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Liapunov Functions and Equiboundedness in Functional Differential Equations

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The equiboundedness of the solutions of ordinary and functional differential equations are mainly discussed by using Liapunov's second method. For a theorem on equiboundedness in ordinary differential equations in [V. Lakshmikantham and S. Leela, Math. Systems Theory, 10 (1976), 85–90], an example is presented. Moreover the theorem is extended to functional differential equations with infinite delay by using Liapunov-Razumikhin method, and an application is presented.

§ 1. Introduction

In many theorems on boundedness of the solutions of ordinary and functional differential equations, Liapunov functions (or functionals) play important roles. It is well known that in proving uniform boundedness of the solutions of ordinary differential equations by using Liapunov functions, it is sufficient to impose conditions in the complement of a bounded set in R^n ([6], [10]). But, in the case of equiboundedness, we usually need to impose conditions everywhere in R^n. In [7], some efforts are made to overcome such a deficiency.

The purpose of this paper is to study equiboundedness of the solutions of ordinary and functional differential equations by employing Liapunov's second method. In §3, we present an example for a theorem in [7], which concerns equiboundedness in ordinary differential equations. In §4, we extend the result for ordinary differential equations in §3 to functional differential equations by using the theory of Liapunov-Razumikhin type. The comparison method used in §4 is similar to those found in [1–6] and [9]. In §5, we present an application of the result obtained in §4.

§ 2. Notations and preliminary results

Let I and R denote the intervals $0 \leqq t < \infty$ and $-\infty < t < \infty$, respectively. R^n denotes the Euclidean space and $C(A, B)$ the class of continuous functions from A to B. For any set $E \subset R^n$, we denote by E^c and $\bar{E}$, the complement and the closure of E, respectively. For any $\rho > 0$, let $S(\rho) = \{x \in R^n : |x| < \rho\}$, where $|\cdot|$ denotes any convenient norm in R^n. BC denotes the Banach space of bounded continuous functions

ϕ: $(-\infty, 0]\rightarrow R^n$, with the uniform norm, $\|\phi\|=\sup\{|\phi(s)|: s\leqq 0\}$. For any bounded continuous function $x(s)$ defined on $-\infty<s<T(0<T\leqq\infty)$ and any fixed t, $0\leqq t<T$, x_t is defined by $x_t(\theta)=x(t+\theta)$, $\theta\leqq 0$.

Consider the following differential equations

$$\dot{x}=f(t, x), \tag{1}$$

$$\dot{u}=g(t, u), \tag{2}$$

$$\dot{v}=h(t, v), \tag{3}$$

$$\dot{x}(t)=F(t, x_t), \tag{4}$$

where the dot denotes the right-hand derivative, $f\in C(I\times R^n, R^n)$, g, $h\in C(I\times I, R)$, and $F\in C(I\times BC, R^n)$. Moreover, we assume that $g(t, u)$ and $h(t, u)$ are locally Lipschitzian with respect to u for $u>0$. For T with $0<T\leqq\infty$, x: $(-\infty, T)\rightarrow R^n$ is said to be a solution of Equation (4) through (t_0, ϕ) on $[t_0, T)$ if $x(t)$ is continuous on $(-\infty, T)$, $x(t)$ satisfies Equation (4) for $t_0\leqq t<T$, and $x_{t_0}=\phi$. For $t_0\in I$ and $x_0\in R^n$, $x(t, t_0, x_0)$ denotes a solution of Equation (1) through (t_0, x_0). Similarly, $u(t, t_0, u_0)$, $v(t, t_0, v_0)$ and $x(t, t_0, \phi)$ denote solutions of Equations (2), (3) and (4) through (t_0, u_0), (t_0, v_0) and (t_0, ϕ), respectively. In what follows, we assume the local existence of a solution $x(t, t_0, \phi)$ of Equation (4) for any $t_0\in I$ and any $\phi\in BC$. Moreover, it is assumed that if $x(t)$ is a solution of Equation (4) on $(-\infty, T)$, $0<T<\infty$, then either $x(t)$ is continuable past T or $\|x_t\|\rightarrow\infty$ as $t\rightarrow T-$.

In this paper, we study equiboundedness of the solutions of Equations (1) and (4) by utilizing Liapunov functions. For Equation (4), we use Razumikhin method. Now let E be a compact subset of R^n.

DEFINITION 1. $U(t, x)\in C(I\times\overline{E^c}, I)$ (or $C(R\times\overline{E^c}, I)$) is said to be a Liapunov function if $U(t, x)$ is locally Lipschitzian with respect to x.

For a Liapunov function $U(t, x)$, we define $U'_{(1)}(t, x)$ by

$$U'_{(1)}(t, x)=\limsup_{h\rightarrow 0+}\frac{1}{h}\{U(t+h, x+hf(t, x))-U(t, x)\}.$$

Similarly, $U'_{(4)}(t, \phi)$ is defined by taking $\phi(0)$ and $F(t, \phi)$ instead of x and $f(t, x)$, respectively. For a solution $x(t)$ of Equation (1) (or (4)), $U'_{(1)}(t, x(t))$ (or $U'_{(4)}(t, x_t)$) is equal to the following derivative of U along the solution $x(t)$.

$$\limsup_{h\rightarrow 0+}\frac{1}{h}\{U(t+h, x(t+h))-U(t, x(t))\}.$$

Let $x(t)$ be a solution of Equation (1) (or (4)). We study equiboundedness of the solutions of Equation (1) (or (4)) through the behavior of the scalar functions $U(t, x(t))$ and $U(t, x(t))+V(t, x(t))$. To evaluate the values of these two scalar functions, we employ differential inequalities with respect to Equations (2) and (3).

We first state a fundamental theorem concerning the estimate of the value of $U(t, x(t, t_0, \phi))$ with respect to Equation (2). Similar result holds for $U(t, x(t, t_0, \phi))+V(t, x(t, t_0, \phi))$ and Equation (3).

THEOREM 1. *Let* $U(t, x) \in C(R \times \overline{E^c}, I)$ *be a Liapunov function, and* $L(t, u) \in C(I \times I, I)$ *be nondecreasing in u for each fixed t and that*

$$L(t, u) > u \quad for \quad t \geqq 0, u > 0,$$

or

$$L(t, u) \equiv u \quad for \quad t \geqq 0, u > 0.$$

Suppose that there exists a nonnegative function $u(t)$ *defined on* $(-\infty, T)$ *for some* T *with* $0 < T \leqq \infty$ *such that* $u(t)$ *is a positive solution of Equation* (2) *on* $[\tau, T)$ *for some* $\tau \in [0, T)$ *which satisfies* $u(t+\theta) \leqq L(t, u(t))$ *for* $\tau \leqq t < T$, $\theta \leqq 0$, *and that we have*

$$U'_{(4)}(t, \phi) \leqq g(t, U(t, \phi(0)))$$

for all functions $\phi \in BC$ *with the properties that*

$$\phi(0) \in \overline{E^c},\ U(t+\theta, \phi(\theta)) \leqq L(t, U(t, \phi(0))) \qquad for \quad \theta \leqq 0 \quad with \quad \phi(\theta) \in \overline{E^c}.$$

Then, for $\phi \in BC$ *such that* $\phi(0) \in \overline{E^c}$, $U(\tau+\theta, \phi(\theta)) \leqq u(\tau+\theta)$ *for* $\theta \leqq 0$ *with* $\phi(\theta) \in \overline{E^c}$, *we have* $U(t, x(t, \tau, \phi)) \leqq u(t)$ *for* $\tau \leqq t < T$ *as long as* $x(t, \tau, \phi) \in \overline{E^c}$.

Since the proof is essentially the same as one for Theorem 3.2 in [1], we omit the proof. In [1], $L(t, u)$ is independent of t and the time delay is fixed and finite.

§ 3. Equiboundedness in ordinary differential equations

In this section, we discuss equiboundedness of the solutions of the ordinary differential equation (1). The following definitions can be found in [10].

DEFINITION 2. The solutions of Equation (1) are equibounded, if for any $t_0 \in I$ and any $\delta > 0$, there exists a $B = B(t_0, \delta) > 0$ such that if $|x_0| < \delta$, $|x(t, t_0, \phi)| < B$ for $t \geqq t_0$.

DEFINITION 3. The solutions of Equation (1) are uniformly bounded, if the B in Definition 2 is independent of t_0.

For these definitions, the corresponding definitions for the solutions of Equations (2), (3) and (4) are similarly obtained by taking $u_0 \geqq 0$, $v_0 \geqq 0$ and $\phi \in BC$ instead of x_0, respectively.

Now we state a theorem, which is a special case of Theorem 1 in [7] by Lakshmikantham and Leela.

THEOREM 2. *Suppose that the following conditions hold.*

(i) *$U(t, x) \in C(I \times R^n, I)$ is a Liapunov function which satisfies*

$$U'_{(1)}(t, x) \leqq g(t, U(t, x)), \quad (t, x) \in I \times R^n. \tag{5}$$

(ii) *For a constant $\rho > 0$, $V \in C(I \times S^c(\rho), I)$ is a Liapunov function which satisfies*

$$a(|x|) \leqq V(t, x) \leqq b(|x|), \quad (t, x) \in I \times S^c(\rho), \tag{6}$$

where $a, b \in C([\rho, \infty), I)$ are nondecreasing and $a(u) \to \infty$ as $u \to \infty$, and for $(t, x) \in I \times S^c(\rho)$ with $|x| > \rho$,

$$U'_{(1)}(t, x) + V'_{(1)}(t, x) \leqq h(t, U(t, x) + V(t, x)). \tag{7}$$

(iii) *The solutions of Equations* (2) *and* (3) *are equibounded and uniformly bounded, respectively.*

Then the solutions of Equation (1) *are equibounded.*

For the proof, see [7].

In Theorem 2, it is remarkable that equiboundedness is concluded by combining two Liapunov functions $U(t, x)$ and $V(t, x)$, though the conditions for $V(t, x)$ are imposed on the restricted region $I \times S^c(\rho)$ in $I \times R^c$. Since no example is shown for Theorem 1 in [7], here we give an example for Theorem 2.

To construct an example for Theorem 2, consider the scalar equation

$$\dot{x} = (6t\sin t - 2t)x, \quad (t, x) \in I \times I, \tag{8}$$

which can be found in [8]. The function $w(t) = \exp(6\sin t - 6t\cos t - t^2)$ is clearly bounded and continuously differentiable on I, and a solution of Equation (8) through (0, 1) on I. For this $w(t)$ and some positive constant w_0, define a function $w_0(t)$ on I by

$$w_0(t) = w_0 + \int_0^t \min\{0, -\dot{w}(s)\} ds.$$

Then $w_0(t)$ is continuously differentiable and $\dot{w}_0(t) \leqq 0$ on I. Moreover $w_0(t)$ is positive for a sufficiently large w_0, since the total variation of $w(t)$ on I is clearly finite. Here we assume that w_0 is taken sufficiently large so that $w_0(t) \geqq w(t)$ on I. For this $w_0(t)$ and $w(t)$, define a function $U(t, x)$ by

$$U(t, x) = \frac{w_0^2(t)}{w^2(t)} x^2, \quad (t, x) \in I \times R.$$

Next, define another function $V(t, x)$ by

$$V(t, x) = x^2, \quad (t, x) \in I \times S^c(1).$$

Then, clearly $U(t, x)$ and $V(t, x)$ are Liapunov functions. Now we show that all condistions in Theorem 2 are satisfied for $a(u)\equiv b(u)\equiv u^2$, $\rho=1$, $f(t, x)=(6tsint-2t)x$, and $g(t, u)\equiv h(t, u)\equiv 0$.

Differentiating U along the solutions of Equation (8) we have

$$U'_{(8)}(t, x)=\frac{2w_0(t)\,(\dot{w}_0(t)w(t)-w_0(t)\dot{w}(t))}{w^3(t)}x^2+\frac{2w_0^2(t)f(t, x)x}{w^2(t)}$$

$$=\frac{2w_0(t)\,(\dot{w}_0)(t)-(6tsint-2t)w_0(t))}{w^2(t)}x^2+\frac{2(6tsint-2t)w_0^2(t)}{w^2(t)}x^2$$

$$=\frac{2w_0(t)\dot{w}_0(t)}{w^2(t)}x^2\leqq 0,$$

and hence Condition (5) holds for $g(t, u)\equiv 0$. Similarly we obtain

$$U'_{(8)}(t, x)+V'_{(8)}(t, x)$$

$$=\frac{2w_0(t)\,(\dot{w}_0(t)w(t)-w_0(t)\dot{w}(t))}{w^3(t)}x^2+\frac{2(w_0^2(t)+w^2(t))\,f(t, x)x}{w^2(t)}$$

$$=\frac{2w_0(t)\,(\dot{w}_0(t)-(6tsint-2t)w_0(t))}{w^2(t)}x^2+\frac{2(w_0^2(t)+w^2(t))(6tsint-2t)}{w^2(t)}x^2$$

$$=\frac{2}{w^2(t)}(w_0(t)\dot{w}_0(t)+w(t)\dot{w}(t))x^2\leqq\frac{2}{w(t)}(\dot{w}_0(t)+\dot{w}(t))\leqq 0,$$

as long as $|x|>1$, and hence Condition (7) holds for $h(t, v)\equiv 0$. Since clearly Condition (6) holds for $a(u)\equiv b(u)\equiv u^2$, and the solutions of each equation of $\dot{u}=0$ and $\dot{v}=0$ are uniformly bounded, all conditions in Theorem 2 hold, and consequently we can conclude that the solutions of Equation (8) are equibounded.

§4. Equiboundedness in functional differential equations

In this section, we discuss equiboundedness of the solutions of Equation (4) by employing Liapunov-Razumikhin method. Corresponding to Theorem 2, we have the following theorem on equiboundedness of the solutions of Equation (4).

THEOREM 3. *Let $L(t, u)$ satisfy the conditions in Theorem 1 and an additional condition that $L(t, u)$ is nondecreasing in t for each fixed u. Suppose that the following conditions hold.*

(i) *For any $t_0\geqq 0$ and any $\eta\geqq 0$, there exists a $u_0=u_0(t_0, \eta)>\eta$ such that $u(t)=u(t, t_0, u_0)$ exists for $t\geqq t_0$, and satisfies $u(t+\theta)\leqq L(t, u(t))$ for $t\geqq t_0$, $t_0-t\leqq\theta\leqq 0$.*

(ii) *$U(t, x)\in C(R\times R^n, I)$ is a Liapunov function which satisfies*

$$U(t, x)\leqq c(s, |x|),\quad t\leqq s,\ x\in R^n,$$

where $c(s, u) \in C(I \times I, I)$ *is nondecreasing in s for each fixed u. Moreover,* $U(t, x)$ *satisfies*

$$U'_{(4)}(t, \phi) \leqq g(t, U(t, \phi(0)))$$

for all functions $\phi \in BC$ *with the property that*

$$U(t+\theta, \phi(\theta)) \leqq L(t, U(t, \phi(0))) \quad \textit{for} \quad \theta \leqq 0.$$

(iii) *For a constant* $\rho > 0$, $V(t, x) \in C(R \times S^c(\rho), I)$ *is a Liapunov function which satisfies*

$$a(|x|) \leqq V(t, x) \leqq b(|x|), \quad (t, x) \in I \times S^c(\rho)$$

where $a, b \in C([\rho, \infty), I)$ *are nondecreasing and* $a(u) \to \infty$ *as* $u \to \infty$.

(iv) *For a constant* $K > 0$, *any* $t_0 \geqq 0$, *and any* $\eta \geqq K$, *we can choose a* $v_0 = v_0(\eta) > \eta$ *such that* $v(t) = v(t, t_0, v_0)$ *exists for* $t \geqq t_0$, *and* $v(t+\theta) \leqq L(t, v(t))$ *for* $t \geqq t_0$, $t_0 - t \leqq \theta \leqq 0$ *as long as* $v(t) \geqq K$.

(v) $U(t, x)$ *and* $V(t, x)$ *satisfy*

$$U'_{(4)}(t, \phi) + V'_{(4)}(t, \phi) \leqq h(t, U(t, \phi(0)) + V(t, \phi(0)))$$

for all functions $\phi \in BC$ *with the properties that*

$$U(t, \phi(0)) + V(t, \phi(0)) > K, \quad |\phi(0)| > \rho,$$

$$U(t+\theta, \phi(\theta)) + V(t+\theta, \phi(\theta)) \leqq L(t, U(t, \phi(0)) + V(t, \phi(0))) \quad \textit{for} \quad \theta \leqq 0 \textit{ with } |\phi(\theta)| > \rho.$$

Then the solutions of Equation (4) *are equibounded, if the solutions of Equations* (2) *and* (3) *are equibounded and uniformly bounded, respectively.*

PROOF. For any $t_0 \geqq 0$ and any $\alpha \geqq \rho$, let $\eta = \max\{c(t_0, u): 0 \leqq u \leqq \alpha\}$, and let $u_0 = u_0(t_0, \eta) > \eta$ be a number in the condition (i). Since the solutions of Equation (2) are equibounded, for $t_0 \geqq 0$ and $u_0 > 0$, there exists a $B_0 = B_0(t_0, u_0) > 0$ such that

$$u(t, t_0, u_0) < B_0, \quad t \geqq t_0. \tag{9}$$

Let $\alpha_0 = \max\{B_0 + b(\alpha), K\}$. For any $\alpha_1 > \alpha_0$, let $v_0 = v_0(\alpha_1) > \alpha_1$ be a number in the condition (iv). Since the solutions of Equation (3) are uniformly bounded, for this v_0, there exists a $B_1(\alpha_1) > 0$ such that for any $t_0 \geqq 0$, $v(t, t_0, v_0) < B_1(\alpha_1)$ for $t \geqq t_0$. As $a(u) \to \infty$ with $u \to \infty$, we can choose a $B = B(t_0, \alpha) > \alpha$ such that

$$a(B) > B_1(\alpha_1). \tag{10}$$

We now claim that for any $t_0 \geqq 0$ and any $\phi \in BC$ with $\|\phi\| < \alpha$, any solution $x(t, t_0, \phi)$ of Equation (4) satisfies $|x(t, t_0, \phi)| < B$ for $t \geqq t_0$. If this is not true, there exists a solution $x(t, t_0, \phi)$ of Equation (4) with $\|\phi\| < \alpha$ such that for some $t^* > t_0$,

$|x(t^*, t_0, \phi)|=B$, $|x(t, t_0, \phi)|<B$ for $t_0 \leqq t<t^*$. First we show that

$$U(t, x(t, t_0, \phi))<B_0 \qquad \text{for} \quad t_0 \leqq t \leqq t^*. \tag{11}$$

If we define a function $u(t)$ by $u(t)=u_0$ for $t<t_0$, $u(t)=u(t, t_0, u_0)$ for $t \geqq t_0$, then $u(t)$ satisfies the conditions in Theorem 1 for $\tau=t_0$, and we have $U(\tau+\theta, \phi(\theta)) \leqq \max\{c(t_0, u): 0 \leqq u \leqq \|\phi\|\} \leqq \eta<u(t_0+\theta)$ for $\theta \leqq 0$. Thus, if we employ Theorem 1 with the case that E is empty, we obtain

$$U(t, x(t, t_0, \phi)) \leqq u(t)<B_0 \qquad \text{for} \quad t_0 \leqq t \leqq t^*$$

from (9), and hence we have (11).

Next, we consider a function $u^*(t)$ defined on $(-\infty, t^*]$. For $t_1=\sup\{t \in [t_0, t^*]: v(s, t_0, v_0)>\alpha_1$ for $t_0 \leqq s \leqq t\}$, we define a function $u^*(t)$ on $(-\infty, t_1)$ by $u^*(t)=v_0$ for $t<t_0$ and $u^*(t)=v(t, t_0, v_0)$ for $t_0 \leqq t<t_1$. If $t_1<t^*$, then for $t_2=\sup\{t \in [t_1, t^*]: v(s, t_1, v_0)>\alpha_1$ for $t_1 \leqq s \leqq t\}$, we similarly define $u^*(t)$ by $u^*(t)=v(t, t_1, v_0)$ for $t_1 \leqq t<t_2$. By repeating this process, we obtain a sequence $t_1<t_2<\cdots$. Since the solutions of Equation (3) are uniformly bounded, each solution $v(t, t_k, v_0)$ satisfies $v(t, t_k, v_0)< B_1(\alpha_1)$ for $t \geqq t_k$. From this and the fact that $\max\{|h(t, v)|: t_0 \leqq t \leqq t^*, |v| \leqq B_1(\alpha_1)\}$ is finite, there exists an integer κ such that $t_\kappa=t^*$. If we define $u^*(t^*)$ by $u^*(t^*)=v(t^*, t_{\kappa-1}, v_0)$, then $u^*(t)$ is a function defined on $(-\infty, t^*]$ which satisfies

$$\alpha_1 \leqq u^*(t)<B_1(\alpha_1) \qquad \text{for} \quad t \leqq t^*. \tag{12}$$

Let $x(t)=x(t, t_0, \phi)$ and $w(t)=U(t, x(t))+V(t, x(t))$ for $t_0 \leqq t \leqq t^*$. We now show that

$$w(t) \leqq u^*(t) \qquad \text{for} \quad t_{k-1} \leqq t \leqq t_k \quad \text{with} \quad |x(t)| \geqq \rho \tag{13}$$

holds for $k=1,\dots, \kappa$. First, let $k=1$. There are two cases to consider.

(i) The case when $|x(t, t_0, \phi)|>\rho$ for $t_0 \leqq t \leqq t_1$. If we define a function $u(t)$ by $u(t)=u^*(t)$ for $t \leqq t_1$, then $u(t)$ satisfies the conditions in Theorem 1 for $\tau=t_0$, and we have

$$U(t_0+\theta, \phi(\theta))+V(t_0+\theta, \phi(\theta)) \leqq B_0+b(\|\phi\|) \leqq B_0+b(\alpha)<u(t_0+\theta) \qquad \text{for} \quad \theta \leqq 0.$$

Thus Theorem 1 with $E=\overline{S(\rho)}$ implies that (13) holds for $k=1$.

(ii) The case when $|x(t, t_0, \phi)| \leqq \rho$ for some $t \in [t_0, t_1)$. First we show that the sequence $\{[r_k, s_k]\}_{k \geqq 1}$ of disjoint intervals, which satisfy the following conditions, is a finite sequence if it exists.

$$\begin{aligned} &t_0 \leqq r_k<s_k \leqq t_1, \quad |x(r_k)|=\rho, \quad w(s_k)=u^*(s_k), \\ &|x(t)|>\rho, \quad w(t)<u^*(t) \qquad \text{for} \quad r_k<t<s_k, \\ &r_k<s_k<r_{k+1} \quad \text{or} \quad r_{k+1}<s_{k+1}<r_k, \ k \geqq 1. \end{aligned} \tag{14}$$

Suppose that $\{[r_k, s_k]\}$ is an infinite sequence, and $r_k < s_k < r_{k+1}$, $k \geqq 1$. Then, r_k and s_k converge to some $\sigma \in [t_0, t_1]$, and clearly we have $|x(\sigma)| = \rho$ and $u^*(t) \to w(\sigma)$ as $t \to \sigma -$. Thus we obtain

$$w(\sigma) \leqq B_0 + b(\rho) \leqq B_0 + b(\alpha) < \alpha_1 .$$

On the other hand, we have $w(\sigma) \geqq \alpha_1$ from (12). This is a contradiction. In the case when $r_{k+1} < s_{k+1} < r_k$, $k \geqq 1$, we arrive at the same contradiction. Thus, $\{[r_k, s_k]\}$ which satisfies Condition (14) is a finite sequence if it exists.

Next, if we define a set T by $T = \{t \in (t_0, t_1): |x(t)| > \rho\}$, then T is expressed by a union of disjoint open intervals. If (t_0, τ) with some $\tau \in (t_0, t_1)$ is an interval constructing T, discussing as in the case (i), we have $w(t) \leqq u^*(t)$ for $t_0 \leqq t \leqq \tau$, since $w(t_0) < u^*(t_0)$ in case $|x(t_0)| = \rho$.

In case $\{[r_k, s_k]\}$ is empty, let (r, s), $r > t_0$ be any open interval constructing T. Then we obtain

$$w(t) < u^*(t) \quad for \quad r < t < s.$$

Moreover, since we have

$$w(t) \leqq B_0 + b(\rho) < u^*(t) \quad for \quad t \in [t_0, t_1] \quad \text{with} \quad |x(t)| = \rho$$

from (11), we can conclude that (13) holds for $k=1$ in this case.

On the other hand, in case $\{[r_k, s_k]\}$ is not empty, we can assume that $r_1 < s_1 < \cdots < r_m < s_m$ by changing the number if necessary. Discussing as in the above, we obtain

$$w(t) \leqq u^*(t) \quad for \quad t_0 \leqq t \leqq s_1 \quad with \quad |x(t)| \geqq \rho.$$

Thus, by employing Theorem 1 for $(s_1, x_{s_1}(t_0, \phi))$ and $u(t)$ in the case (i), we have

$$w(t) \leqq u^*(t) \quad for \quad t > s_1 \quad as\ long\ as\ |x(t)| > \rho.$$

By continuing the same process, we obtain (13) for $k=1$.

If $\kappa = 1$, then $t_1 = t^*$ and the relations (10), (12) and (13) imply

$$a(B) \leqq w(t^*) \leqq u^*(t^*) < B_1(\alpha_1) < a(B) \tag{15}$$

which is a contradiction.

If $\kappa \geqq 2$, define a function $u(t)$ by

$$u(t) = u^*(t), \; t \leqq t_k \quad (2 \leqq k \leqq \kappa).$$

Then $u(t)$ satisfies the conditions in Theorem 1 for $\tau = t_{k-1}$ $(2 \leqq k \leqq \kappa)$. Thus, by discussing similarly as in the above, we again arrive at the contradiction (15). This proves that for any $t_0 \geqq 0$ and any $\phi \in BC$ with $\|\phi\| < \alpha$, any solution $x(t, t_0, \phi)$ of Equation (4) satisfies $|x(t, t_0, \phi)| < B$ for $t \geqq t_0$ if $\alpha \geqq \rho$.

For $\alpha<\rho$, we take $B(t_0, \alpha)=B(t_0, \rho)$. Thus the proof is complete.

Remark. If $F(t, \phi)=f(t, \phi(0))$ for some $f \in C(I\times R^n, R^n)$, then Equation (4) is reduced to the ordinary differential equation (1). For Liapunov functions $U(t, x)$ and $V(t, x)$ which satisfy the conditions in Theorem 2, we can extend $U(t, x)$ and $V(t, x)$ for $t<0$ by defining $U(t, x)=U(0, x)$ and $V(t, x)=V(0, x)$ for $t<0$. Then, for these Liapunov functions and $L(t, u)\equiv u$, it is clearly seen that all conditions in Theorem 3 hold. First we note that θ in Theorem 3 is zero in the case of ordinary differential equations. Now the conditions (i) and (iv) hold from the assumptions for Equations (2) and (3). Moreover, the condition (ii) holds with $c(s, u)=\max\{U(t, x): 0\leqq t\leqq s, |x|\leqq u\}$, and the conditions (iii) and (v) are contained in the condition (ii) of Theorem 2. Thus, Theorem 3 is an extension of Theorem 2.

Now we present a theorem on uniform boundedness of the solutions of Equation (4), which can be proved by similar arguments as in the proof of Theorem 3.

THEOREM 4. *Let L be as in Theorem* 3. *In addition to* (*iii*) *and* (*iv*) *in Theorem* 3, *suppose that we have*

$$V'_{(4)}(t, \phi)\leqq h(t, V(t, \phi(0)))$$

for all functions $\phi \in BC$ with the properties that

$$V(t, \phi(0))>K, \quad |\phi(0)|>\rho,$$

$$V(t+\theta, \phi(\theta))\leqq L(t, V(t, \phi(0))) \quad \text{for} \quad \theta\leqq 0 \quad \text{with} \quad |\phi(\theta)|>\rho.$$

Then the solutions of Equation (4) *are uniformly bounded, if the solutions of Equation* (3) *are uniformly bounded.*

PROOF. If we take $U(t, x)\equiv 0$ and $g(t, u)\equiv 0$ in Theorem 3, we can choose the B_0 in (10) independent of t_0. Thus uniform boundedness of the solutions of Equation (3) implies uniform boundedness of the solutions of Equation (4).

§5. Application of Theorem 3

In this section, we present an application of Theorem 3. Consider the scalar delay equation

$$\dot{x}(t)=(6t\sin t-2t)x(t)-c(t)x(t)+F(t, x_t), \tag{16}$$

where $c \in C(I, I)$, and $F \in C(I\times BC, R)$ satisfies

$$|F(t, \phi)|\leqq d(t)\|\phi\|, \quad t\geqq 0, \quad |\phi(0)|\geqq \rho,$$

where $d \in C(I, I)$ and ρ is a nonnegative constant. We assume that for any $t_0 \in I$

and any $\phi \in BC$, the solution $x(t, t_0, \phi)$ of Equation (16) exists locally, and is continuable as long as it is bounded. For the functions $w(t)$ and $w_0(t)$ in §3, we define the function $\rho(t)$ and the number ω by

$$\rho(t) = \frac{1}{w(t)} \sup_{0 \leqq s < t} w(s),$$

$$\omega = \sup_{0 \leqq s < t} \left(\frac{w_0^2(t) + w^2(t)}{w_0^2(s) + w^2(s)} \right)^{\frac{1}{2}},$$

where it is easily seen that ω satisfies $1 < \omega \leqq \sqrt{2}$. Moreover, we assume that

$$d(t) \leqq \frac{c(t)}{\omega \rho(t)}.$$

Then we have the proposition.

PROPOSITION. *Suppose that all hypothesis above for* $c(t)$ *and* $F(t, \phi)$ *are satisfied. Then the solutions of Equation* (16) *are equibounded.*

PROOF. If we define $U(t, x)$ and $V(t, x)$ by $U(t, x) = (w_0^2(t)/w^2(t))x^2$ for $t \geqq 0$, $U(t, x) = (w_0^2(0)/w^2(0))x^2$ for $t < 0$, and $V(t, x) = x^2$ for $|x| \geqq 1$, then U and V are Liapunov functions. Differentiating U along the solutions of Equation (16) we have

$$\begin{aligned} U'_{(16)}(t, x_t) &= \frac{2w_0(t)(\dot{w}_0(t)w(t) - w_0(t)\dot{w}(t))}{w^3(t)} x^2(t) + \frac{2w_0^2(t)(6t\sin t - 2t)}{w^2(t)} x^2(t) \\ &\quad + \frac{2w_0^2(t)}{w^2(t)} (F(t, x_t) - c(t)x(t))x(t) \\ &= \frac{2w_0(t)\dot{w}_0(t)}{w^2(t)} x^2(t) + \frac{2w_0^2(t)}{w^2(t)} (F(t, x_t) - c(t)x(t))x(t) \\ &\leqq \frac{2w_0^2(t)}{w^2(t)} (F(t, x_t) - c(t)x(t))x(t) \end{aligned}$$

by the similar calculation as in §3. For $L(t, u) \equiv u$, if $U(s, x(s)) \leqq L(t, U(t, x(t))$ for $s \leqq t$, then we obtain $\|x_t\| \leqq \rho(t)|x(t)|$ and

$$\begin{aligned} U'_{(16)}(t, x_t) &\leqq \frac{2w_0^2(t)}{w^2(t)} \left(\frac{|F(t, x_t)|}{\|x_t\|} \frac{\|x_t\|}{|x(t)|} - c(t) \right) x^2(t) \\ &\leqq \frac{2w_0^2(t)}{w^2(t)} (d(t)\rho(t) - c(t))x^2(t) \leqq 0 \end{aligned}$$

if $|x(t)| > 1$. Thus the condition (ii) holds with $g(t, u) \equiv 0$. Next, differentiating $U + V$ along the solutions of Equation (16) we have

$$U'_{(16)}(t, x_t)+V'_{(16)}(t, x_t)$$
$$\leqq U'_{(16)}(t, x_t)+2x(t)((6sint-2t)x(t)-c(t)x(t)+F(t, x_t))$$
$$\leqq \frac{2x^2(t)}{w^2(t)}(w_0(t)\dot{w}_0(t)+w(t)\dot{w}(t))+2\Big(\frac{w_0^2(t)}{w^2(t)}+1\Big)x(t)(F(t, x_t)-c(t)x(t))$$
$$\leqq 2\Big(\frac{w_0^2(t)}{w^2(t)}+1\Big)x(t)(F(t, x_t)-c(t)x(t))$$

by the similar calculation as in §3, if $|x(t)|>1$. If $U(s, x(s))+V(s, x(s))\leqq U(t, x(t))+V(t, x(t))$ for $s\leqq t$ with $|x(s)|>1$ and $|x(t)|>1$, then we obtain $\|x_t\|\leqq \omega\rho(t)\,|x(t)|$ and

$$U'_{(16)}(t, x_t)+V'_{(16)}(t, x_t)\leqq 2\Big(\frac{w_0^2(t)}{w^2(t)}+1\Big)\Big(\frac{|F(t, x_t)|}{\|x_t\|}\frac{\|x_t\|}{|x(t)|}-c(t)\Big)x^2(t)$$
$$\leqq 2\Big(\frac{w_0^2(t)}{w^2(t)}+1\Big)(\omega d(t)\rho(t)-c(t))x^2(t)\leqq 0.$$

Thus the condition (v) holds with $h(t, v)\equiv 0$ and $K=\rho=1$. Moreover it is easily seen that other conditions in Theorem 3 are satisfied, and consequently we can conclude from Theorem 3 that the solutions of Equation (16) are euqibounded.

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