Tracy-Widom method for Janossy densities

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Shimane University

"Structured Random Matrices in Down Under"
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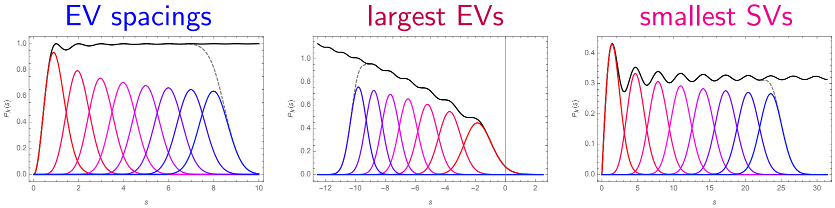
1 Introduction

Extremal EV distribution of RMs

Gap probability & integrability

$$\mathrm{Prob}(\text{no EV in } (a_1, a_2)) = \mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(a_1,a_2)})$$

Jimbo-et al. 1980 : $\beta = 2$ spec.bulk : $\mathbf{K}_{\mathrm{sinc}}|_{(0,s)} \to$ Painleve V
Tracy-Widom 1993 : $\beta = 2$ soft edge : $\mathbf{K}_{\mathrm{Airy}}|_{(s,\infty)} \to$ Painleve II
Tracy-Widom 1993 : $\beta = 2$ hard edge : $\mathbf{K}_{\mathrm{Bessel}}|_{(0,s)} \to$ Painleve III'
Tracy-Widom 1996 : $\beta = 1, 4$ (Pf) related to $\beta = 2$ (Det)
+ finite-N kernels (e.g. CUE$_N$: $\mathbf{K}_{\sin/\sin} \to$ Painleve VI), beyond-Airy, ...



Extremal EV distribution of RMs

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Tracy–Widom Distribution

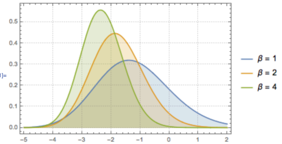
Tracy–Widom distribution is the limiting distribution of the scaled largest eigenvalue of a random matrix belonging to Gaussian ensembles. It also appears in various different disciplines, such as counting problems, random growth models, phase transitions, etc., and gives accurate predictions.

Tracy–Widom distribution consists of three classes $\beta = 1, 2$, and 4. Each corresponds to different Gaussian ensembles; see the corresponding PDFs.

⊟ hide input

```
In[1]:= Plot[Table[PDF[TracyWidomDistribution[β], x], {β, {1, 2, 4}}] // Evaluate,
  {x, -5, 2}, Filling → Axis, PlotLegends → {"β = 1", "β = 2", "β = 4"},
  ImageSize → Medium, PlotTheme → "Detailed"]
```

Out[1]=



Use MatrixPropertyDistribution to represent the scaled largest eigenvalue of a matrix from GUE.

```
In[2]:= ev𝒟[2, n_] := MatrixPropertyDistribution[
   (Max[Eigenvalues[x]] - 2 Sqrt[n]) n^(1/6),
   x ≈ GaussianUnitaryMatrixDistribution[n]]
```

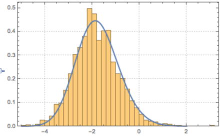
Sample from the distribution and compare the histogram with the PDF.

```
In[3]:= sample = RandomVariate[ev𝒟[2, 250], 2000];
```

⊟ hide input

```
In[4]:= Show[Histogram[sample, {0.2}, PDF, PlotTheme → "Detailed"],
  Plot[PDF[TracyWidomDistribution[2], x], {x, -5, 2}, PlotStyle → Thick,
   PlotTheme → "Detailed", PlotLegends → None], ImageSize -> Medium]
```

Out[4]=

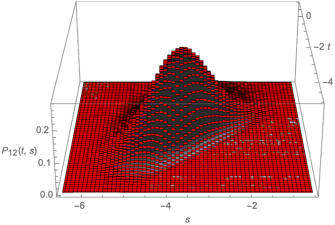


Tracy–Widom distribution can be well approximated by gamma distribution in the central region.

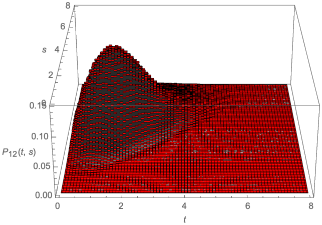
Extremal EV distribution of RMs

What about **joint** distributions?

2 largest EVs



2 smallest SVs



$N = 128$, #samples $= 10^7$

Joint distr. $P_{12}(t,s)$ **of 1st & 2nd EVs** $\to$ **Isomonodromic Systems**

Forrester-Witte 2007 (70 pages) : hard edge $\to$ IS for Painleve III'

Witte-Bornemann-Forrester 2013 (29 pages) : soft edge $\to$ IS for Painleve II

Perret-Schehr 2014 (34 pages) : soft edge $\to$ Lax pair for Painleve XXXIV

Extremal EV distribution of RMs

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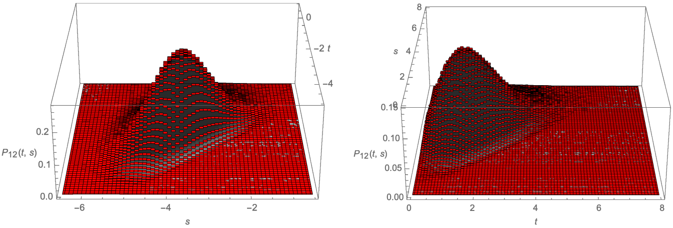

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Tracy-Widom method

Revisit this problem for

- More **user-friendly** analytical formulation!
- Universally applicable to $K(x,y) = \dfrac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}$
- Easily generalizable to $P_{12\cdots p}(s_1, \ldots, s_p)$

Solution : apply T-W method

- if
$$m(x)\frac{d}{dx}\begin{bmatrix}\varphi(x)\\ \psi(x)\end{bmatrix} = \begin{bmatrix}A(x) & B(x)\\ -C(x) & -A(x)\end{bmatrix}\begin{bmatrix}\varphi(x)\\ \psi(x)\end{bmatrix}$$ with **polynomials** m, A, B, C,
- then $\partial_{a_i} \log \mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(a_1,a_2)})$ satisfy a system of PDEs containing **coefficients** of m, A, B, C

to the kernel with conditioned EVs

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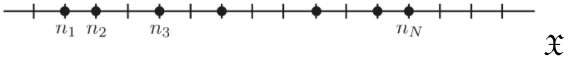
2 Janossy density in DPP

Determinantal point process

Consider DPP on a countable set



Joint prob of N pts : $P(n_1, \ldots, n_N) = \dfrac{1}{N!} \det \left[K(n_i, n_j)\right]_{i,j=1}^N$

with kernel $\mathbf{K} = \left[K(n, n')\right]_{n,n' \in \mathfrak{X}} = \mathbf{K}^t = \mathbf{K} \cdot \mathbf{K}, \quad \operatorname{tr} \mathbf{K} = N$

$\Downarrow$

Joint prob of k pts : $\rho_k(n_1, \ldots, n_k) = \det \left[K(n_i, n_j)\right]_{i,j=1}^k$

Gap probability $= \det(\mathbb{I} - \mathbf{K}_I), \quad \mathbf{K}_I = \left[K(n, n')\right]_{n,n' \in I}$

everything carries over to DPPs on continuum

$\rho_k(\{n\}) \to \rho_k(\{x\}) dx_1 \cdots dx_k$, $\det \to \mathrm{Det}$

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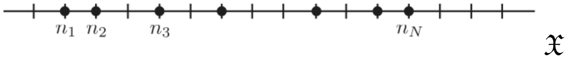

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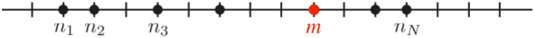
Conditional probability

fix a point m and define 'modified kernel'

$$\tilde{K}(n,n') = K(n,n') - \frac{K(n,\boldsymbol{m})K(\boldsymbol{m},n')}{K(\boldsymbol{m},\boldsymbol{m})}$$



- satisfies $\tilde{\mathbf{K}} = \left[\tilde{K}(n,n')\right]_{n,n'\in\mathfrak{X}} = \tilde{\mathbf{K}}^t = \tilde{\mathbf{K}}\cdot\tilde{\mathbf{K}}, \quad \mathrm{tr}\,\tilde{\mathbf{K}} = N-1$
- conditional joint prob of k pts, with $\boldsymbol{m}$ already occupied:

$$\tilde{\rho}_1(n|\boldsymbol{m}) = \frac{\rho_2(n,\boldsymbol{m})}{\rho_1(\boldsymbol{m})} = \frac{K(n,n)K(\boldsymbol{m},\boldsymbol{m}) - K(n,\boldsymbol{m})K(\boldsymbol{m},n)}{K(\boldsymbol{m},\boldsymbol{m})} = \tilde{K}(n,n)$$

$$\tilde{\rho}_2(n_1,n_2|\boldsymbol{m}) = \frac{\rho_3(n_1,n_2,\boldsymbol{m})}{\rho_1(\boldsymbol{m})}$$

$$= \frac{K(n_1,n_1)K(n_2,n_2)K(\boldsymbol{m},\boldsymbol{m}) \pm (5\text{ terms})}{K(\boldsymbol{m},\boldsymbol{m})} = \det\left[\tilde{K}(n_i,n_j)\right]_{i,j=1}^{2}, \text{ etc.}$$

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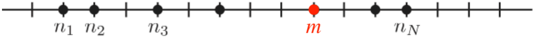

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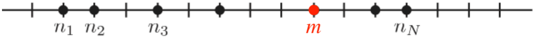

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thus the modified kernel

$$\tilde{K}(n,n') = K(n,n') - \frac{K(n,\boldsymbol{m})K(\boldsymbol{m},n')}{K(\boldsymbol{m},\boldsymbol{m})}$$

corresponds to a DPP, governing the conditional joint prob

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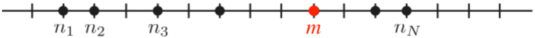

now fix more points one by one. by induction it generalizes to ...

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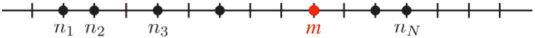

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Lemma (1)

fix p distinct points $\boldsymbol{m}_1, \dots, \boldsymbol{m}_p$ and let

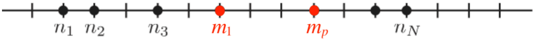
$$\boldsymbol{\kappa} = [K(\boldsymbol{m}_i, \boldsymbol{m}_j)]_{i,j=1}^{p}, \ \boldsymbol{k} = [K(\boldsymbol{m}_i, n)]_{i=1,\dots,p}^{n\in\mathfrak{X}}, \ \mathbf{K} = \left[K(n, n')\right]_{n,n'\in\mathfrak{X}}$$

then

$$\tilde{\mathbf{K}} = \mathbf{K} - \boldsymbol{k}^t \boldsymbol{\kappa}^{-1} \boldsymbol{k}$$

governs the conditional joint prob with $\boldsymbol{m}_1, \dots, \boldsymbol{m}_p$ already occupied:

$$\tilde{\rho}_k(n_1, \dots, n_k | \boldsymbol{m}_1, \dots, \boldsymbol{m}_p) = \det\left[\tilde{K}(n_i, n_j)\right]_{i,j=1}^{k}$$



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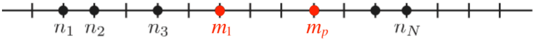

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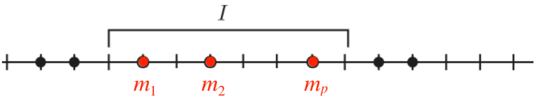
Janossy density

Lemma (2)

Probability that a subset $I \subset \mathfrak{X}$ is otherwise empty under the condition that $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_p \in I$ are already occupied is given by

$$\tilde{J}_p(I|\boldsymbol{m}_1, \ldots, \boldsymbol{m}_p) = \det(\mathbb{I} - \tilde{\mathbf{K}}_I) \ , \quad \tilde{\mathbf{K}}_I = \left[\tilde{K}(n, n')\right]_{n,n' \in I}$$



Janossy density = *Probability that a subset $I \subset \mathfrak{X}$ contains exactly p points at $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_p$ is given by*

$$J_p(I; \boldsymbol{m}_1, \ldots, \boldsymbol{m}_p) = \det \boldsymbol{\kappa} \cdot \det(\mathbb{I} - \tilde{\mathbf{K}}_I)$$

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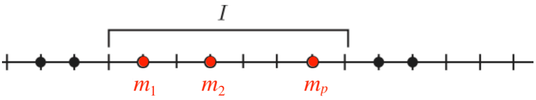

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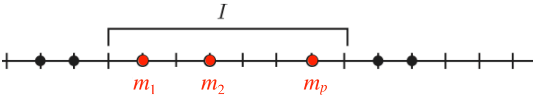


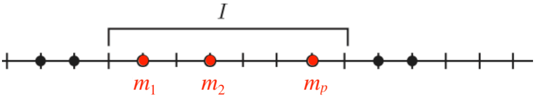
so far, nothing new actually.

using $\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{D}| \cdot \left|\mathbf{A} - \mathbf{C}\mathbf{D}^{-1}\mathbf{B}\right|$, Janossy density allows for 3 disguises

$$\begin{aligned} J_p(I; m_1, \ldots, m_p) &= \det \boldsymbol{\kappa} \cdot \det\left(\mathbb{I} - (\mathbf{K} - \boldsymbol{k}^t \boldsymbol{\kappa}^{-1} \boldsymbol{k})_I\right) \\ &= (-1)^k \det \begin{vmatrix} -\boldsymbol{\kappa} & -\boldsymbol{k} \\ -\boldsymbol{k}^t & \mathbb{I} - \mathbf{K}_I \end{vmatrix} \\ &= \det(\mathbb{I} - \mathbf{K}_I) \cdot \det\left[\langle m_i | \mathbf{K}_I (\mathbb{I} - \mathbf{K}_I)^{-1} | m_j \rangle\right]_{i,j=1}^p \end{aligned}$$

- 3rd line is listed e.g. in textbook of Daley-Vere Jones (1988), p.140
- 1st line is suited for applying T-W method to $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_I)$

Janossy density



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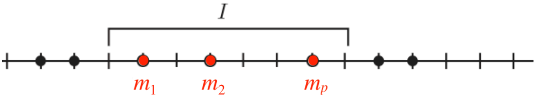

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$$\begin{aligned} J_p(I; m_1, \ldots, m_p) &= \det \boldsymbol{\kappa} \cdot \det \left(\mathbb{I} - (\mathbf{K} - \boldsymbol{k}^t \boldsymbol{\kappa}^{-1} \boldsymbol{k})_I\right) \\ &= (-1)^k \det \begin{vmatrix} -\boldsymbol{\kappa} & -\boldsymbol{k} \\ -\boldsymbol{k}^t & \mathbb{I} - \mathbf{K}_I \end{vmatrix} \\ &= \det(\mathbb{I} - \mathbf{K}_I) \cdot \det \left[\langle m_i | \mathbf{K}_I (\mathbb{I} - \mathbf{K}_I)^{-1} | m_j \rangle\right]_{i,j=1}^p \end{aligned}$$

- 3rd line is listed e.g. in textbook of Daley-Vere Jones (1988), p.140
- 1st line is suited for applying T-W method to $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_I)$

3 Applicability of T-W method

T-W method for $\tilde{\mathrm{K}}$

Tracy-Widom criteria (1994) :

1. kernel is of Christoffel-Darboux form $K(x,y) = \dfrac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}$
2. 2 functions satisfy linear DEs

$$m(x)\frac{d}{dx}\begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} A(x) & B(x) \\ -C(x) & -A(x) \end{bmatrix}\begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix}$$

with **polynomials** m, A, B, C,

then $\partial_{a_i} \log \mathrm{Det}(\mathbb{I} - \mathbf{K}|_{(a_1,a_2)})$ is determined from a system of PDEs in a_i

now comes the punchline:

Theorem

If a kernel $\mathbf{K}$ satisfies the T-W criteria, so does the modified kernel $\tilde{\mathbf{K}}$

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If a kernel $\mathbf{K}$ satisfies the T-W criteria, so does the modified kernel $\tilde{\mathbf{K}}$

Proof of Theorem

by induction, sufficient to prove for fixing one point at t.

1. $\tilde{\mathbf{K}}$ is of Christoffel-Darboux form

almost trivial, because for $\mathbf{K}$ consisting of polynomials orthogonal by weight $w(x)$
$\tilde{\mathbf{K}}$ consists of polynomials orthogonal by weight $\tilde{w}(x) = (x-t)^2 w(x)$.

more explicitly,

$$\tilde{K}(x,y) = \underbrace{\frac{\varphi(x)\psi(y)-\psi(x)\varphi(y)}{x-y}}_{K(x,y)} - \underbrace{\frac{\varphi(x)\psi(t)-\psi(x)\varphi(t)}{x-t}}_{K(x,t)} \underbrace{\frac{1}{\rho_1(t)}}_{K(t,t)^{-1}} \underbrace{\frac{\varphi(t)\psi(y)-\psi(t)\varphi(y)}{t-y}}_{K(t,y)}$$

$$= \frac{\tilde{\varphi}(x)\tilde{\psi}(y)-\tilde{\psi}(x)\tilde{\varphi}(y)}{x-y}, \quad \text{where}$$

$$\tilde{\varphi}(x) := \varphi(x) - \frac{b(a\varphi(x)-b\psi(x))}{x-t} \qquad a = \frac{\psi(t)}{\sqrt{\rho_1(t)}}, \quad b = \frac{\varphi(t)}{\sqrt{\rho_1(t)}}$$

$$\tilde{\psi}(x) := \psi(x) - \frac{a(a\varphi(x)-b\psi(x))}{x-t}$$

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Proof of Theorem

2. $\tilde{\varphi}(x), \tilde{\psi}(x)$ satisfy linear DEs of T-W form

nontrivial. differentiate them to get

$$m(x)\frac{d}{dx}\begin{bmatrix} \tilde{\varphi}(x) \\ \tilde{\psi}(x) \end{bmatrix} = \begin{bmatrix} \tilde{A}(x) & \tilde{B}(x) \\ -\tilde{C}(x) & -\tilde{A}(x) \end{bmatrix}\begin{bmatrix} \tilde{\varphi}(x) \\ \tilde{\psi}(x) \end{bmatrix}, \quad \text{with}$$

$$\tilde{A}(x) = A(x) + \frac{a^2B(x) - b^2C(x)}{x-t} - \frac{ab\left(2abA(x) + a^2B(x) + b^2C(x) - m(x)\right)}{(x-t)^2}$$

$$\tilde{B}(x) = B(x) - \frac{2b(bA(x) + aB(x))}{x-t} + \frac{b^2\left(2abA(x) + a^2B(x) + b^2C(x) - m(x)\right)}{(x-t)^2}$$

$$\tilde{C}(x) = C(x) + \frac{2a(aA(x) + bC(x))}{x-t} + \frac{a^2\left(2abA(x) + a^2B(x) + b^2C(x) - m(x)\right)}{(x-t)^2}$$

since m, A, B, C are polymomials in x, so are $\tilde{A}, \tilde{B}, \tilde{C}$ after redefinition $(x-t)^2m(x) \mapsto m(x)$, $(x-t)^2\tilde{A}(x) \mapsto \tilde{A}(x)$, etc

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Remarks

- $\tilde{K}(x,y)$ vanishes at $x, y = t$:

$$\tilde{K}(t,y) = K(t,y) - \frac{K(t,t)K(t,y)}{K(t,t)} = 0 \quad \Rightarrow$$

$$(\tilde{\mathbf{K}}_I \cdot f)(t) = 0 \quad \Rightarrow \quad ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot f)(t) = f(t)$$

- $\tilde{\varphi}(x), \tilde{\psi}(x)$ vanish at $x = t$ by def. thus, for $j \in \mathbb{N}$

$$q_j(t) := ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot x^j \tilde{\varphi})(t) = t^j \tilde{\varphi}(t) = 0$$

$$p_j(t) := ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot x^j \tilde{\psi})(t) = t^j \tilde{\psi}(t) = 0$$

... will be used for consistency check of the solution $q_j(s), p_j(s)$ obtained from BC imposed at $s = \infty$ or $s = 0$

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Remarks

- **Fixing EVs on $t_j \Leftrightarrow$ Multiplying the measure by $\prod_j \det(H-t_j)^2$**

$$\prod_{i=1}^{N} w(x_i)\cdot\prod_{i>j}^{N}(x_i-x_j)^2\Big|_{x_N=t} \propto \prod_{i=1}^{N-1}\underbrace{w(x_i)(x_i-t)^2}_{\tilde{w}(x_i)}\cdot\prod_{i>j}^{N-1}(x_i-x_j)^2$$

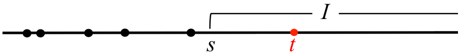
conditional prob. $\tilde{\rho}_k(x_1,\ldots,x_k|t_1,\ldots,t_p)$, $\tilde{J}_p(I|t_1,\ldots,t_p)$ for weight $w(x)$
$=$ unconditional prob. $\rho_k(x_1,\ldots,x_k)$, $J_0(I)$ for weight $\tilde{w}(x)$

"QCD with pairwise-degenerated quarks
with masses $m_j^2=-t_j$"

4 Joint distributions of extremal EVs

Janossy density for $\mathrm{K}_{\mathrm{Airy}}$

Evaluate $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}|_{(s,\infty)}) = J_1((s,\infty);t)$



$\varphi(x) = \mathrm{Ai}(x)$, $\psi(x) = \mathrm{Ai}'(x)$ satisfy LDEs with

$m(x) = 1$, $A(x) = 0$, $B(x) = 1$, $C(x) = -x$

$$\Downarrow$$

$\tilde{\varphi}(x)$, $\tilde{\psi}(x)$ satisfy LDEs with

$m(x) = (x-t)^2$

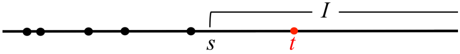
$$\tilde{A}(x) = -ab(a^2-1) - a^2 t + \left(a^2 + ab^3 - b^2 t\right) x + b^2 x^2 := \sum_{j=0}^{2} \alpha_j x^j$$

$$\tilde{B}(x) = b^2(a^2-1) + 2abt + t^2 - \left(2ab + b^4 + 2t\right) x + x^2 := \sum_{j=0}^{2} \beta_j x^j$$

$$\tilde{C}(x) = a^2(a^2-1) - (ab-t)^2 x - 2(ab-t)x^2 - x^3 := \sum_{j=0}^{3} \gamma_j x^j$$

Janossy density for $\mathrm{K}_{\mathrm{Airy}}$

Evaluate $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}|_{(s,\infty)}) = J_1((s,\infty);t)$



$$R(s) = \partial_s \log \mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{(s,\infty)}) = p_0(s)q_0'(s) - q_0(s)p_0'(s)$$

$$q_k(s) = ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot x^k \varphi)(s), \quad p_k(s) = ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot x^k \psi)(s)$$

$$u_k(s) = \int_I dx\, \varphi(x)\, x^k ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot \varphi)(x), \quad v_k(s) = \int_I dx\, \psi(x)\, x^k ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot \varphi)(x)$$

$$\tilde{v}_k(s) = \int_I dx\, \varphi(x)\, x^k ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot \psi)(x), \quad w_k(s) = \int_I dx\, \psi(x)\, x^k ((\mathbb{I} - \tilde{\mathbf{K}}_I)^{-1} \cdot \psi)(x)$$

satisfy a closed system of ODEs in s, with coefficients $\alpha_{0,1,2}, \beta_{0,1,2}, \gamma_{0,1,2,3}(t)$:

Janossy density for $\mathrm{K}_{\mathrm{Airy}}$

T-W system of ODEs

$$
\begin{aligned}
(s-t)^2 q_0' &= \sum_{j=0}^{2}\Big(\alpha_j + \sum_{k=0}^{1}\alpha_{j+k+1}v_k + \sum_{k=0}^{2}\gamma_{j+k+1}u_k\Big)q_j - v_0 q_0 \\
&\quad + \sum_{j=0}^{2}\Big(\beta_j + \sum_{k=0}^{1}\alpha_{j+k+1}u_k + \sum_{k=0}^{1}\beta_{j+k+1}v_k\Big)p_j + u_0 p_0 \\
(s-t)^2 p_0' &= \sum_{j=0}^{3}\Big(-\gamma_j + \sum_{k=0}^{1}\alpha_{j+k+1}w_k + \sum_{k=0}^{2}\gamma_{j+k+1}\tilde{v}_k\Big)q_j - w_0 q_0 \\
&\quad + \sum_{j=0}^{2}\Big(-\alpha_j + \sum_{k=0}^{1}\alpha_{j+k+1}\tilde{v}_k + \sum_{k=0}^{1}\beta_{j+k+1}w_k\Big)p_j + \tilde{v}_0 p_0
\end{aligned}
$$

$$u_0' = -q_0 q_0,\ u_1' = -q_0 q_1,\ u_2' = -q_0 q_2,\ v_0' = -q_0 p_0,\ v_1' = -q_0 p_1,\ v_2' = -q_0 p_2$$

$$w_0' = -p_0 p_0,\ w_1' = -p_0 p_1$$

$$q_1 = s\,q_0 - v_0 q_0 + u_0 p_0,\ q_2 = s^2 q_0 - v_0 q_1 - v_1 q_0 + u_0 p_1 + u_1 p_0$$

$$q_3 = s^3 q_0 - v_0 q_2 - v_1 q_1 - v_2 q_0 + u_0 p_2 + u_1 p_1 + u_2 p_0$$

$$p_1 = s\,p_0 - w_0 q_0 + \tilde{v}_0 p_0,\ p_2 = s^2 p_0 - w_0 q_1 - w_1 q_0 + \tilde{v}_0 p_1 + \tilde{v}_1 p_0$$

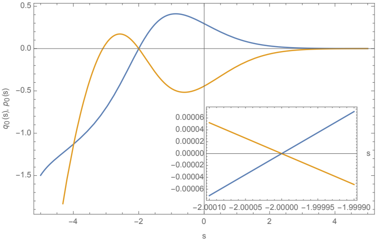
$$\tilde{v}_0 = v_0,\ \tilde{v}_1 = v_1 - v_0\tilde{v}_0 + u_0 w_0,\ \tilde{v}_2 = v_2 - v_0\tilde{v}_1 - v_1\tilde{v}_0 + u_0 w_1 + u_1 w_0$$

with appropriate BCs at $s \gg 1$

Janossy density for $\mathrm{K}_{\mathrm{Airy}}$

$q_0(s)$ and $p_0(s)$

$$R(s) = \partial_s \log \mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_{(s,\infty)}) = p_0(s) q_0'(s) - q_0(s) p_0'(s)$$



`NDSolve` from $q_0(8) = \mathrm{Ai}(8),\ p_0(8) = \mathrm{Ai}'(8)$ for $t = -2$

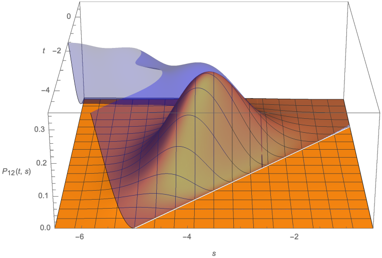
$\Rightarrow \quad q_0(-2) = p_0(-2) = 0.000000\ldots$

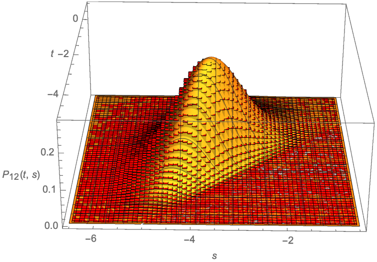
numerically very stable!

Janossy density for $\mathrm{K}_{\mathrm{Airy}}$

Joint distribution of 1st & 2nd largest EVs

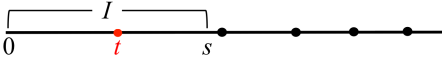
$$P_{12}(t,s) = \Theta(t-s)\rho_1(t)\partial_s \exp\left(-\int_s^\infty ds' R(s')\right)$$





Janossy density for $\mathrm{K}_{\mathrm{Bessel}}$

Evaluate $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}|_{(0,s)}) = J_1((0,s);t)$



$$\varphi(x) = J_\nu(\sqrt{x}), \quad \psi(x) = \frac{\sqrt{x}}{4}\left(J_{\nu-1}(\sqrt{x}) - J_{\nu+1}(\sqrt{x})\right)$$

$$m(x) = x, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = \frac{1}{4}(x - \nu^2)$$

$$\Downarrow$$

$$\boldsymbol{m}(x) = x(x-t)^2$$

$$\tilde{A}(x) = -ab(a^2-1) - a^2 t + \frac{\nu^2 b^2}{4}(ab - t) + \left(a^2 - \frac{ab^3}{4} + \frac{b^2 t}{4} + \frac{\nu^2 b^2}{4}\right)x - \frac{b^2}{4}x^2$$

$$\tilde{B}(x) = b^2(a^2-1) + 2abt + t^2 - \frac{\nu^2 b^4}{4} + \left(-2ab + \frac{b^4}{4} - 2t\right)x + x^2$$

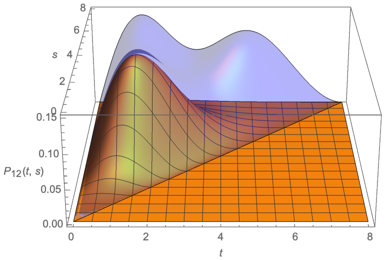
$$\tilde{C}(x) = a^2(a^2-1) - \frac{\nu^2}{4}(ab-t)^2 + \left(\frac{1}{4}(ab-t)^2 - \frac{\nu^2}{2}(ab-t)\right)x + \left(\frac{1}{2}(ab-t) - \frac{\nu^2}{4}\right)x^2 + \frac{x^3}{4}$$

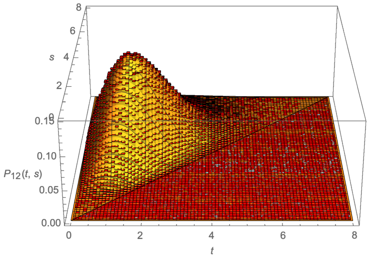
$\Rightarrow$ repeat the same tedious procedure ...

Janossy density for $\mathrm{K}_{\mathrm{Bessel}}$

Joint distribution of 1st & 2nd smallest SVs

$$P_{12}(t,s) = -\Theta(s-t)\rho_1(t)\partial_s \exp\left(-\int_0^s ds' R(s')\right)$$





Conclusion

Summary

- T-W method is proven to be directly applicable to Janossy density / Joint EV distribution for a $\mathbf{K}$ if it is applicable to its gap probability.
- Evaluated $P_{12}(t, s)$ for $\mathbf{K}_{\mathrm{Airy}}$ & $\mathbf{K}_{\mathrm{Bessel}}$ by brute force of `NDSolve[···]`. Results are in agreement with quadrature approx of $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_I)$.
- **Pros**: Universal method, applicable to $^{\forall}q$-orthogonal, finite-N,... kernels of your choice.
- **Cons**: Zero elegance. Lacks beauty of [Forrester-Witte 2007]. Hard to see relationship with PII & PIII' and associated isomonodromic systems. Not suited for asymptotic analysis.

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Perspectives

- $P_{1\cdots k}(s_1, \ldots, s_k)$ can be obtained by $k\times$repeating the procedure. Orders of polynomials $\tilde{A}(x)$ etc. increase by $2k$, but Mathematica® cares not.
- Although analytical application of our strategy to quaternion kernels of $\beta = 1, 4$ or transitive ensembles does not look promising, quadrature approx of $\mathrm{Det}(\mathbb{I} - \tilde{\mathbf{K}}_I)^{1/2}$ works perfectly well.
  $\Rightarrow$ Individual distributions $P_k(s; m; \mu)$ of staggered Dirac EVs of LQCD with $N_\mathrm{C} = 2, N_\mathrm{F} = 4, 8$ at finite chem. pot. can be predicted from the chGSE-chGUE transitive kernel of [Forrester-Nagao-Honner 1999].
  Precise determination of $\Sigma(m, \mu), F_\pi(m, \mu)$ by fitting Dirac spectra is ongoing [Kanamori-SN 2021].
- Should we press Wolfram to include the WBF distribution $A(x) = \int ds\, P_{12}(x + s, s)$ to basic Mathematica® commands?

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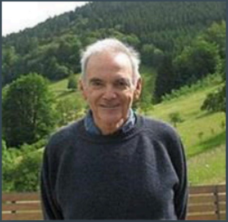
Harold Widom

1932 - 2021



BORN

1932

DIED

2021

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Harold Widom
September 23, 1932 - January 20, 2021
Santa Cruz, California

Harold Widom passed away peacefully at home with his family on Wednesday, January 20, 2021 due to complications of COVID-19.
Harold was born on September 23, 1932 to Morris and Rebecca Widom in Newark, New Jersey, the younger of their two sons. His father, a dentist,