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Canonical Connections of Homogeneous Lie Loops and 3-Webs

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A differentiable left I. P. loop (G, μ) admits on the tangent space at the unit element two kinds of bilinear operations $d\mu$ and dL which are induced from the multiplication μ and left inner mappings. In this paper, after recalling some formulas of the Chern connection of a local 3-web of a differentiable loop, relations between this connection and bilinear operations above are investigated in differentiable left I. P. loops. The results are applied to homogeneous Lie loops and it is shown that the bilinear- and trilinear products of the tangent Lie triple algebras are given by the torsion and the curvature of the Chern connections.

§1. Introduction

A quasigroup (G, μ) with the multiplication $xy=\mu(x, y)$ for x, y in G is called a *loop* if it has a unit element e in G (cf. [1], [7]). We denote by L_x the left translation of a loop G by an element x. A loop G has the *left inverse property* if, for every x in G, there exists a two-sided inverse x^{-1} of x such that $L_{x^{-1}}L_x=1_G$ (the identity map on G). Such a loop will be called a *left I. P. loop*. For any two elements a, b of a loop G, the permutation of G given by $L_{a,b}=L_{ab}^{-1}L_aL_b$ is called a *left inner mapping* of G. A *homogeneous loop* G is a left I. P. loop in which all left inner mappings are automorphisms of G (cf. [12], [13], [25]).

Let G be a left I. P. loop. For x, y, z in G, we set

$$\eta(x, y, z)=x((x^{-1}y)(x^{-1}z)).$$

Then, the ternary system $\eta\colon G\times G\times G\to G$ satisfies the following relations;

$$(\mathrm{H}_1)\qquad \eta(x, x, y)=y,$$

$$(\mathrm{H}_2)\qquad \eta(x, y, x)=y,$$

$$(\mathrm{H}_3')\qquad \eta(x, e, \eta(e, x, y))=\eta(e, x, \eta(x, e, y))=y,$$

where e denotes the unit element of G. In this case, for any fixed x in G, the multiplication μ_x given by $\mu_x(y, z)=\eta(x, y, z)$ makes G a left I. P. loop with the unit element x. An isomorphism $(G, \mu)\cong(G, \mu_x)$ holds for every x and, especially, $\mu_e=\mu$. Assume that (G, μ) is a homogeneous loop. Then, (G, μ_x) is homogeneous, and the ternary operation η on G satisfies the additional relations (cf. [15]);

$$(\mathrm{H}_3)\qquad \eta(x, y, \eta(y, x, z))=z,$$

$$(\mathrm{H}_4)\qquad \eta(x, y, \eta(u, v, w))=\eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)).$$

In this case, the permutation $\eta(x, y)$ of G given by $\eta(x, y)z=\eta(x, y, z)$ induces an isomorphism $(G, \mu_x)\cong(G, \mu_y)$ for any x, y in G.

In general, a ternary system $\eta: G\times G\times G\to G$ satisfying the relations (H_1), (H_2), (H_3) and (H_4) is called a *homogeneous system* on G, and the permutations $\eta(x, y)$ of G are called *displacements* of (G, η) (cf. [15], [16]). It has been shown that the displacement $\eta(x, y)$ from x to any point y of a homogeneous system (G, η) has exactly the same properties as those of the left translations of the homogeneous loop (G, μ_x) at x given above.

In 1930's, it was found that the concept of loops is closely connected with the concept of 3-webs (cf. [6], [24]). Let (G, μ) be a loop. We consider three families $\sigma^{(\rho)}$, $\rho=1, 2, 3$, of subsets of $W=G\times G$ as follows: For any $g\in G$, we set $F_{(1)}(g)=\{(g, v)|v\in G\}$, $F_{(2)}(g)=\{(u, g)|u\in G\}$ and $F_{(3)}(g)=\{(u, v)|\mu(u, v)=g\}$, and we call them *vertical lines*, *horizontal lines* and *transversal lines* of W, respectively. Then, $\sigma^{(\rho)}=\{F_{(\rho)}(g)|g\in G\}$, $\rho=1, 2, 3$, satisfy the following axioms (W_1), (W_2) of 3-web on W (cf. [5], [6]);

(W_1) Each point in W is contained in exactly one line of every $\sigma^{(\rho)}$, $\rho=1, 2, 3$.

(W_2) Two lines of different families have exactly one point in common.

Moreover, if (G, μ) is non-trivial, we have

(W_3) There exist three lines $F_{(\rho)}$, $\rho=1, 2, 3$, which contain no point in common.

In general, a set W with three families $\sigma^{(\rho)}$, $\rho=1, 2, 3$, of subsets $F_{(\rho)}$ satisfying (W_1) and (W_2) is called a 3-*web* and it is said to be *non-degenerate* if (W_3) is satisfied. Let W be a non-degenerate 3-web with families $\sigma^{(\rho)}$, $\rho=1, 2, 3$, of '*lines*' in W. For any fixed vertical line $G=F_{(1)}$, choose a point e of G fixed. Then, we can define a multiplication on G in the following manner (Fig. 1): Let G' be the horizontal line through e. For any two points $x, y\in G$, let $x'\in G'$ be the intersection of the transversal line through x with G', and let $P(x, y)$ be the point of intersection of the vertical line through x' and the horizontal line through y. Then, we get the point $\mu(x, y)$ of G as the intersection of the transversal line through $P(x, y)$ with the vertical line G. We can check easily that (G, μ) is a loop with the unit element e, and that the 3-web of this loop constructed on $G\times G$ is equivalent to the given 3-web W. Here, 3-webs W and W' are *equivalent* if there exists a bijection $\alpha: W\to W'$ under which the vertical-, horizontal- and transversal lines are preserved, respectively. In our case, the equivalence is given by $\mathrm{P}: G\times G\to W$ sending (x, y) into the point $P(x, y)$.

Let (G, μ) be a loop represented on the vertical line $\{e\}\times G$ of the 3-web $W=G\times G$. In [6], it was shown that each element x of the loop (G, μ) has a two-sided inverse x^{-1} if and only if the corresponding 3-web W is *hexagonal* at e (Fig. 2), that is,

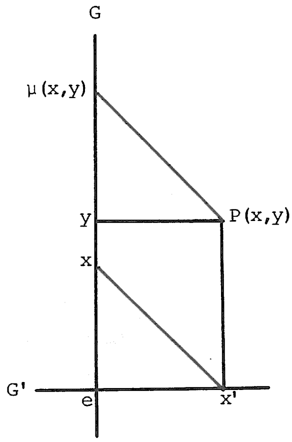

Fig. 1

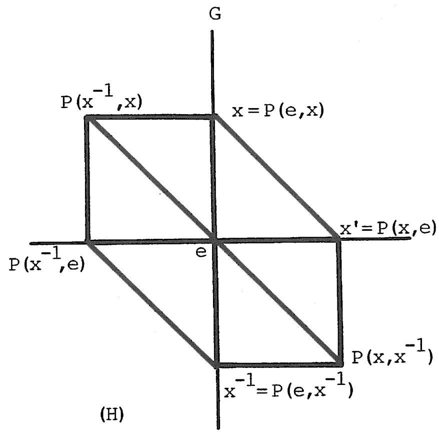

Fig. 2

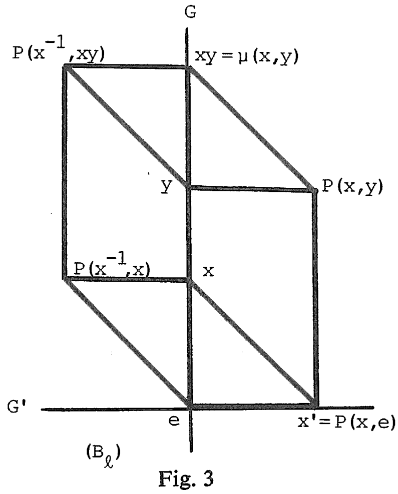

Fig. 3

the hexagon (H) in Fig. 2 with the center e is a closed figure, and shown that (G, μ) has the left inverse property if and only if W satisfies the *B_l-closure condition of Bol* along the vertical line $\{e\} \times G$, that is, the figure (B_l) in Fig. 3 is closed. Note that these conditions are weaker than the corresponding closure conditions appeared in [6] which are to hold at any place in W. The closed figure which characterizes homogeneous loops is rather complicated and we omit to illustrate it. However, here is a special class of homogeneous loops, K-homogeneous loops, which can be characterized by the closed figure (K) in Fig. 4. A *K-homogeneous loop* is a left I. P. loop with the following property: (K) For any x and y in G, there exists a unique element z in G such that $L_x L_y = L_z L_x$. In fact, we can show that the left inner mappings of a left

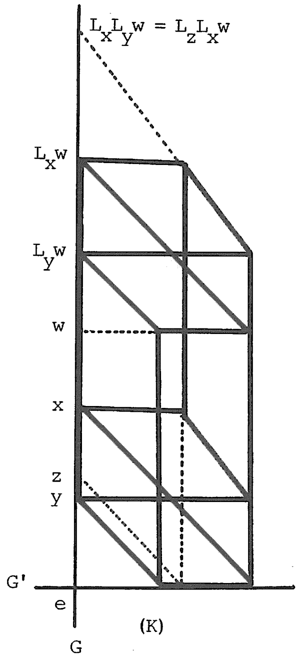

Fig. 4

I. P. loop G are automorphisms of G if the condition (K) is satisfied (Proposition 4).

In 1936, S. S. Chern introduced in his thesis [9] a differential geometric method to the theory of 3-webs in differentiable manifolds whose 'lines' are given by differential systems, and found an affine connection on 3-webs invariant under differentiable equivalences of 3-webs, which we will call the Chern connection in this paper. He characterized closure conditions of some figures in 3-webs by means of relations of the torsion and the curvature of the Chern connection.

In this paper, after recalling the formulas for the torsion and the curvature of the Chern connection, we will apply them to 3-webs of differentiable left I. P. loops and of differentiable homogeneous loops (homogeneous Lie loops), and then clarify the interrelation between the canonical connection of homogeneous Lie loops and the bilinear operations on the tangent spaces at the unit elements, induced by multiplications and left inner mappings of the loops. On our way to this investigation, we will get the relation of tangent Lie triple algebras of homogeneous Lie loops and their Akivis algebras (cf. [3] or Ch. IX of [8]). Recently, K. H. Hofmann-K. Strambach have treated it in [12]. We will also show that geodesic K-homogeneous Lie loops are reduced to Lie groups.

§ 2. Chern connections of 3-webs

In this section, we will recall the results about Chern connections of differentiable 3-webs in [9]. Let M be a $2r$-dimensional differentiable manifold of class C^∞. A *differentiable* 3-*web* W of dimension r (codimension r) in M is a triple of foliations $\sum^{(\rho)}$, $\rho=1, 2, 3$, of codimension r on an open subset W of M with the following properties: Given any point p in W, there exists exactly one leaf $F_{(\rho)}$ of $\sum^{(\rho)}$ through p, for each $\rho=1, 2, 3$, such that the tangent spaces $T_p(F_{(\rho)})$ of $F_{(\rho)}$ at p satisfy $\langle T_p(F_{(\rho)}) \cap T_p(F_{(\tau)})\rangle=\{0\}$ and $\langle T_p(F_{(\rho)}) \cup T_p(F_{(\tau)})\rangle=T_p(W)$ for $\rho\neq\tau$, $1\leqq\rho, \tau\leqq 3$, with $\langle\ \rangle$ denoting the linear span in the tangent space $T_p(W)$. This can be described in terms of differential systems as follows: A differentiable 3-web W with foliations $\sum^{(\rho)}$, $\rho=1, 2, 3$, is given by three families of involutive differential systems $\omega^1_{(\rho)},\dots, \omega^r_{(\rho)}$ on W with which the foliation $\sum^{(\rho)}$ is defined by the equations $\omega^k_{(\rho)}=0$, $k=1,\dots, r$, where each of the families of 1-forms $\{\omega^1_{(1)},\dots, \omega^r_{(1)}; \omega^1_{(2)},\dots, \omega^r_{(2)}\}$, $\{\omega^1_{(1)},\dots, \omega^r_{(1)}; \omega^1_{(3)},\dots, \omega^r_{(3)}\}$ and $\{\omega^1_{(2)},\dots, \omega^r_{(2)}; \omega^1_{(3)},\dots, \omega^r_{(3)}\}$ is linearly independent at each point of W. In this case, we can choose the forms $\omega^k_{(\rho)}$ to satisfy

$$\omega^k_{(3)}=\omega^k_{(1)}+\omega^k_{(2)}, \quad k=1,\dots, r, \tag{2.1}$$

without loss of generality.

Let W be a differentiable 3-web of codimension r in a differentiable manifold M of dimension $2r$ and assume that it is given by involutive differential systems $\{\omega^k_{(\rho)}\}$, $\rho=1, 2, 3$, satisfying (2.1). Then there exist 1-forms $\theta^{(\rho)k}_i$ on W such that

$$d\omega^k_{(\rho)}=\omega^i_{(\rho)}\wedge\theta^{(\rho)k}_i. \tag{2.2}$$

If we put

$$\begin{aligned}\theta^{(3)k}_i&=\theta^{(2)k}_i+A^k_{ij}\omega^j_{(1)}+B^k_{ij}\ \omega^j_{(2)}\\ &=\theta^{(1)k}_i+C^k_{ij}\ \omega^j_{(1)}+D^k_{ij}\omega^j_{(2)},\end{aligned} \tag{2.3}$$

we can show the relations $B^k_{ij}=B^k_{ji}$, $C^k_{ij}=C^{\ k}_{ji}$ and $A^k_{ij}-A^k_{ji}=D^k_{ij}-D^k_{ji}$, by using $d\omega^k_{(3)}=d\omega^k_{(1)}+d\omega^k_{(2)}$. In the following, we denote

$$a^k_{ij}=A^k_{ij}-A^k_{ji}. \tag{2.4}$$

Also, we can show that the forms $\theta^{(1)k}_i-\theta^{(2)k}_i$ are described as follows:

$$\theta^{(1)k}_i-\theta^{(2)k}_i=A^{(1)k}_{ij}\ \omega^j_{(1)}+A^{(2)k}_{ij}\omega^j_{(2)}, \tag{2.5}$$

where $A^{(1)k}_{ij}-A^{(1)k}_{ji}=A^{(2)k}_{ij}-A^{(2)k}_{ji}=a^k_{ij}$. Now, from (2.5) we can consider 1-forms ω^k_i given by

$$\omega^k_i=\theta^{(k)}_i-A^{(1)k}_{ij}\omega^j_{(1)}=\theta^{(2)k}_i+A^{(2)k}_{ij}\ \omega^j. \tag{2.6}$$

From (2.2) and (2.3) the following formulas follow:

$$(2.7)\qquad \begin{aligned} d\omega^k_{(1)} &= \omega^i_{(1)} \wedge \omega^k_i + a^k_{ij}\, \omega^i_{(1)} \wedge \omega^j_{(1)} \\ d\omega^k_{(2)} &= \omega^i_{(2)} \wedge \omega^k_i - a^k_{ij}\, \omega^i_{(2)} \wedge \omega^j_{(2)}, \end{aligned}$$

$$(2.8)\qquad d\omega^k_j = \omega^i_j \wedge \omega^k_i + b^k_{j\,lm}\omega^l_{(1)} \wedge \omega^m_{(2)},$$

with differentiable functions a^k_{ij} and $b^k_{j\,lm}$ satisfying, $a^k_{ij}+a^k_{ji}=0$. The *Chern connection* of the 3-web W is, by definition, the affine connection on W whose connection forms $\tilde{\omega}^\alpha_\beta$, $1\leqq\alpha$, $\beta\leqq 2r$, are given by $\tilde{\omega}^k_i=\tilde{\omega}^{r+k}_{r+i}=\omega^k_i$ and $\tilde{\omega}^k_{r+i}=\tilde{\omega}^{r+k}_i=0$, $1\leqq i, k\leqq r$, with respect to the linearly independent 1-forms $\{\omega^1_{(1)},\dots,\omega^r_{(1)};\ \omega^r_{(2)},\dots,\omega^1_{(2)}\}$. The formulas (2.7) and (2.8) show that the torsion tensor $\tilde{T}^\alpha_{\beta\gamma}$ and the curvature tensor $\tilde{R}^\alpha_{\beta\gamma\delta}$ with respect to the base forms $\{\omega^k_{(1)};\ \omega^k_{(2)}\}$ are given by*)

$$\tilde{T}^k_{ij} = -\tilde{T}^{r+k}_{r+i\,r+j} = -2a^k_{ij} \text{ and } \tilde{T}^\alpha_{\beta\gamma}=0 \text{ otherwise};$$

$$\tilde{R}^k_{j\,l\,r+m} = -\tilde{R}^k_{j\,r+m\,l} = \tilde{R}^{r+k}_{r+j\,l\,r+m} = -\tilde{R}^{r+k}_{r+j\,r+l\,m} = -b^k_{j\,lm},$$

$$\tilde{R}^\alpha_{\beta\gamma\delta}=0 \text{ otherwise}.$$

For brevity, we will call the functions a^k_{ij} and $b^k_{j\,lm}$ the *torsion* and the *curvature* of the Chern connection of the 3-web. One of the most significant facts is that the Chern connection is invariant under any diffeomorphism which induces an equivalence of 3-webs.

REMARK. In [10] and [11], V. Goldberg introduced another connection on 3-web by choosing $\theta^{(3)k}_i$ as the connection forms with respect to the base forms $\{\omega^k_{(1)};\ \omega^k_{(2)}\}$, and he generalized it on d-webs $w(d, n, r)$ of codimension r in nr-dimensional manifolds (cf. Ch. X of [8]).

Now, we show some relations of the torsion and the curvature of the Chern connection of a differentiable 3-web, which owe to S. S. Chern [9]. By substituting the first equation of (2.7) to the equation $dd\omega^k_{(1)}=0$, we have

$$\omega^i_{(1)} \wedge (d\omega^k_i - \omega^j_i \wedge \omega^k_j) = (\nabla^{(1)}_j a^k_{lm} + a^k_{im}a^i_{jl} + a^k_{li}a^i_{jm})\omega^j_{(1)} \wedge \omega^l_{(1)} \wedge \omega^m_{(1)} + \nabla^{(2)}_j a^k_{lm}\omega^l_{(1)} \wedge \omega^m_{(1)} \wedge \omega^j_{(2)},$$

where

$$\nabla^{(\alpha)}_j a^k_{lm}\omega^j_{(\alpha)} = \partial^{(\alpha)}_j a^k_{lm}\omega^j_{(\alpha)} + a^j_{lm}\omega^{(\alpha)k}_j - a^k_{jm}\,\omega^{(\alpha)j}_l - a^k_{lj}\omega^{(\alpha)j}_m,$$

$$da^k_{lm} = \partial^{(1)}_j a^k_{lm}\omega^j_{(1)} + \partial^{(2)}_j a^k_{lm}\omega^j_{(2)},$$

$$\omega^k_j = \omega^{(1)k}_j + \omega^{(2)k}_j,\quad \omega^{(\alpha)}_j = \Gamma^{(\alpha)k}_{ij}\omega^i_{(\alpha)},\ \alpha=1, 2.$$

* In this paper, we adopt the opposite signs of torsion and curvature of affine connection to those appeared in usual bibliography.

Then, from (2.8), we obtain

$$(\nabla_j^{(1)} a^k_{lm} + a^k_{im} a^i_{jl} - a^k_{il} a^i_{jm})\omega^j_{(1)} \wedge \omega^l_{(1)} \wedge \omega^m_{(2)} = 0,$$

$$(\nabla_j^{(2)} a^k_{lm} - b^k_{lmj})\omega^l_{(1)} \wedge \omega^m_{(1)} \wedge \omega^j_{(2)} = 0,$$

which imply the following formulas:

$$\text{(2.9)} \qquad \begin{aligned} &\mathfrak{S}_{j,l,m}(\nabla_j^{(1)} a^k_{lm} + 2a^k_{im} a^i_{jl}) = 0, \\ &2\nabla_j^{(2)} a^k_{lm} = b^k_{lmj} - b^k_{mlj}, \end{aligned}$$

where $\mathfrak{S}_{j,l,m}$ denotes the cyclic summation with respect to j, l, m. In the same way, by the equation $dd\omega^k_{(2)} = 0$, we get

$$\text{(2.10)} \qquad \begin{aligned} &\mathfrak{S}_{j,l,m}(\nabla_j^{(2)} a^k_{lm} + 2a^k_{il} a^i_{jm}) = 0, \\ &2\nabla_j^{(1)} a^k_{lm} = b^k_{ljm} - b^k_{mjl}. \end{aligned}$$

By the formulas (2.9) and (2.10) we have

$$\text{(2.11)} \qquad \begin{aligned} &\mathfrak{S}_{j,l,m}(b^k_{jlm} - b^k_{ljm}) = 4\mathfrak{S}_{j,l,m} a^k_{ij} a^i_{lm}, \\ &2\nabla a^k_{lm} = (b^k_{ljm} - b^k_{mjl})\omega^j_{(1)} + (b^k_{lmj} - b^k_{mlj})\omega^j_{(2)}. \end{aligned}$$

On the other hand, by using (2.7) and (2.8), the following formulas are obtained from the equation $dd\omega^k_j = 0$:

$$\text{(2.12)} \qquad \begin{aligned} &\nabla_p^{(1)} b^k_{jlm} - \nabla_l^{(1)} b^k_{jpm} = 2\, b^k_{jim} a^i_{lp}, \\ &\nabla_p^{(2)} b^k_{jlm} - \nabla_m^{(2)} b^k_{jlp} = 2 b^k_{jli} a^i_{pm}, \end{aligned}$$

where

$$\nabla^{(\alpha)} b^k_{jlm} = \partial_p^{(\alpha)} b^k_{jlm} \omega^p_{(\alpha)} + b^i_{jlm} \omega_i^{(\alpha)k} - b^k_{ilm} \omega_j^{(\alpha)i}$$
$$- b^k_{jim} \omega_l^{(\alpha)i} - b^k_{jli} \omega_m^{(\alpha)i}, \quad \alpha = 1, 2.$$

§3. 3-Webs of differentiable left I. P. loops

Let (G, μ) be a differentiable loop of dimension r with unit element e. As considered in §1, a differentiable 3-web of (G, μ) on $G \times G$ is given by the following three families $\sigma^{(\rho)}$, $\rho = 1, 2, 3$, of r-dimensional submanifolds $F_{(\rho)}(g)$, $g \in G$, of $G \times G$, i.e., vertical lines $F_{(1)}(g) = \{g\} \times G$, horizontal lines $F_{(2)}(g) = G \times \{g\}$ and transversal lines $F_{(3)}(g) = \{(u, v) | \mu(u, v) = g\}$. Let U be a coordinate neighborhood of e and choose a neighborhood V of e such that $\mu(V, V)$ is contained in U. Then, for the coordinate neighborhood $W = V \times V$ of (e, e) in $G \times G$ with coordinates $(u^1, \ldots, u^r; v^1, \ldots, v^r)$, the 3-web $\{\sigma^{(\rho)}\}$ on $G \times G$ induces a local 3-web on W defined by the following differential systems; $\sum^{(1)}=\sigma^{(1)}|_W$, $du^k=0$; $\sum^{(2)}=\sigma^{(2)}|_W$, $dv^k=0$; $\sum^{(3)}=\sigma^{(3)}|_W$, $d\mu^k=0$. We set $\omega^k_{(1)}=P^k_i du^i$, $\omega^k_{(2)}=Q^k_i dv^i$ and $\omega^k_{(3)}=d\mu^k$ for $P^k_i=\partial_{u^i}\mu^k$ and $Q^k_i=\partial_{v^i}\mu^k$, where $\mu^k(u,v)=\mu^k(u^1,\dots,u^r;v^1,\dots,v^r)$. Then the 3-web $\{\sum^{(\rho)}\}$ on W is described by the equations $\omega^k_{(\rho)}=0$ with $\omega^k_{(3)}=\omega^k_{(1)}+\omega^k_{(2)}$, so that we can apply the results in §2. The $r\times r$-matrices (P^k_i) and (Q^k_i) are nonsingular on W whose inverse matrices will be denoted by $(\tilde{P}^i_k)$ and $(\tilde{Q}^i_k)$, respectively. Since $d\omega^k_{(1)}=\tilde{P}^i_j dP^k_i\wedge\omega^j_{(1)}$ and $d\omega^k_{(2)}=\tilde{Q}^i_j dQ^k_i\wedge\omega^j_{(2)}$, we can write the 1-forms $\theta^{(1)k}_j$ and $\theta^{(2)k}_j$ given in (2.2) by the following manner;

$$\begin{aligned}\theta^{(1)k}_j&=(\beta^k_{ji}-\tilde{P}^p_j\tilde{P}^q_i\partial_{u^q}P^k_p)\omega^i_{(1)}-\tilde{P}^p_j\tilde{Q}^q_i\partial_{v^q}P^k_p\omega^i_{(2)},\\ \theta^{(2)k}_j&=-\tilde{P}^q_i\tilde{Q}^p_j\partial_{u^q}Q^k_p\omega^i_{(1)}+(\gamma^k_{ij}-\tilde{Q}^q_i\tilde{Q}^p_j\partial_{v^q}Q^k_p)\omega^i_{(2)},\end{aligned}\tag{3.1}$$

where β^k_{ij} and γ^k_{ij} are suitable functions satisfying $\beta^k_{ij}=\beta^k_{ji}$ and $\gamma^k_{ij}=\gamma^k_{ji}$. From (3.1) we can obtain the torsion a^k_{ij} of the Chern connection in the following;

$$a^k_{ij}=\frac{1}{2}(\tilde{P}^p_j\tilde{Q}^q_i-\tilde{P}^p_i\tilde{Q}^q_j)\partial_{u^p}\partial_{v^q}\mu^k.\tag{3.2}$$

For the connection form ω^k_i of the Chern connection, we have

$$\omega^k_j=\tilde{\Gamma}^k_{ij}\omega^i_{(1)}+\tilde{\Gamma}^k_{ji}\omega^i_{(2)},\quad \tilde{\Gamma}^k_{ij}=-\tilde{P}^p_i\tilde{Q}^q_j\partial_{u^p}\partial_{v^q}\mu^k.\tag{3.3}$$

A straightforward calculation of the 2-form $d\omega^k_j-\omega^i_j\wedge\omega^k_i$ shows the followings:

$$b^k_{jlm}=\partial^{(1)}_l\tilde{\Gamma}^k_{jm}-\partial^{(2)}_m\tilde{\Gamma}^k_{lj}+\tilde{\Gamma}^k_{ij}\tilde{\Gamma}^i_{lm}+\tilde{\Gamma}^k_{li}\tilde{\Gamma}^i_{jm}-\tilde{\Gamma}^k_{ji}\tilde{\Gamma}^k_{lm}-\tilde{\Gamma}^k_{im}\tilde{\Gamma}^i_{lj},\tag{3.4}$$

and

$$\begin{aligned}\partial^{(1)}_l\tilde{\Gamma}^k_{mj}-\partial^{(1)}_m\tilde{\Gamma}^k_{lj}&=\tilde{\Gamma}^k_{mi}\tilde{\Gamma}^i_{lj}-\tilde{\Gamma}^k_{li}\tilde{\Gamma}^i_{mj}\\ \partial^{(2)}_l\tilde{\Gamma}^k_{jm}-\partial^{(2)}_m\tilde{\Gamma}^k_{jl}&=\tilde{\Gamma}^k_{im}\tilde{\Gamma}^i_{jl}-\tilde{\Gamma}^k_{il}\tilde{\Gamma}^i_{jm},\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}\partial^{(1)}_l\tilde{\Gamma}^k_{ij}&=\tilde{P}^m_l\partial_{u^m}\tilde{\Gamma}^k_{ij}\\ \partial^{(2)}_l\tilde{\Gamma}^k_{ij}&=\tilde{Q}^m_l\partial_{v^m}\tilde{\Gamma}^k_{ij}.\end{aligned}\tag{3.6}$$

Together with these formulas we get a local expression of the curvature b^k_{jlm} of the Chern connection as follows:

$$\begin{aligned}b^k_{jlm}=&-\tilde{P}^p_l\tilde{P}^q_j\tilde{Q}^r_m\,\partial_{u^p}\partial_{u^q}\partial_{v^r}\mu^k\\ &+\tilde{P}^p_l\tilde{Q}^q_j\tilde{Q}^r_m\partial_{u^p}\partial_{v^q}\partial_{v^r}\mu^k\\ &+\tilde{P}^p_i\tilde{Q}^q_m\partial_{u^p}\partial_{v^q}\mu^k(\tilde{P}^r_l\tilde{P}^s_j\partial_{u^r}\partial_{u^s}\mu^i-\tilde{P}^r_l\tilde{Q}^s_j\partial_{u^r}\partial_{v^s}\mu_i)\\ &+\tilde{P}^p_l\tilde{Q}^q_i\partial_{u^p}\partial_{v^q}\mu^k(\tilde{P}^r_j\tilde{Q}^s_m\partial_{u^r}\partial_{v^s}\mu^i-\tilde{Q}^r_j\tilde{Q}^s_m\partial_{v^r}\partial_{v^s}\mu^i).\end{aligned}\tag{3.7}$$

Hence, we have the following;

PROPOSITION 1. *Let (G, μ) be a differentiable loop of dimension r. Choose a coordinate neighborhood of the unit element e. Then, for a neighborhood V of e, the coefficients of the Chern connection, its torsion and curvature of the local 3-web on $W=V\times V$ are evaluated at (e, e) as follows:*

$$\tilde{\Gamma}^k_{ij}(e, e)=-\partial_{u^i}\partial_{v^j}\mu^k(e, e) \tag{3.8}$$

$$a^k_{ij}(e, e)=\frac{1}{2}(\partial_{u^j}\partial_{v^i}\mu^k(e, e)-\partial_{u^i}\partial_{v^j}\mu^k(e, e)), \tag{3.9}$$

$$\begin{aligned} b^k_{jlm}(e, e)&=\partial_{u^l}\partial_{v^j}\partial_{v^m}\mu^k(e, e)-\partial_{u^l}\partial_{u^j}\partial_{v^m}\mu^k(e, e)\\ &\quad+\tilde{\Gamma}^k_{li}(e, e)\tilde{\Gamma}^i_{jm}(e, e)-\tilde{\Gamma}^k_{im}(e, e)\tilde{\Gamma}^i_{lj}(e, e), \end{aligned} \tag{3.10}$$

where the components are indexed with respect to the base forms $\{\omega^k_{(1)}; \omega^k_{(2)}\}$ with $\omega^k_{(1)}=\partial_{u^i}\mu^k du^i$ and $\omega^k_{(2)}=\partial_{v^i}\mu^k dv^i$.

PROOF. To show the formulas, we are only to notice the following facts in the formulas (3.2), (3.3) and (3.7); $P^k_i(e, e)=Q^k_i(e, e)=\delta^k_i$, $\tilde{P}^i_k(e, e)=\tilde{Q}^i_k(e, e)=\delta^i_k$ and $\partial_{u^i}\partial_{u^j}\mu^k(e, e)=\partial_{v^i}\partial_{v^j}\mu^k(e, e)=0$. q.e.d.

In the rest of this section, we assume that the differentiable loop (G, μ) has the left inverse property, that is, there exists an inverse x^{-1} of each x in G such that $x^{-1}(xy)=y$ for $y\in G$. We use the notation J, L_a and $L_{a,b}$ for the transformations of G defined by the inversion $J(x)=x^{-1}$, left translations $L_a(x)=ax$ and left inner mappings $L_{a,b}=L^{-1}_{ab}L_aL_b$. Choose a coordinate neighborhood U of e and denote by $\mu^k(u^1,\dots, u^r; v^1,\dots, v^r)$ the coordinates of $\mu(u, v)$ when $\mu(u, v)\in U$. We investigate the relations of the torsion and the curvature of the Chern connection at the point (e, e) of the 3-web $W=V\times V$. The following is seen immediately from $\mu(u, e)=u$ and $\mu(e, v)=v$:

$$\begin{aligned} &\partial_{u^i}\mu^k(u, e)=\delta^k_i,\ \partial_{v^i}\mu^k(e, v)=\delta^k_i\\ &\partial_{u^i}\partial_{u^j}\cdots\partial_{u^m}\mu^k(u, e)=0,\\ &\partial_{v^i}\partial_{v^j}\cdots\partial_{v^m}\mu^k(e, v)=0. \end{aligned} \tag{3.11}$$

By the relation

$$\mu(x, x^{-1})=\mu(x^{-1}, x)=e, \tag{3.12}$$

we have

$$\begin{aligned} &\partial_iJ^k(e)=-\delta^k_i,\\ &\partial_i\partial_jJ^k(e)=\partial_{u^i}\partial_{v^j}\mu^k(e, e)+\partial_{u^j}\partial_{v^i}\mu^k(e, e). \end{aligned} \tag{3.13}$$

Furthermore, from the third order partial derivatives of the equations (3.12), the followings are obtained:

$$\partial_j\partial_l\partial_m J^k(e)=\mathfrak{S}_{j,l,m}(\partial_{u^j}\partial_{v^l}\partial_{v^m}\mu^k-\partial_{u^l}\partial_{u^m}\partial_{v^j}\mu^k-\tilde{\Gamma}^k_{ij}(\tilde{\Gamma}^i_{lm}+\tilde{\Gamma}^i_{ml}))(e, e)$$
$$=\mathfrak{S}_{j,l,m}(\partial_{u^j}\partial_{u^l}\partial_{v^m}\mu^k-\partial_{u^j}\partial_{v^l}\partial_{v^m}\mu^k-\tilde{\Gamma}^k_{ji}(\tilde{\Gamma}^i_{lm}+\tilde{\Gamma}^i_{ml}))(e, e), \tag{3.14}$$

where $\tilde{\Gamma}^k_{ij}(e, e)=-\partial_{u^i}\partial_{v^j}\mu^k(e, e)$ are the components of the Chern connection of the 3-web evaluated at (e, e). By (3.14) and (3.10) of Proposition 1, we have;

PROPOSITION 2. (Cf. [9]) *If the 3-web of a differentiable loop (G, μ) is hexagonal at e (Fig. 2 in §1), that is, if each x has an inverse x^{-1} such that $x^{-1}x=xx^{-1}=e$, then the curvature $b^k_{j\,lm}$ of the Chern connection satisfies the following relation at (e, e):*

$$\sum_{j,l,m} b^k_{j\,lm}(e, e)=0. \tag{3.15}$$

Now, we derive some formulas from the left inverse property; $\mu(u^{-1}, \mu(u, v))=v$. Differentiating this equation in u and v, we get

$$\partial_{u^i}\partial_{v^p}\mu^k(u^{-1}, uv)\partial_j J^i(u)\partial_{v^l}\mu^p(u, v)$$
$$+\partial_{v^i}\partial_{v^p}\mu^k(u^{-1}, uv)\partial_{v^j}\mu^i(u, v)\partial_{v^l}\mu^p(u, v) \tag{3.16}$$
$$+\partial_{v^i}\mu^k(u^{-1}, uv)\partial_{u^j}\partial_{v^l}\mu^i(u, v)=0.$$

By differentiating (3.16) once again in u and evaluating it at (e, e) we have

$$(2\partial_{u^j}\partial_{u^l}\partial_{v^m}\mu^k-\partial_{u^j}\partial_{v^l}\partial_{v^m}\mu^k-\partial_{u^l}\partial_{v^j}\partial_{v^m}\mu^k)\ (e, e)$$
$$=(\tilde{\Gamma}^k_{ij}\tilde{\Gamma}^i_{lm}+\tilde{\Gamma}^k_{li}\tilde{\Gamma}^i_{jm}-\tilde{\Gamma}^k_{im}\ (\tilde{\Gamma}^i_{jl}+\tilde{\Gamma}^i_{lj}))\ (e, e). \tag{3.17}$$

Hence, from (3.10) of Proposition 1, we obtain

$$b^k_{j\,lm}(e, e)+b^k_{l\,jm}(e, e)=(\partial_{u^l}\partial_{v^j}\partial_{v^m}\mu^k+\partial_{u^j}\partial_{v^l}\partial_{v^m}\mu^k$$
$$-2\partial_{u^l}\partial_{u^j}\partial_{v^m}\mu^k)\ (e, e)+(\tilde{\Gamma}^k_{li}\tilde{\Gamma}^i_{jm}+\tilde{\Gamma}^k_{ji}\ \tilde{\Gamma}^i_{lm}$$
$$-\tilde{\Gamma}^k_{im}(\tilde{\Gamma}^i_{lj}+\tilde{\Gamma}^i_{jl}))(e, e)=0.$$

Thus, we have;

PROPOSITION 3. (Cf. [2]) *If the 3-web of a differentiable loop (G, μ) satisfies the (B_l)-closure condition along the vertical line $\{e\}\times G$* (Fig. 3 *in* §1), *that is, if (G, μ) has the left inverse property, then the curvature $b^k_{j\,lm}$ of the Chern connection satisfies the following relation at (e, e):*

$$b^k_{j\,lm}(e, e)+b^k_{l\,jm}(e, e)=0. \tag{3.18}$$

For any elements a, b in G, the left inner mapping $L_{a,b}$ is a diffeomorphism of G onto itself and it leaves the unit element e fixed. Let $\mathfrak{G}=T_e(G)$ be the tangent space of G at e, $L^*_{a,b}(e)$ the linear tansformation of $\mathfrak{G}$ induced from $L_{a,b}$. Choose a coordinate neighborhood U of e. Then, $L^*_{a,b} \in GL(\mathfrak{G})$ has the following matrix-represention with respect to the natural coordinate basis at e:

$$(3.19) \qquad (L^*_{a,b}(e))^k_j = \partial_{v^p}\mu^k((ab)^{-1}, ab)\partial_{v^q}\mu^p(a, b)\partial_{v^j}\mu^q(b, e).$$

The differentiable map $L^*: G\times G \to GL(\mathfrak{G})$ given by $L^*(a, b) = L^*_{a,b}(e)$ induces a bilinear map dL: $\mathfrak{G}\times\mathfrak{G}\to \mathrm{End}(\mathfrak{G})$ in the following way:

$$(3.20) \qquad dL(X, Y)^k_j = X^p Y^q \partial_{a^p}\partial_{b^q} L^{*k}_j(e, e)$$

for $X = X^i\partial_i(e)$ and $Y = Y^i\partial_i(e)$ in $\mathfrak{G}$. Then, we can show the following equation

$$\begin{aligned}(3.21) \qquad \partial_{a^l}\partial_{b^m} L^{*k}_j(e, e) &= \partial_{u^l}\partial_{u^m}\partial_{v^j}\mu^k(e, e)\\ &\quad - \partial_{u^m}\partial_{v^l}\partial_{v^j}\mu^k(e, e)\\ &\quad + \partial_{u^i}\partial_{v^j}\mu^k(e, e)\partial_{u^m}\partial_{v^l}\mu^i(e, e)\\ &\quad - \partial_{u^m}\partial_{v^i}\mu^k(e, e)\partial_{u^l}\partial_{v^j}\mu^i(e, e).\end{aligned}$$

We consider another bilinear map $d\mu$: $\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ at e induced from the multiplication μ of the loop, that is, in any local coordinates around e, we set

$$(3.22) \qquad d\mu(X, Y) = X^i Y^j \partial_{u^i}\partial_{v^j}\mu^k(e, e)\partial_k(e)$$

for $X = X^i\partial_i(e)$ and $Y = Y^i\partial_i(e)$.

THEOREM 1. *Let (G, μ) be a differentiable left I. P. loop, $\mathfrak{G}$ the tangent space of G at the unit element e. The bilinear maps dL: $\mathfrak{G}\times\mathfrak{G}\to \mathrm{End}(\mathfrak{G})$ and $d\mu$: $\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ given by* (3.20) *and* (3.22), *respectively, are described with respect to any local coordinates around e by the curvature and the torsion of the Chern connection in the following way;*

$$(3.23) \qquad dL(X, Y)^k_j = -X^l Y^m b^k_{lmj}(e, e)$$

$$(3.24) \qquad d\mu(X, Y)^k - d\mu(Y, X)^k = -2X^i Y^j a^k_{ij}(e, e)$$

for any $X = X^i\partial_i(e)$ and $Y = Y^i\partial_i(e)$ in $\mathfrak{G}$, where a^k_{ij} and b^k_{lmj} are indexed with respect to the base forms $\{\omega^k_{(1)}; \omega^k_{(2)}\}$ given by $\omega^k_{(1)} = P^k_i du^i$ and $\omega^k_{(2)} = Q^k_i dv^i$.

PROOF. Comparing the equation (3.10) of Proposition 1 with (3.21), we get (3.23). The equation (3.24) is obtained directly from (3.9) of Proposition 1 and the definition (3.22) of $d\mu$. q. e. d.

From Proposition 3 and the equation (3.23), we have

COROLLARY 1. *$dL(X, Y) + dL(Y, X) = 0$ for $X, Y \in \mathfrak{G}$.*

In his work [3], M. Akivis introduced a tangent algebra of a differentiable loop, which was given the name 'Akivis algebra' by K. H. Hofmann-K. Strambach in [8], [12]. The Akivis algebra of a differentiable loop (G, μ) is, by definition, the tangent space $\mathfrak{G}$ at e with the bilinear operation $[X, Y]$ and the trilinear operation $\langle X, Y, Z\rangle$ given by (cf. Ch. X of [8])

$$[X, Y]^k = -2X^i Y^j a^k_{ij},$$

$$\langle X, Y, Z\rangle^k = X^l Y^m Z^j b^k_{jlm}$$

for $X = X^i\partial_i(e)$, $Y = Y^i\partial_i(e)$ and $Z = Z^i\partial_i(e)$ with respect to any local coordinates at e. It satisfies the following axioms;

$$(A_1) \qquad [X, X] = 0$$

$$(A_2) \qquad \mathfrak{S}(\langle X, Y, Z\rangle - \langle Y, X, Z\rangle) = \mathfrak{S}[[X, Y], Z],$$

where $\mathfrak{S}$ denotes the cyclic summation with respect to X, Y, Z. The condition (A_2) is assured by the first equation of the formula (2.11) of S. S. Chern.

COROLLARY 2. *The Akivis algebra* $(\mathfrak{G}, [\ ,\ ], \langle\ ,\ ,\ \rangle)$ *of a differentiable left I. P. loop is given by*

$$[X, Y] = d\mu(X, Y) - d\mu(Y, X),$$

$$\langle X, Y, Z\rangle = dL(X, Z)Y$$

for $X, Y, Z \in \mathfrak{G}$.

§4. Canonical connections of homogeneous Lie loops and Chern connections

Let (G, μ) be an r-dimensional differentiable left I. P. loop of class C^∞. If (G, μ) satisfies the condition; (L) each left inner mapping $L_{a,b}$ is an automorphism of (G, μ); it is called a *homogeneous Lie loop* ([13]). In this section, we consider the differentiable homogeneous system η of a homogeneous Lie loop (G, μ) and describe the relations of the torsion and the curvature tensor of the canonical connection with those of the Chern connection of the 3-web of (G, μ). The concept of canonical connections has been introduced in [13] for homogeneous Lie loops and for differentiable homogeneous systems in [16–I]. More generally, we define the canonical connection on a differentiable left I. P. loop (G, μ) as follows: Let η be the ternary system on G defined by $\eta(x, y, z) = x((x^{-1}y)(x^{-1}z))$ (cf. §1). Then η is differentiable and it satisfies (H_1), (H_2) and (H'_3) in §1. Since each left translation L_x is a diffeomorphism of G, any displacement $\eta(x, y)\colon G \to G$ given by $\eta(x, y)z = \eta(x, y, z)$ is a diffeomorphism of G. We denote by $d\eta(x, y)$ the linear map of $T_x(G)$ to $T_y(G)$ induced by the displacement $\eta(x, y)$. At an arbitrary point x in G, we choose a coordinate neighborhood U and we denote by $E_i^0=\partial_i(x)$, $i=1, 2,\ldots, r$, the natural basis of the tangent space $T_x(G)$ at x with respect to the local coordinates (u^i). Put $E_i^*(y)=d\eta(x, y)E_i^0$ for $y \in U$. Then, we have differentiable vector fields $E_1^*,\ldots, E_r^*$ on U which are linearly independent at each point in U. For any differentiable vector field Y on G, we can set $Y=Y^iE_i^*$ on U with differentiable functions Y^i on U. Let X be another differentiable vector field on G. We can define the covariant differentiation of Y at x with respect to X in the following;

$$(\nabla_X Y)_x=(XY^i)(x)E_i^0. \tag{4.1}$$

The connection on G defined by the operator of the covariant differentiation ∇ above will be called the *canonical connection of the differentiable left I. P. loop* (G, μ). From the definition above, we can see that the canonical connection ∇ of (G, μ) coincides with one introduced in [13] if (G, μ) is a homogeneous Lie loop. By the property (H_2), there exists a neighborhood V of x such that $\eta(a, b, c) \in U$ for $a, b, c \in V$. We denote the coordinates of $\eta(a, b, c)$ and their partial derivatives by $\eta^k(a^i, b^j, c^m)$, $\partial_{a^i}\eta^k$, $\partial_{b^j}\eta^k$, $\partial_{c^m}\eta^k$, and so on. The properties (H_1) and (H_2) imply the followings;

$$\begin{aligned} &\partial_{c^i}\eta^k(a, a, c)=\delta_i^k, \quad \partial_{b^i}\eta^k(a, b, a)=\delta_i^k, \\ &\partial_{c^i}\partial_{c^j}\cdots\partial_{c^m}\eta^k(a, a, c)=\partial_{b^i}\partial_{b^j}\cdots\partial_{b^m}\eta^k(a, b, a)=0. \end{aligned} \tag{4.2}$$

Since $E_j^*(y)=d\eta(x, y)E_j^0=(\partial_{u^j}\eta^k(x, y, u))_{u=x}\partial_k(y)=\partial_{c^j}\eta^k(x, y, x)\partial_k(y)$ for $y \in U$, we can set

$$\partial_k(y)=Y_k^j(y)E_j^*(y),$$

where $(Y_k^j(y))$ is the inverse matrix of $(\partial_{c^j}\eta^k(x, y, x))$. By the equation

$$\partial_{y^i}Y^k{}_j\partial_{c^k}\eta^m(x, y, x)+Y_j^k(y)\partial_{b^i}\partial_{c^k}\eta^m(x, y, x)=0$$

and by the definition (4.1) of the covariant derivatives, we get the components of the canonical connection;

$$\Gamma_{ij}^k(x)=-\partial_{b^i}\partial_{c^j}\eta^k(x, x, x), \tag{4.3}$$

where $\nabla_{\partial_i}\partial_j=\Gamma_{ij}^k\partial_k$ in the coordinate neighborhood U. Hence, we have

$$T_{ij}^k(x)=\partial_{b^i}\partial_{c^j}\eta^k(x, x, x)-\partial_{b^j}\partial_{c^i}\eta^k(x, x, x), \tag{4.4}$$

where $T_{ij}^k=\Gamma_{ji}^k-\Gamma_{ij}^k$ are the components of the torsion tensor of the canonical connection ∇.

In the following, we assume that (G, μ) is a homogeneous Lie loop. Then, we have seen in §1 that the associated homogeneous system η satisfies (H_3) and (H_4). Differentiate the both sides of the equation

(H$_4$) $\eta(e, x, \eta(e, u, v))=\eta(x, \eta(e, x, u), \eta(e, x, v))$

in u and v, in a neighborhood of e contained in U. Then, we get

$$\begin{aligned}(4.5)\qquad &\partial_{b^q}\partial_{c^r}\eta^k(x, \eta(e, x, u), \eta(e, x, v))\partial_{c^m}\eta^q(e, x, u)\partial_{c^j}\eta^r(e, x, v)\\ &=\partial_{c^q}\partial_{c^r}\eta^k(e, x, \eta(e, u, v))\partial_{b^m}\eta^q(e, u, v)\partial_{c^j}\eta^r(e, u, v)\\ &\quad+\partial_{c^i}\eta^k(e, x, \eta(e, u, v))\partial_{b^m}\partial_{c^j}\eta^i(e, u, v).\end{aligned}$$

We use the notation in §3 for $\mu(u, v)=\eta(e, u, v)$; i.e.,

$$P^k_j(x, y)=\partial_{b^j}\eta^k(e, x, y), \quad Q^k_j(x, y)=\partial_{c^j}\eta^k(e, x, y)$$

and $(\tilde{P}^j_k)$, $(\tilde{Q}^j_k)$ are the inverse matrices of (P^k_j) and (Q^k_j), respectively. The equation (4.5) evaluated at $u=v=e$ implies

$$(4.6)\qquad \Gamma^k_{ij}(x)=\Gamma^h_{pq}(e)Q^k_h(x, e)\tilde{Q}^p_i(x, e)\tilde{Q}^q_j(x, e)-\partial_{v^p}\partial_{v^q}\mu^k(x, e)\tilde{Q}^p_i(x, e)\tilde{Q}^q_j(x, e)$$

for $x\in U$. From the equation (4.5), we can obtain

$$\begin{aligned}(4.7)\qquad &(\partial_{a^p}\partial_{b^q}\partial_{c^r}\eta^k(x, x, x)+\partial_{b^p}\partial_{b^q}\partial_{c^r}\eta^k(x, x, x)\\ &\quad+\partial_{b^q}\partial_{c^p}\partial_{c^r}\eta^k(x, x, x))Q^q_m(x, e)Q^r_j(x, e)\\ &=\partial_{b^p}\partial_{c^m}\partial_{c^j}\eta^k(e, x, e)+\Gamma^k_{ir}(x)\partial_{b^p}\partial_{c^m}\eta^i(e, x, e)Q^r_j(x, e)\\ &\quad+\Gamma^k_{qi}(x)\partial_{b^p}\partial_{c^j}\eta^i(e, x, e)Q^q_m(x, e)\\ &\quad-\Gamma^i_{mj}(e)\partial_{b^p}\partial_{c^i}\eta^k(e, x, e).\end{aligned}$$

Evaluating the both sides of the equation (4.7) at $x=e$ and substituting the equation

$$(4.8)\qquad -\partial_l\Gamma^k_{mj}(e)=\partial_{a^l}\partial_{b^m}\partial_{c^j}\eta^k(e, e, e)+\partial_{b^l}\partial_{b^m}\partial_{c^j}\eta^k(e, e, e)+\partial_{b^m}\partial_{c^l}\partial_{c^j}\eta^k(e, e, e),$$

we get

$$\begin{aligned}(4.9)\qquad R^k_{j\,lm}(e)&=\partial_{b^l}\partial_{c^m}\partial_{c^j}\eta^k(e, e, e)-\partial_{b^m}\partial_{c^l}\partial_{c^j}\eta^k(e, e, e)\\ &\quad+\Gamma^k_{li}(e)\Gamma^i_{mj}(e)-\Gamma^k_{mi}(e)\Gamma^i_{lj}(e)+\Gamma^k_{ij}(e)T^i_{lm}(e),\end{aligned}$$

where $R^k_{j\,lm}=-\partial_l\Gamma^k_{mj}+\partial_m\Gamma^k_{lj}-\Gamma^k_{li}\Gamma^i_{mj}+\Gamma^k_{mi}\Gamma^i_{lj}$ are the components of the curvature tensor of the canonical connection.

By the way, the equations (4.7) and (4.8) also imply the following;

$$(4.10)\qquad \partial_l T^k_{mj}(e)=-\Gamma^k_{li}(e)T^i_{mj}(e)+\Gamma^i_{lm}(e)T^k_{ij}(e)+\Gamma^i_{lj}(e)T^k_{mi}(e),$$

that is, $\nabla T(e)=0$, which has been shown in [13] to hold at each point of the homogeneous Lie loop G.

The Chern connection of the local 3-web of the homogeneous Lie loop (G, μ) has the components $\tilde{\Gamma}^k_{ij}(u, v)$ written in the form (3.3). Hence, we have

$$\begin{aligned}\partial_{vj}\tilde{\Gamma}^k_{lm} = &\tilde{P}^h_l\tilde{P}^p_i\tilde{Q}^q_m\partial_{u^p}\partial_{v^q}\mu^k\partial_{u^h}\partial_{vj}\mu^i \\ &+\tilde{P}^p_l\tilde{Q}^h_m\tilde{Q}^q_i\partial_{u^p}\partial_{v^q}\mu^k\partial_{v^h}\partial_{vj}\mu^i \\ &-\tilde{P}^p_l\tilde{Q}^q_m\partial_{u^p}\partial_{v^q}\partial_{vj}\mu^k,\end{aligned}$$

and, evaluating it at (e, e), we have

$$\partial_{vj}\tilde{\Gamma}^k_{lm}(e, e) = \tilde{\Gamma}^k_{im}(e, e)\tilde{\Gamma}^i_{lj}(e, e) - \partial_{u^l}\partial_{v^m}\partial_{vj}\mu^k(e, e).$$

Since $\tilde{\Gamma}^k_{lm}(e, e) = \Gamma^k_{lm}(e)$, we obtain from (4.8) and (4.10)

$$\begin{aligned}(4.11)\qquad \partial_{vj}(\tilde{\Gamma}^k_{ml} - \tilde{\Gamma}^k_{lm})(e, e) = &R^k_{j\,lm}(e) + \partial_j T^k_{lm}(e) \\ &+ T^k_{ij}(e)T^i_{lm}(e) + T^i_{jm}(e)T^k_{li}(e) \\ &+ T^i_{jl}(e)T^k_{im}(e).\end{aligned}$$

THEOREM 2. *Let (G, μ) be a homogeneous Lie loop. Denote by $d\mu$ and dL the bilinear maps on the tangent space $\mathfrak{G} = T_e(G)$ at the unit element e, induced from the multiplication μ and left inner mappings, respectively (cf. (3.20), (3.22)). Then, the torsion tensor T and the curvature tensor R of the canonical connection of (G, μ) admit the following experessions:*

$$(4.12)\qquad T_e(X, Y) = d\mu(X, Y) - d\mu(Y, X),$$

$$(4.13)\qquad R_e(X, Y) = 2\, dL(X, Y)$$

for X, Y in $\mathfrak{G}$. The components of T and R with respect to any local coordinates (u^i) around e are related with the Chern connection of the local 3-web as follows;

$$(4.14)\qquad T^k_{ij}(e) = -2a^k_{ij}(e, e),$$

$$(4.15)\qquad R^k_{j\,lm}(e) = -2b^k_{lmj}(e, e),$$

where a^k_{ij} and b^k_{lmj} are the torsion and the curvature of the Chern connection with respect to the base forms $\{\omega^k_{(1)};\ \omega^k_{(2)}\}$ given by $\omega^k_{(1)} = P^k_i du^i$ and $\omega^k_{(2)} = Q^k_i dv^i$.

PROOF. Since $\mu^k(u, v) = \eta^k(e, u, v)$, we have $\Gamma^k_{ij}(e) = -\partial_{u^i}\partial_{vj}\mu^k(e, e)$, which implies (4.12) and (4.14) by virtue of (3.22), (3.24) and (4.4). On the other hand, by using the components of $dL(X, Y)$ given in (3.20) and (3.21), we get

$$X^l Y^m R^k_{j\,lm}(e) = dL(X, Y)^k_j - dL(Y, X)^k_j.$$

Then, Corollary 1 to Theorem 1 and the equation (3.23) in Theorem 1 lead us to the equations (4.13) and (4.15). q.e.d.

In [13] we have introduced the concept of tangent Lie triple algebra of a homogeneous Lie loop (G, μ) as follows: Let $\mathfrak{G}=T_e(G)$ be the tangent space of G at the unit e. We set $XY=T_e(X, Y)$ and $D(X, Y)=R_e(X, Y)$ for $X, Y\in\mathfrak{G}$, where T and R denote the torsion and the curvature of the canonical connection of (G, μ). We have seen that the bilinear product XY and the trilinear product $D(X, Y)Z$ satisfy the axiom of *Lie triple algebra* (general Lie triple system of K. Yamaguti [26]);

(L_1) $\qquad XX=0$

(L_2) $\qquad D(X, X)=0$

(L_3) $\qquad \mathfrak{S}(D(X, Y)Z+(XY)Z)=0$

(L_4) $\qquad \mathfrak{S}D(XY, Z)=0$

(L_5) $\qquad D(X, Y)(UV)=(D(X, Y)U)V+U(D(X, Y)V)$

(L_6) $\qquad [D(X, Y), D(U, V)]=D(D(X, Y)U, V)+D(U, D(X, Y)V)$,

where $\mathfrak{S}$ denotes the cyclic summation with respect to X, Y and Z. We have developed in the series of articles [13]–[23] an analogy of the theory of Lie groups and Lie algebras for homogeneous Lie loops or, in more general case, for differentiable homogeneous systems. Now, from Theorem 2 and Corollary 2 to Theorem 1, it follows that the tangent Akivis algebra and the tangent Lie triple algebra coincide with each other for homogeneous Lie loops, up to the order and a scalar multiple of the trilinear product, that is,

COROLLARY. *Let $\mathfrak{G}$ be the tangent space of a homogeneous Lie loop (G, μ) at the unit element e. Then, the tangent Lie triple algebra $\{\mathfrak{G}; XY, D(X, Y)Z\}$ and the tangent Akivis algebra $\{\mathfrak{G}; [X, Y], \langle X, Y, Z\rangle\}$ are related in the following;*

$$XY=[X, Y], \quad D(X, Y)Z=2\langle X, Z, Y\rangle \quad \text{for} \quad X, Y, Z\in\mathfrak{G}.$$

REMARK. The same result has been shown recently in [12].

Finally, we consider the canonical connection of a K-homogeneous Lie loop.

PROPOSITION 4. *Let (G, μ) be a left I. P. loop. If it satisfies the condition* (K) *in* §1, *then it is a homogeneous loop.*

PROOF. The condition (K) implies that, for any x, y, u in G, there exists an element w in G such that

$$L_{xy}^{-1}L_xL_yL_uL_y^{-1}L_x^{-1}L_{xy}=L_w.$$

Operating both sides of this equation to the unit e, we get $w=L_{x,y}u$. Therefore, we have $L_{x,y}L_u=L_wL_{x,y}$, that is,

$$L_{x,y}(uv)=(L_{x,y}u)(L_{x,y}v). \qquad \text{q.e.d.}$$

PROPOSITION 5. *A homogeneous loop (G, μ) is K-homogeneous if and only if the homogeneous system η: $G\times G\times G\to G$ satisfies*

(K′) *For any x, $y\in G$, there exists an element $z\in G$ such that, $\eta(x, y)=\eta(e, z)$.*

PROOF. For any x, $y\in G$, set $u=x^{-1}y$. By (H_3) and (H_4), the equation $L_xL_uL_x^{-1}=L_z$ can be rewritten as $\eta(x, xu)=\eta(e, z)$. q.e.d.

Now, we assume that (G, μ) is a K-homogeneous Lie loop. By the proposition above, the differentiable homogeneous system η of (G, μ) must satisfy the condition (K′). Let x be a fixed point in G and set $\eta(x, y, v)=\eta(e, u, v)$ with $y=\mu(u, x)$ for y, $v\in G$. In a coordinate neighborhood around e, we can see

$$\partial_{y^i}u^k(y)=\tilde{P}^k_i(u, x),$$

$$\partial_{b^i}\partial_{c^j}\eta^k(x, y, v)=\partial_{b^m}\partial_{c^j}\eta^k(e, u, v)\tilde{P}^m_i(u, x).$$

Hence, the components $\tilde{\Gamma}^k_{ij}$ of the Chern connection can be evaluated at (e, x) as follows:

$$\begin{aligned}\tilde{\Gamma}^k_{ij}(e, x)&=-\tilde{P}^h_i(e, x)\tilde{Q}^m_j(e, x)\partial_{u^h}\partial_{v^m}\mu^k(e, x)\\&=-\tilde{P}^h_i(e, x)\partial_{u^h}\partial_{v^j}\mu^k(e, x)\\&=-\partial_{b^i}\partial_{c^j}\eta^k(x, x, x),\end{aligned}$$

that is, $\tilde{\Gamma}^k_{ij}(e, x)=\Gamma^k_{ij}(x)$.
Thus, we have

PROPOSITION 6. *The canonical connection of a K-homogeneous Lie loop satisfies*

$$\Gamma^k_{ij}(x)=\tilde{\Gamma}^k_{ij}(e, x)$$

in a neighborhood of the unit e, where $\tilde{\Gamma}^k_{ij}$ are the components of the Chern connection with respect to the base forms $\{\omega^k_{(1)};\ \omega^k_{(2)}\}$.

THEOREM 3. *Let (G, μ) be a K-homogeneous Lie loop. Then, the curvature R of the canonical connection vanishes identically on G.*

PROOF. Since $T^k_{ij}(x)=\tilde{\Gamma}^k_{ji}(e, x)-\tilde{\Gamma}^k_{ij}(e, x)$ by Proposition 6, we can use (4.11) with $\partial_jT^k_{lm}(x)=\partial_{v^j}(\tilde{\Gamma}^k_{ml}-\tilde{\Gamma}^k_{lm})(e, x)$ and we can see

$$R^k_{jlm}(e)+\mathfrak{S}_{j,l,m}T^k_{ij}(e)T^i_{lm}(e)=0.$$

From the condition (L_3) of the tangent Lie triple algebra, which is equivalent to the condition (A_2) of Akivis algebra, we can show

$$\begin{aligned}0&=\mathfrak{S}_{j,l,m}(R^k_{jlm}(e)+T^k_{ij}(e)T^i_{lm}(e))\\&=4\mathfrak{S}_{j,l,m}T^k_{ij}(e)T^i_{lm}(e),\end{aligned}$$

Hence, we have $R^k_{jlm}(e)=0$, and this holds at every point of G since (G, μ) is homogeneous. q. e. d.

By Theorem 1.1 in [16–II] we have immediately the following;

COROLLARY. *Let (G, μ) be a connected analytic K-homogeneous Lie loop. If (G, μ) is geodesic, then it is reduced to a Lie group.*

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