

in terms of the *zeta function* $\zeta(z):=\sum_{n=1}^{\infty}\frac{1}{n^z}$, the *Dirichlet eta function* $\eta(z):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}$, the *Dirichlet beta function* $\beta(z):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^z}$, for $\Re(z)>0$. The *Dirichlet lambda function* defined by:

$$\lambda(p):=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^p}=\frac{1}{2}\{\zeta(p)+\eta(p)\}=(1-2^{-p})\,\zeta(p)$$

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is also relevant here. We will also need the digamma and the polygamma functions:

$$\psi(z) = \frac{d}{dz}\ln(\Gamma(z));\ \psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}k!\sum_{i=0}^{\infty}\frac{1}{(z+i)^{k+1}}.$$

The multiplication formula for the polygamma function can be written as:

$$\psi^{(k)}(pz) = \delta_{p,0}\ \ln p + \frac{1}{p^{k+1}}\sum_{j=0}^{p-1}\psi^{(k)}\left(z+\frac{j}{p}\right) \tag{3}$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. The Polylogarithmic function

$$\mathrm{Li}_t(z) = \sum_{n\geq 1}\frac{z^n}{n^t},$$

for $t \in \mathbb{C}$ when $|z| < 1; \Re(t) > 1$ when $|z| = 1$, where $\mathbb{C}$ is the set of complex numbers. As usual, the harmonic number and the generalized harmonic number of order r are denoted by:

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}; \qquad H_n^{(r)} := 1 + \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{n^r}.$$

These numbers can be extended to non-integral index and have this connection with the polygamma functions:

$$H_z^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(z+1),\ z \neq -1, -2, -3, \ldots$$

We may also recall here the function introduced by Thomas Clausen(1832)

$$\mathrm{Cl}_2(\varphi) = \sum_{n=1}^{\infty}\frac{\sin(n\varphi)}{n^2},$$

whose generalization is associated with the Dirichlet beta function:

$$\beta(2q) = \mathrm{Cl}_{2q}\left(\frac{\pi}{2}\right) = 2^{2q-1}\left\{\mathrm{Cl}_{2q}\left(\frac{\pi}{4}\right) - \mathrm{Cl}_{2q}\left(\frac{3\pi}{4}\right)\right\}.$$

In the two papers, [4] and [5], Elaissaoui and Guennoun showed that for any square-integrable function f on $(0, \pi/2)$, the integral

$$L(f) := \int_0^{\frac{\pi}{2}} f(x)\log(\tan x)\ dx$$

can be approximated by a finite sum involving the Riemann zeta function at odd positive integers. The authors noted that besides number theory, integrals involving log-tangent functions have important applications in various fields of mathematics. In physics, logarithmic-trigonometric integrals also have some applications in the evaluation of classical, semi-classical and quantum entropies of position and momentum.

We shall investigate integrals of the type (1) and (2) to be represented by Euler sums and hence in terms of special functions such as the Riemann zeta function.

Through a search of the current literature, we found some examples for the representation of the log tangent integral in terms of Euler sums. (see [1], [4], [5],[10]). Two papers, [16] and [19], also examined some integrals in terms of Euler sums. Three excellent books [6], [21] and [22] are important sources of material on log-tangent integrals and Euler sums. Other useful references for the representation of Euler sums in terms of special functions include [2], [3], [13], [14], [15], [16], [17]. We shall give examples highlighting specific cases of the integrals, some of which are not amenable to a computer mathematical package.

2. Analysis of integrals: Part 1

2.1. General case.

Lemma 2.1. *For* $t \in \left[0, \frac{\pi}{4}\right]$

$$\int_0^t \ln(\tan\theta)\, d\theta = t\ln(\tan t) - \frac{1}{4}\sum_{j\geq 0} \frac{(2\sin(2t))^{2j+1}}{(2j+1)^2 \binom{2j}{j}}. \tag{4}$$

Proof. Integrating by parts gives

$$\int_0^t \ln(\tan\theta)\, d\theta = t\ln(\tan t) - \int_0^t \frac{2\theta}{\sin(2\theta)}\, d\theta,$$

and re-scaling the limits changes the integral on the right hand side into

$$\int_0^{\sin(2t)} \frac{\arcsin\varphi}{2\varphi\sqrt{1-\varphi^2}}\, d\varphi.$$

The Taylor series expansion

$$\frac{2t\arcsin t}{\sqrt{1-t^2}} = \sum_{j\geq 1} \frac{(2t)^{2j}}{j\binom{2j}{j}}$$

therefore

$$\begin{aligned} \int_0^{\sin(2t)} \frac{\arcsin\varphi}{2\varphi\sqrt{1-\varphi^2}} d\varphi &= \int_0^{\sin(2t)} \sum_{j\geq 1} \frac{(2\varphi)^{2j-2}}{j\binom{2j}{j}} d\varphi = \sum_{j\geq 0} \frac{(2\sin(2t))^{2j+1}}{2(j+1)(2j+1)\binom{2j+2}{j+1}} \\ &= \sum_{j\geq 0} \frac{(2\sin(2t))^{2j+1}}{4(2j+1)^2\binom{2j}{j}}, \end{aligned}$$

replacing the variable we have (4). ∎

Note that from the definition of the Clausen function we also have

$$\int_0^t \ln(\tan\theta)\, d\theta = -\frac{1}{2}\mathrm{Cl}_2(2t) - \frac{1}{2}\mathrm{Cl}_2(\pi - 2t).$$

Lemma 2.2. *For $q \in \mathbb{N}$*

$$\int_0^{\frac{\pi}{4}} \theta^q \ln(\tan\theta)\, d\theta = -\frac{2}{q+1}\int_0^{\frac{\pi}{4}} \frac{\theta^{q+1}}{\sin(2\theta)} d\theta = 2\sum_{n\geq 0} \frac{(-1)^{n+1} 2^{2n} \left(2^{2n-1}-1\right)}{(2n)!\,(2n+q+1)} B_{2n} \left(\frac{\pi}{4}\right)^{2n+q+1} \tag{5}$$

where B_{2n} are the Bernoulli numbers. We also have the result, for $t \in \left[0, \frac{\pi}{2}\right]$

$$\int_0^t \ln(\tan\theta)\, d\theta = -\sum_{j\geq 0} \frac{\sin(2(2j+1)t)}{(2j+1)^2}. \tag{6}$$

Proof. For $\theta \in [-\pi, \pi]$ the Laurent series for the $\csc\theta$ function is

$$\csc\theta = \sum_{n\geq 0} \frac{2(-1)^{n+1}\left(2^{2n-1}-1\right)}{(2n)!} B_{2n}\, \theta^{2n-1}$$

therefore

$$\int_0^{\frac{\pi}{4}} \frac{\theta^{q+1}}{\sin(2\theta)} d\theta = \int_0^{\frac{\pi}{4}} \sum_{n\geq 0} \frac{2^{2n}(-1)^{n+1}\left(2^{2n-1}-1\right) B_{2n}}{(2n)!} \theta^{2n+q}\, d\theta = \sum_{n\geq 0} \frac{(-1)^{n+1} 2^{2n}\left(2^{2n-1}-1\right)}{(2n)!\,(2n+q+1)} B_{2n} \left(\frac{\pi}{4}\right)^{2n+q+1}$$

and (5) follows. For $x \in \left[0, \frac{\pi}{4}\right]$ the Fourier series for

$$\ln(\tan x) = -2\sum_{j\geq 0} \frac{\cos((4j+2)x)}{2j+1}$$

therefore

$$\int_0^t \ln(\tan x)\, dx = -2\int_0^t \sum_{j\geq 0} \frac{\cos((4j+2)x)}{2j+1} dx = -\sum_{j\geq 0} \frac{\sin(2(2j+1)t)}{(2j+1)^2}.$$

∎

The master theorem is enunciated below before we examine special cases.

Theorem 2.3. *Let $I(a,q)$ be as defined in (1); $a \in [-1,\infty), q \in \mathbb{N}$. Then,*

$$I(a,q) = \int_0^1 \frac{x^a \ln(x)}{1+x^2} \arctan^q(x)\, dx$$

$$= \frac{q!}{16} \sum_{n_0 \geq 1} \frac{(-1)^{n_0+1} \left(H^{(2)}_{\frac{2n_0+a+q-5}{4}} - H^{(2)}_{\frac{2n_0+a+q-3}{4}} \right)}{(2n_0+q-2)} \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{1}{2n_k+q-2-k}$$

where $H^{(2)}_{\frac{2n_0+a+q-5}{4}}$ are harmonic numbers of order 2.

Proof. We know from the work of [12] and [20] that for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\arctan^q(x) = q! \sum_{n_0 \geq 1} \frac{(-1)^{n_0+1} x^{2n_0+q-2}}{(2n_0+q-2)} \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{1}{2n_k+q-2-k} \tag{7}$$

therefore

$$I(a,q) = q! \int_0^1 \sum_{n_0 \geq 1} \sum_{j \geq 0} \frac{(-1)^{n_0+1+j}}{(2n_0+q-2)} \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{x^{2n_0+2j+a+q-2} \ln x}{2n_k+q-2-k} dx.$$

In the domain $x \in (0,1)$ convergence of the series is assured and interchanging the order of summation and integration, which is permissible by the dominated convergence theorem, we have,

$$\begin{aligned}
I(a,q) &= q! \sum_{n_0 \geq 1} \sum_{j \geq 0} \frac{(-1)^{n_0+1+j}}{(2n_0+q-2)} \\
&\quad \times \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{1}{2n_k+q-2-k} \int_0^1 x^{2n_0+2j+a+q-2} \ln(x)\, dx \\
&= q! \sum_{n_0 \geq 1} \sum_{j \geq 0} \frac{(-1)^{n_0+j}}{(2n_0+q-2)(2n_0+2j+a+q-1)^2} \\
&\quad \times \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{1}{2n_k+q-2-k} \\
&= \frac{q!}{16} \sum_{n_0 \geq 1} \frac{(-1)^{n_0+1} \left(\psi'\left(\frac{2n_0+a+q+1}{4}\right) - \psi'\left(\frac{2n_0+a+q-1}{4}\right) \right)}{(2n_0+q-2)} \\
&\quad \times \left(\prod_{k=1}^{q-1} \sum_{n_k=1}^{n_{k-1}} \right) \frac{1}{2n_k+q-2-k}
\end{aligned}$$

$$= \frac{q!}{16}\sum_{n_0\geq 1}\frac{(-1)^{n_0+1}\left(H^{(2)}_{\frac{2n_0+a+q-5}{4}}-H^{(2)}_{\frac{2n_0+a+q-3}{4}}\right)}{(2n_0+q-2)}$$
$$\times\left(\prod_{k=1}^{q-1}\sum_{n_k=1}^{n_{k-1}}\right)\frac{1}{2n_k+q-2-k},$$

hence (2.3) is achieved. ∎

2.2. **Illustrative examples.**

Example 2.4. Let $a=1, q=2$ then

$$I(1,2)=\int_0^1\frac{x\ln x}{1+x^2}\arctan^2(x)\,dx=\int_0^{\frac{\pi}{4}}\theta^2\tan\theta\ln(\tan\theta)\,d\theta \tag{8}$$

where $x=\tan\theta$.
Integration by parts produces:

$$\int_0^1\frac{x\ln x}{1+x^2}\arctan^2(x)\,dx=-\frac{1}{3}\int_0^1\arctan^3(x)\,dx-\frac{1}{3}\int_0^1\ln x\arctan^3(x)\,dx. \tag{9}$$

From [19] Corollary 8, we have the result

$$\int_0^1\arctan^3(x)\ dx=\frac{63\zeta(3)}{64}+\frac{\pi^3}{64}+\frac{3\pi^2\log(2)}{32}-\frac{3\pi G}{4}, \tag{10}$$

where $G=\beta(2)=-\int_0^1\frac{\log(x)}{1+x^2}dx$ is Catalan's constant and given an approximate value, Paul Levrie used the PSLQ algorithm to obtain

$$\begin{aligned}\int_0^1\ln(x)\arctan^3(x)\ dx=&\frac{3\pi G}{4}-\frac{63\zeta(3)}{64}-\frac{21\zeta(3)\log(2)}{16}-\frac{\pi^3}{64}\\&-\frac{59\pi^4}{2560}-\frac{3\log^4(2)}{16}+\frac{3\pi^2\log^2(2)}{32}-\frac{3\pi^2\log(2)}{32}\\&+3\pi\Im\left[\text{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right]-\frac{9}{2}\text{Li}_4\left(\frac{1}{2}\right).\end{aligned} \tag{11}$$

Using the results (10) and (11) in (9) produces the result for (8), namely

$$\begin{aligned}I(1,2)\ =\ &\int_0^1\frac{x\ln x}{1+x^2}\arctan^2(x)\,dx=\frac{59\pi^4}{7680}-\frac{\pi^2\log^2(2)}{32}+\frac{7\zeta(3)\log(2)}{16}\\&+\frac{\log^4(2)}{16}+\frac{3}{2}\text{Li}_4\left(\frac{1}{2}\right)-\pi\Im\left[\text{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right],\end{aligned}$$

where the imaginary part of $\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)$,

$$\Im\left[\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right]=\sum_{n\geq1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right),$$

see [20]. From Theorem 2.3, for $a=1, q=2$ we also have the representation

$$\begin{aligned}
I(1,2) &= \int_0^1 \frac{x\ln x}{1+x^2}\arctan^2(x)\,dx \\
&= \frac{2}{16}\sum_{n\geq1}\frac{(-1)^{n+1}\left(H^{(2)}_{\frac{n}{2}-\frac{1}{2}}-H^{(2)}_{\frac{n}{2}}\right)}{2n}\sum_{j=1}^{n}\frac{1}{2j-1} \\
&= \frac{1}{16}\sum_{n\geq1}\frac{(-1)^{n+1}\left(H^{(2)}_{\frac{n}{2}-\frac{1}{2}}-H^{(2)}_{\frac{n}{2}}\right)\left(H_{2n}-\frac{1}{2}H_n\right)}{2n},
\end{aligned}$$

using the multiplication identity (3) and putting $h_n=H_{2n}-\frac{1}{2}H_n=\sum_{j=1}^{n}\frac{1}{2j-1}$, we have

$$\begin{aligned}
I(1,2) &= \frac{1}{16}\sum_{n\geq1}\frac{(-1)^{n+1}\left(4H^{(2)}_n-2\zeta(2)-2H^{(2)}_{\frac{n}{2}}\right)h_n}{n} \\
&= -\frac{45}{64}\zeta(4)+\frac{1}{4}\sum_{n\geq1}\frac{(-1)^{n+1}\left(H^{(2)}_n-\frac{1}{2}H^{(2)}_{\frac{n}{2}}\right)h_n}{n} \\
&= \frac{59\pi^4}{7680}-\frac{\pi^2\log^2(2)}{32}+\frac{7\zeta(3)\log(2)}{16} \\
&\quad +\frac{\log^4(2)}{16}+\frac{3}{2}\mathrm{Li}_4\left(\frac{1}{2}\right)-\pi\Im\left[\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right].
\end{aligned}$$

In a related example, consider the case, $a=-1, q=2$ then:

$$\begin{aligned}
I(-1,2) &= \int_0^1 \frac{\ln x}{x(1+x^2)}\arctan^2(x)\,dx=\int_0^{\frac{\pi}{4}}\theta^2\cot\theta\ln(\tan\theta)\,d\theta \\
&= \int_0^1\frac{\ln x}{x}\arctan^2(x)\,dx-I(1,2) \\
&= \frac{1}{4}\sum_{n\geq1}(-1)^n\frac{h_n}{n^3}-I(1,2).
\end{aligned}$$

The sum can be evaluated by referring to [20], producing the value

$$
\begin{aligned}
I(-1,2) &= \int_0^1 \frac{\ln x}{x(1+x^2)} \arctan^2(x)\,dx = \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\log^4(2) \\
&\quad + \frac{7}{8}\zeta(3)\log(2) - \frac{1}{4}\zeta(2)\log^2(2) - \frac{151}{128}\zeta(4) - I(1,2) \\
&= \frac{7}{16}\zeta(3)\log(2) + \pi\Im\left[\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right] - \frac{1}{2}\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1}{48}\log^4(2) \\
&\quad - \frac{1}{16}\zeta(2)\log^2(2) - \frac{479}{256}\zeta(4).
\end{aligned}
$$

2.3. **Special case a=0.** We now concentrate our analysis on the case $a = 0$ to show that the resulting integral

$$I(0,q) = \int_0^1 \frac{\ln x}{1+x^2} \arctan^q(x)\,dx$$

will reduce to a $\log(\tan\theta)$ type integral, and that it may be represented as a linear combination of the product of zeta functions and Dirichlet beta functions. We also observe that for the case $q = 1$, Ramanujan [11] investigated the resulting integral. We shall also prove that the following integral on the positive half line, $x \geq 0$

$$J(0,q) = \int_0^\infty \frac{\ln x}{1+x^2} \arctan^q(x)\,dx$$

can be represented as a linear combination of zeta and Dirichlet beta functions at all integral points of q.

Theorem 2.5. *For* $q \in \mathbb{N} \cup \{0\}$,

$$I(0,q) = \int_0^1 \frac{\ln x}{1+x^2} \arctan^q(x)\,dx \tag{12}$$

$$= -\frac{1}{1+q} \int_0^1 \frac{\arctan^{q+1}(x)}{x} dx \tag{13}$$

$$= \int_0^{\frac{\pi}{4}} \theta^q \ln(\tan\theta)\,d\theta \tag{14}$$

$$
\begin{aligned}
&= (-1)^{\frac{q(q-1)}{2}} (1-(-1)^q) \frac{q!}{2^{q+1}} \lambda(q+2) \\
&\quad + \sum_{r=0}^{\left[\frac{q}{2}\right]} \frac{(-1)^{r+1} q!}{4^r (q-2r)!} \left(\frac{\pi}{4}\right)^{q-2r} \beta(2r+2)
\end{aligned} \tag{15}
$$

where $[x]$ *is the integer part of* x.

Proof. Consider

$$I(0,q)=\int_0^1 \frac{\ln x}{1+x^2}\arctan^q(x)\,dx$$

and integrate by parts, so that

$$I(0,q)=-\frac{1}{1+q}\int_0^1 \frac{\arctan^{q+1}(x)}{x}dx$$

which satisfies (13). In (12) make the substitution $x=\tan\theta$ and we obtain (14). Before deriving (15) we detail two other representations of (12). From (2.3)

$$I(0,q)=\int_0^1 \frac{\ln x}{1+x^2}\arctan^q(x)\,dx$$

$$=\frac{q!}{16}\sum_{n_0\geq 1}\frac{(-1)^{n_0+1}\left(H^{(2)}_{\frac{2n_0+q-5}{4}}-H^{(2)}_{\frac{2n_0+q-3}{4}}\right)}{(2n_0+q-2)}\left(\prod_{k=1}^{q-1}\sum_{n_k=1}^{n_{k-1}}\right)\frac{1}{2n_k+q-2-k}.$$

From (13) and using (5)

$$\begin{aligned}I(0,q) &= -\frac{1}{1+q}\int_0^1 \frac{\arctan^{q+1}(x)}{x}dx=-\frac{2}{1+q}\int_0^{\frac{\pi}{4}}\theta^{q+1}\csc(2\theta)\,d\theta\\ &= -\frac{2}{1+q}\int_0^{\frac{\pi}{4}}\sum_{n\geq 0}\frac{2^{2n}(-1)^{n+1}(2^{2n-1}-1)}{(2n)!}B_{2n}\theta^{2n+q}d\theta\end{aligned}$$

where B_{2n} are the Bernoulli numbers, therefore

$$I(0,q)=-\frac{2}{1+q}\sum_{n\geq 0}\frac{(-1)^{n+1}2^{2n}(2^{2n-1}-1)B_{2n}}{(2n+q+1)(2n)!}\left(\frac{\pi}{4}\right)^{2n+q+1}.$$

From (14)

$$I(0,q)=\int_0^{\frac{\pi}{4}}\theta^q\ln(\tan\theta)\,d\theta$$

we use (6) and apply repeated integration by parts, so that

$$\begin{aligned} I(0,q) &= -\left[\theta^q \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{(2n+1)^2}\right]_0^{\frac{\pi}{4}} + q\int_0^{\frac{\pi}{4}} \theta^{q-1} \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{(2n+1)^2} d\theta \\ &= -\left(\frac{\pi}{4}\right)^q \beta(2) + \frac{q(q-1)}{2^2}\left(\frac{\pi}{4}\right)^{q-2}\beta(4) \\ &\quad -q(q-1)(q-2)\int_0^{\frac{\pi}{4}} \theta^{q-3} \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{2^2(2n+1)^4} d\theta. \end{aligned}$$

In general

$$\begin{aligned} I(0,q) &= -\left(\frac{\pi}{4}\right)^q \beta(2) + \sum_{j=1}^{k} (-1)^{j+1}\left(\frac{\pi}{4}\right)^{q-2j} \frac{q(q-1)\cdots(q-2j+1)}{2^{2j}}\beta(2j+2) \\ &\quad + (-1)^k q(q-1)\cdots(q-2k)\int_0^{\frac{\pi}{4}} \theta^{q-2k-1} \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{(2n+1)^{2k+2}} d\theta \end{aligned}$$

for $q \geq 2k+1$, therefore

$$= -\left(\frac{\pi}{4}\right)^q \beta(2) + \frac{q(q-1)}{2^2}\left(\frac{\pi}{4}\right)^{q-2}\beta(4) + \cdots\cdots + \begin{cases} \frac{(-1)^{\frac{(q+1)(q-2)}{2}}}{2^q} q!\beta(q+2), \text{for } q \text{ even} \\ \\ \frac{(-1)^{\frac{q(q-1)}{2}}}{2^q} q!\lambda(q+2), \text{for } q \text{ odd} \end{cases}$$

collecting terms and taking into account q even or odd for the last term, we have the general formula, for $q \in \mathbb{N}$

$$I(0,q) = \sum_{r=0}^{\left[\frac{q}{2}\right]} \frac{(-1)^{r+1}}{4^r} q^{\underline{2r}} \left(\frac{\pi}{4}\right)^{q-2r} \beta(2r+2) + (-1)^{\left[\frac{3q+1}{2}\right]}(1-(-1)^q)\frac{q!}{2^{q+1}}\lambda(q+2)$$

The falling factorial $a^{\underline{b}}$, for $a \in \mathbb{R}$, $b \in \mathbb{N}$

$$\begin{aligned} a^{\underline{b}} &= a(a-1)\cdot\cdots\cdot(a+1-b) = \prod_{j=0}^{b-1}(a-j) = \frac{a!}{(a-b)!} \\ &= \sum_{j=0}^{b} s(b,j)\, a^j \end{aligned}$$

where $s(n,j)$ are signed Stirling numbers of the first kind. Hence, we obtain

$$I(0,q) = (-1)^{\frac{q(q-1)}{2}}(1-(-1)^q)\frac{q!}{2^{q+1}}\lambda(q+2) + \sum_{r=0}^{\left[\frac{q}{2}\right]} \frac{(-1)^{r+1} q!}{4^r(q-2r)!}\left(\frac{\pi}{4}\right)^{q-2r}\beta(2r+2).$$

∎

It is interesting to note that Ramanujan [11] obtained various interesting series for the case $q = 0$. In particular in section 2, Ramanujan obtained the result of lemma 2.1. We note that (13) can be generalized to $\int_0^1 \frac{\arctan^q(x)}{x^r}dx$, for positive integers q, r where $q \geq r$, and using (7) can be analytically expressed in terms of the Riemann zeta and Dirichlet beta functions. The case of $\int_0^1 \frac{\arcsin^q(x)}{x^r}dx$ has been analyzed in the paper [7].

2.3.1. *Illustrative examples.*

Example 2.6. For $q = 5$ and 6

$$I(0,5) = \int_0^1 \frac{\ln(x)}{1+x^2}\arctan^5(x)\,dx = \int_0^{\frac{\pi}{4}} \theta^5 \ln(\tan\theta)\,d\theta$$

$$= \frac{1905}{512}\zeta(7) - \frac{1}{1024}\pi^5 G + \frac{5\pi^3}{64}\beta(4) - \frac{15\pi}{8}\beta(6).$$

$$I(0,6) = \int_0^1 \frac{\ln(x)}{1+x^2}\arctan^6(x)\,dx = \int_0^{\frac{\pi}{4}} \theta^6 \ln(\tan\theta)\,d\theta$$

$$= -\frac{1}{4096}\pi^6 G + \frac{15\pi^4}{512}\beta(4) - \frac{45\pi^2}{32}\beta(6) + \frac{45}{4}\beta(8)$$

$$= -\frac{2}{7}\sum_{n\geq 0} \frac{(-1)^{n+1} 2^{2n}(2^{2n-1}-1)B_{2n}}{(2n+7)(2n)!}\left(\frac{\pi}{4}\right)^{2n+7}$$

$$= 45\sum_{n_0\geq 1} \frac{(-1)^{n_0+1}\left(H^{(2)}_{\frac{2n_0+1}{4}} - H^{(2)}_{\frac{2n_0+3}{4}}\right)}{(2n_0+4)}\left(\prod_{k=1}^{5}\sum_{n_k=1}^{n_{k-1}}\right)\frac{1}{2n_k+4-k}.$$

We may remark that, in general, the association of the three representations for the integral gives us the equality in terms of multiple zeta values, Bernoulli numbers and linear combinations of zeta and Dirichlet beta functions, namely

$$\int_0^1 \frac{\ln(x)}{1+x^2}\arctan^q(x)\,dx = -\frac{1}{1+q}\int_0^1 \frac{\arctan^{q+1}(x)}{x}dx = \int_0^{\frac{\pi}{4}} \theta^q \ln(\tan\theta)\,d\theta$$

$$= -\frac{2}{1+q}\sum_{n\geq 0} \frac{(-1)^{n+1} 2^{2n}(2^{2n-1}-1)B_{2n}}{(2n+q+1)(2n)!}\left(\frac{\pi}{4}\right)^{2n+q+1}$$

$$= \frac{q!}{16}\sum_{n_0\geq 1} \frac{(-1)^{n_0+1}\left(H^{(2)}_{\frac{2n_0+q-5}{4}} - H^{(2)}_{\frac{2n_0+q-3}{4}}\right)}{(2n_0+q-2)}\left(\prod_{k=1}^{q-1}\sum_{n_k=1}^{n_{k-1}}\right)\frac{1}{2n_k+q-2-k}$$

$$= (-1)^{\frac{q(q-1)}{2}} (1-(-1)^q) \frac{q!}{2^{q+1}} \lambda(q+2) + \sum_{r=0}^{\left[\frac{q}{2}\right]} \frac{(-1)^{r+1} q!}{4^r (q-2r)!} \left(\frac{\pi}{4}\right)^{q-2r} \beta(2r+2).$$

3. Analysis of integrals: Part 2

Now we analyze the integral (2)

We define

$$f(q,x) = \frac{\ln(x)}{1+x^2} \arctan^q(x) \tag{16}$$

which is continuous, bounded and differentiable on the interval $x \in (0,1]$, with $\lim_{x\to 0+} f(q,x) = \lim_{x\to 1} f(q,x) = 0$.

Theorem 3.1. *Let $q \in \mathbb{N}$, the following integral,*

$$\begin{aligned} J(q) &= \int_0^\infty f(q,x)\,dx = \int_0^{\frac{\pi}{2}} \theta^q \ln(\tan\theta)\,d\theta \\ &= (-1)^{\frac{q(q-1)}{2}} (1-(-1)^q) \frac{q!}{2^{q+1}} \lambda(q+2) \\ &\quad + \sum_{r=0}^{\left[\frac{q-1}{2}\right]} \frac{(-1)^r q!}{2^{2r+1}(q-2r-1)!} \left(\frac{\pi}{2}\right)^{q-2r-1} \lambda(2r+3) \end{aligned} \tag{17}$$

Similarly

$$J(q) = \int_0^\infty f(q,x)\,dx = \int_0^{\frac{\pi}{2}} \theta^q \ln(\tan\theta)\,d\theta$$

$$= \left(\frac{\pi}{2}\right)^q G + (1-(-1)^q) I(0,q) - \sum_{r=1}^{q-1} (-1)^{q-r} \binom{q}{r} \left(\frac{\pi}{2}\right)^r I(0,q-r), \tag{18}$$

where $I(0,\cdot)$ is obtained from (15).

Proof. We begin with

$$J(q) = \int_0^{\frac{\pi}{2}} \theta^q \ln(\tan\theta)\,d\theta,$$

we integrate by parts and utilize (5) in Lemma 2.2, so that

$$\int_0^{\frac{\pi}{2}} \theta^q \ln(\tan\theta)\,d\theta = -\left[\theta^q \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{(2n+1)^2}\right]_0^{\frac{\pi}{2}} + q\int_0^{\frac{\pi}{2}} \theta^{q-1} \sum_{n\geq 0} \frac{\sin(2(2n+1)\theta)}{(2n+1)^2} d\theta$$

$$= \frac{q}{2}\left(\frac{\pi}{2}\right)^{q-1}\sum_{n\geq 0}\frac{1}{(2n+1)^3} - \frac{q(q-1)(q-2)}{2^3}\left(\frac{\pi}{2}\right)^{q-3}\sum_{n\geq 0}\frac{1}{(2n+1)^5}$$
$$+\frac{q(q-1)(q-2)(q-3)}{2^4}\int_0^{\frac{\pi}{2}}\theta^{q-4}\sum_{n\geq 0}\frac{\cos(2(2n+1)\theta)}{(2n+1)^5}d\theta.$$

In general

$$\int_0^{\frac{\pi}{2}}\theta^q\ln(\tan\theta)\,d\theta = \sum_{j=0}^{k-1}\frac{(-1)^j q!}{2^{2j+1}(q-2j-1)!}\left(\frac{\pi}{2}\right)^{q-2j-1}\lambda(2j+3)$$
$$+\frac{(-1)^k q(q-1)\cdots(q-2k)}{2^{2k}}\int_0^{\frac{\pi}{2}}\theta^{q-2k-1}\sum_{n\geq 0}\frac{\sin(2(2n+1)\theta)}{(2n+1)^{2k+2}}d\theta$$

where $q \geq 2k+1$, therefore

$$= \frac{q}{2}\left(\frac{\pi}{2}\right)^{q-1}\lambda(3) - \frac{q(q-1)(q-2)}{2^3}\left(\frac{\pi}{2}\right)^{q-3}\lambda(5)+\cdots$$
$$\cdots + \begin{cases} \frac{(-1)^{\left[\frac{q-1}{2}\right]}q!}{2^{2\left[\frac{q-1}{2}\right]+1}}\left(\frac{\pi}{2}\right)\lambda\left(2\left[\frac{q-1}{2}\right]+3\right), \text{for } q \text{ even} \\ \frac{(-1)^{\left[\frac{q-1}{2}\right]}q!}{2^{q-1}}\lambda(q+2), \text{for } q \text{ odd} \end{cases}$$

collecting terms and taking into account q even or odd for the last term, we have for the general case $q \in \mathbb{N}$

$$J(q) = (-1)^{\frac{q(q-1)}{2}}(1-(-1)^q)\frac{q!}{2^{q+1}}\lambda(q+2)+\sum_{r=0}^{\left[\frac{q-1}{2}\right]}\frac{(-1)^r q!}{2^{2r+1}(q-2r-1)!}\left(\frac{\pi}{2}\right)^{q-2r-1}\lambda(2r+3).$$

Now put

$$J(q) = \int_0^\infty f(q,x)\,dx = \int_0^1 f(q,x)\,dx + \int_1^\infty f(q,x)\,dx,$$

then

$$\int_0^1 f(q,x)\,dx = \int_0^\infty f(q,x)\,dx - \int_0^1 \frac{1}{y^2}f\left(q,\frac{1}{y}\right)dy$$

where in the third integral we have made the transformation $xy = 1$, now use the relation $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$, for $x > 0$, so that

$$\begin{aligned}
\int_0^\infty f(q,x)\,dx &= \int_0^1 f(q,x)\,dx - \int_0^1 \frac{\ln(x)\left(\frac{\pi}{2} - \arctan(x)\right)^q}{1+x^2}dx \\
&= \int_0^1 f(q,x)\,dx - \sum_{r=0}^{q} (-1)^{q-r}\binom{q}{r}\left(\frac{\pi}{2}\right)^r \int_0^1 f(q-r,x)\,dx
\end{aligned}$$

$$= (1-(-1)^q)\int_0^1 f(q,x)\,dx - \left(\frac{\pi}{2}\right)^q \int_0^1 \frac{\ln(x)}{1+x^2}dx$$

$$-\sum_{r=1}^{q-1} (-1)^{q-r}\binom{q}{r}\left(\frac{\pi}{2}\right)^r \int_0^1 f(q-r,x)\,dx$$

$$= (1-(-1)^q)\int_0^1 f(q,x)\,dx + \left(\frac{\pi}{2}\right)^q G$$

$$-\sum_{r=1}^{q-1} (-1)^{q-r}\binom{q}{r}\left(\frac{\pi}{2}\right)^r \int_0^1 f(q-r,x)\,dx$$

and (18) follows. ∎

3.1. **Illustrative examples.**

Example 3.2. For $q = 5$

$$J(5) = \int_0^\infty \frac{\ln(x)}{1+x^2}\arctan^5(x)\,dx = \int_0^{\frac{\pi}{2}} \theta^5 \ln(\tan\theta)\,d\theta$$

$$= \frac{15}{8}\lambda(7) + \sum_{r=0}^{2} \frac{(-1)^r 5^{2r}}{2^{2r+2}}\left(\frac{\pi}{2}\right)^{4-2r}\lambda(2r+3)$$

$$= \left(\frac{\pi}{2}\right)^5 G + 2I(0,5) + \sum_{r=1}^{4}(-1)^r\binom{5}{r}\left(\frac{\pi}{2}\right)^r I(0,5-r)$$

$$= \frac{35}{256}\pi^4\zeta(3) - \frac{465\pi^2}{256}\zeta(5) + \frac{1905}{256}\zeta(7).$$

For $q = 6$

$$J(6) = \int_0^\infty \frac{\ln(x)}{1+x^2}\arctan^6(x)\,dx = \int_0^{\frac{\pi}{2}} \theta^6 \ln(\tan\theta)\,d\theta$$

$$= \sum_{r=0}^{2} \frac{(-1)^r\, 6^{2r+1}}{2^{2r+1}} \left(\frac{\pi}{2}\right)^{5-2r} \lambda\,(2r+3)$$

$$= \left(\frac{\pi}{2}\right)^{6} G - \sum_{r=1}^{5} (-1)^r \left(\begin{array}{c} 6 \\ r \end{array}\right) \left(\frac{\pi}{2}\right)^{r} I\,(0, 6-r)$$

$$= \frac{21}{256}\pi^5\zeta\,(3) - \frac{465\pi^3}{256}\zeta\,(5) + \frac{5715\pi}{512}\zeta\,(7)\,.$$

4. Log-sine and log-cosine integrals

We may remark here that an analogous result to (17), for the moments of the $\log(\sin\theta)$ integral, has been evaluated in [7].

Proposition 4.1. *For $q \in \mathbb{N}$*

$$\int_0^{\frac{\pi}{2}} \theta^q \ln(\sin\theta)\, d\theta = -\frac{(-1)^{\frac{q(q+1)}{2}} \left\{1-(-1)^q\right\} q!}{2^{q+2}} \zeta(q+2)$$

$$+ \sum_{k=0}^{\left[\frac{q+1}{2}\right]} \frac{(-1)^{k+1} q!}{(q+1-2k)!\, 2^{2k}} \left(\frac{\pi}{2}\right)^{q+1-2k} \eta\,(2k+1)\,. \tag{19}$$

We believe this result is more transparent than that given by Orr [9].

Since $\log(\tan\theta) = \log(\sin\theta) - \log(\cos\theta)$ for $\theta \in [0, \pi/2]$, we subtract (17) from (19) to deduce the following formula:

Corollary 4.2.

$$\int_0^{\frac{\pi}{2}} \theta^q \ln(\cos\theta)\, d\theta = -\frac{\ln(2)}{q+1}\left(\frac{\pi}{2}\right)^{q+1} + \frac{(-1)^{\frac{q(q+1)}{2}} \left\{1-(-1)^q\right\} q!}{2^{q+2}} \eta(q+2)$$

$$- \sum_{k=1}^{\left[\frac{q+1}{2}\right]} \frac{(-1)^{k+1}\, q!}{(q+1-2k)!\, 2^{2k}} \left(\frac{\pi}{2}\right)^{q+1-2k} \zeta(2k+1). \tag{20}$$

On the interval $x \in (0, \pi/4)$ the integral (14) can be written as

$$\int_0^{\frac{\pi}{4}} \theta^q \ln(\tan\theta)\; d\theta = \int_0^{\frac{\pi}{4}} t^q \ln(\sin t)\; dt - \int_0^{\frac{\pi}{4}} t^q \ln(\cos t)\; dt.$$

using trigonometric identities for the $\sin t$ and re-indexing we obtain,

Corollary 4.3.

$$\int_0^{\frac{\pi}{4}} t^q \ln(\sin t)\, dt = -\frac{\log(2)}{(q+1)2^{q+1}} \left(\frac{\pi}{2}\right)^{q+1} - \frac{(-1)^{\frac{q(q+1)}{2}} \{1-(-1)^q\}\, q!}{2^{q+2}} \zeta(q+2)$$

$$- \sum_{k=1}^{\left[\frac{q+2}{2}\right]} \frac{(-1)^{k+1}\, q!}{(q+2-2k)!\, 2^{q+1}} \left(\frac{\pi}{2}\right)^{q+2-2k} \beta(2k)$$

$$+ \sum_{k=1}^{\left[\frac{q+1}{2}\right]} \frac{(-1)^{k+1}\, q!}{(q+1-2k)!\, 2^{q+2+2k}} \left(\frac{\pi}{2}\right)^{q+1-2k} \eta(2k+1)\,. \tag{21}$$

Corollary 4.4.

$$\int_0^{\frac{\pi}{4}} t^q \ln(\cos t)\, dt = -\frac{\log(2)}{(q+1)2^{q+1}} \left(\frac{\pi}{2}\right)^{q+1} + (-1)^{\frac{q(q+1)}{2}} \{1-(-1)^q\} \frac{q!}{2^{q+2}} \eta(q+2)$$

$$+ \sum_{k=1}^{\left[\frac{q+2}{2}\right]} \frac{(-1)^{k+1}\, q!}{2^{q+1}\,(q+2-2k)!} \left(\frac{\pi}{2}\right)^{q+2-2k} \beta(2k)$$

$$+ \sum_{k=1}^{\left[\frac{q+1}{2}\right]} \frac{(-1)^{k+1}\, q!}{(q+1-2k)!\, 2^{q+2+2k}} \left(\frac{\pi}{2}\right)^{q+1-2k} \eta(2k+1)\,. \tag{22}$$

We now give a couple of examples relating to the last two corollaries.

Example 4.5.

$$\int_0^{\pi/4} \theta^4 \log(\sin\theta)\, d\theta = \frac{3\pi^3\zeta(3)}{2048} - \frac{\pi^4 G}{512} - \frac{\pi^5\log(2)}{5120} - \frac{45\pi\zeta(5)}{4096}$$

$$+ \frac{3\pi^2}{32}\beta(4) - \frac{3}{4}\beta(6).$$

Example 4.6.

$$\int_0^{\pi/4} \theta^4 \log(\cos\theta)\, d\theta = \frac{3\pi^3\zeta(3)}{2048} + \frac{\pi^4 G}{512} - \frac{\pi^5\log(2)}{5120} - \frac{45\pi\zeta(5)}{4096}$$

$$- \frac{3\pi^2}{32}\beta(4) + \frac{3}{4}\beta(6).$$

Concluding Remarks: We have shown that integrals containing log and powers of the arctangent function can be expressed in terms of Euler sums. Formulas for log-sine and log-cosine integrals were also derived. We believe that most of our results are new and interesting. We have given many examples some of which are not amenable to a mathematical computer package.

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