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Supplement to the Paper "Construction of Regular $*$-Semigroups"

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This is a continuation of the previous paper "Note on Construction of Regular $*$-Semigroups, Mem. Fac. Sci., Shimane Univ. 15 (1981), 17–22". In this paper, we shall investigate the structure of completely regular [H-compatible regular, strongly H-compatible regular] $*$-semigroups.

§ 1. Preliminary

In the previous paper [2], the author has shown a way of constructing every general regular $*$-semigroup by means of a fundamental regular $*$-semigroup. This paper is a continuation of [2]. Every terminology and notation should be referred to [2], unless otherwise stated. In the previous paper [2], the following has been shown: Let Γ be a fundamental regular $*$-semigroup, and E_Γ and F_Γ the set of idempotents of Γ and the set of projections of Γ respectively. Let $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ a partial groupoid which is a disjoint sum of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ and satisfies the condition (C.0) of [2]. Put $N_M(F_\Gamma)=\Sigma\{S_\tau: \tau\in F_\Gamma\}$, and let $\Delta=\{\bar{\gamma}, C(\delta, \tau)\}_{\delta,\tau,\gamma\in\Gamma}$ a factor set belonging to $\{N_M(F_\Gamma), \Gamma\}$ (see [2]). Then, the $*$-regular product $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ (see [2]) of $N_M(F_\Gamma)$ and Γ becomes a regular $*$-semigroup, and coversely every regular $*$-semigroup can be obtained in this way. In this case, the multiplication and the special involution ($*$-operation) $*$ are given by

$$N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma=\{(a, \gamma): a\in S_{\gamma\gamma^*}, \gamma\in\Gamma\},$$

$$(a, \gamma)(b, \delta)=(ab^{\bar{\gamma}}C(\gamma, \delta), \gamma\delta),$$

$$(a, \gamma)^*=(t, \gamma^*), \text{ where } t \text{ is given by } t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1}.$$

In this paper, we investigate the following two special cases:

Case I. Γ is a fundamental completely regular $*$-semigroup.

Case II. Γ is an H-degenerate regular [orthodox] $*$-semigroup,[1] and M is a partial groupoid [semigroup] which is a disjoint union [a band E_Γ] of groups $\{S_\lambda: \lambda\in E_\Gamma\}$.

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1) A semigroup Γ is called H-degenerate if every H-class of Γ consists of a single element. A regular $*$-semigroup is called an orthodox $*$-semigroup if it is orthodox.

§2. Case I

Let Γ be a fundamental completely regular $*$-semigroup, and $\Delta=\{\bar{\gamma}, C(\delta, \eta)\}_{\gamma,\delta,\eta\in\Gamma}$ a factor set belonging to $\{N_M(F_\Gamma), \Gamma\}$, and assume that Δ satisfies the following condition:

(C.1) For any $\gamma\in\Gamma$, $C(\gamma, \gamma\gamma^{-1})=C(\gamma\gamma^{-1}, \gamma)=e_{\gamma\gamma^*}$, where $e_{\gamma\gamma^*}$ is the identity of $S_{\gamma\gamma^*}$.

Then, the following result can be obtained:

THEOREM 1. *The $*$-regular product $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is a completely regular $*$-semigroup. Further, every completely regular $*$-semigroup can be obtained in this fashion.*

PROOF. The first part: We need only to show that $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is a union of groups. Let $(a, \gamma), (b, \gamma^{-1})\in N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$. Then, $(a, \gamma)(b, \gamma^{-1})=(ab^{\bar{\gamma}}C(\gamma, \gamma^{-1}), \lambda)$, where $\lambda=\gamma\gamma^{-1}\in E_\Gamma$. Now, let c be an arbitrary element of $S_{\lambda\lambda^*}$. Then, $(ab^{\bar{\gamma}}C(\gamma, \gamma^{-1}), \lambda)(c, \lambda)=(ab^{\bar{\gamma}}C(\gamma, \gamma^{-1})c^{\bar{\lambda}}C(\lambda, \lambda), \lambda)$. We can easily see that for any (x, λ), where $x\in S_{\lambda\lambda^*}$ and $\lambda\in E_\Gamma$, $x^{\bar{\lambda}}=x$. In fact: Let $\lambda=\eta\delta$, where $\eta, \delta\in F_\Gamma$ (such elements η and δ exist). Now, $x^{\bar{\delta}\bar{\eta}}=x^{\overline{\delta\eta C(\eta,\delta)}}=C(\eta, \delta)x^{\overline{\eta\delta}}C(\eta, \delta)=e_{\lambda\lambda^*}x^{\bar{\lambda}}e_{\lambda\lambda^*}=x^{\bar{\lambda}}$. Therefore, $x^{\bar{\lambda}}=e_\eta e_\delta x e_\delta e_\eta=e_{\eta\delta}e_{\eta\delta\eta}xe_{\eta\delta\eta}e_{\delta\eta}$ (since $e_{\eta\delta\eta}x=xe_{\eta\delta\eta}=x$) $=e_{\eta\delta\eta}xe_{\eta\delta\eta}$ (since $\eta\delta\mathscr{R}\eta\delta\eta$ and $\eta\delta\eta\mathscr{L}\delta\eta$, where $\mathscr{R}$ and $\mathscr{L}$ are Green's R- and L-relations, it follows from (3) of (C.0) of [2] that $e_{\eta\delta}e_{\eta\delta\eta}=e_{\eta\delta\eta}$ and $e_{\eta\delta\eta}e_{\delta\eta}=e_{\eta\delta\eta}$) $=x$. Therefore, $(ab^{\bar{\gamma}}C(\gamma, \gamma^{-1})c^{\bar{\lambda}}C(\lambda, \lambda), \lambda)=(ab^{\bar{\gamma}}C(\gamma, \gamma^{-1})c, \lambda)$ (since $C(\lambda, \lambda)=e_{\lambda\lambda^*}$). Since c is an arbitrary element of $S_{\lambda\lambda^*}$, there exists $c'\in S_{\lambda\lambda^*}$ such that $(a, \gamma)(b, \gamma^{-1})(c', \lambda)=(e_{\lambda\lambda^*}, \lambda)$. Let $(b, \gamma^{-1})(c', \lambda)=(d, \gamma^{-1})$. Then, $(a, \gamma)(d, \gamma^{-1})=(e_{\lambda\lambda^*}, \lambda)$. Similarly, there exists (e, γ^{-1}) such that $(e, \gamma^{-1})(a, \gamma)=(e_{\lambda\lambda^*}, \lambda)$. Now, $(a, \gamma)(e_{\lambda\lambda^*}, \lambda)=(ae^{\bar{\gamma}}_{\lambda\lambda^*}C(\gamma, \lambda), \gamma)=(a, \gamma)$ (by (C.1)), and $(e_{\lambda\lambda^*}, \lambda)(a, \gamma)=(e_{\lambda\lambda^*}a^{\bar{\lambda}}C(\lambda, \gamma^{-1}\gamma)=(a, \gamma)$ (since $C(\lambda, \gamma)=e_{\gamma\gamma^*}$ and $a^{\bar{\gamma}}=a$). Further, $(d, \gamma^{-1})(e_{\lambda\lambda^*}, \lambda)=(de^{\overline{\gamma^{-1}}}_{\lambda\lambda^*}C(\gamma^{-1}. \lambda), \gamma^{-1})=(de_{\gamma^{-1}(\gamma^{-1})^*}e_{\gamma^{-1}(\gamma^{-1})^*}, \gamma^{-1})$ $=(d, \gamma^{-1})$. Similarly, $(e, \gamma^{-1})(e_{\lambda\lambda^*}, \lambda)=(e, \gamma^{-1})$. Now, $(e, \gamma^{-1})(a, \gamma)(d, \gamma^{-1})=(e, \gamma^{-1})(e_{\lambda\lambda^*}, \lambda)=(e, \gamma^{-1})$. Therefore, $(e_{\lambda\lambda^*}, \lambda)(d, \gamma^{-1})=(e, \gamma^{-1})$. Then, $(a, \gamma)(e_{\lambda\lambda^*}, \lambda)\cdot(d, \gamma^{-1})(e_{\lambda\lambda^*}, \lambda)=(e_{\lambda\lambda^*}, \lambda)$ and $(e_{\lambda\lambda^*}, \lambda)(d, \gamma^{-1})(e_{\lambda\lambda^*}, \lambda)(a, \gamma)=(e, \gamma^{-1})(a, \gamma)=(e_{\lambda\lambda^*}, \lambda)$. Thus, (a, γ) is contained in the maximal subgroup $G_{(e_{\lambda\lambda^*},\lambda)}$ (of $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$) containing $(e_{\lambda\lambda^*}, \lambda)$. Accordingly, $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is a union of groups.

The latter half: Let S be a completely regular $*$-semigroup. Let μ be the maximum idempotent separating congruence on S. Then, $\Gamma=S/\mu$ is a fundamental regular $*$-semigroup (of course, Γ is also a completely regular $*$-semigroup), and the natural homomorphism $\xi: S\to S/\mu$ is a $*$-homomorphism (see [1]). Hence, a $*$-peration $*$ in Γ can be defined by $(a\xi)^*=a^\#\xi$, where $\#$ is the $*$-operation in S. Further, it is obvious that $\lambda\xi^{-1}=S_\lambda$ is a subgroup of S for each $\lambda\in E_\Gamma$ (where E_Γ is the set of idempotents of Γ). Hence, $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ is a partial subgroupoid of S and satisfies (C.0) of [2]. Let $N_M(F_\Gamma)=\Sigma\{S_\lambda: \lambda\in F_\lambda\}$, where F_Γ is the set of projections of Γ, that is, $F_\Gamma=\{\tau\in E_\Gamma: \tau^*=\tau\}$. For any $\gamma\in\Gamma$, let $\gamma\xi^{-1}=S_\gamma$. Let x_γ be a representative of S_γ for each $\gamma\in\Gamma$, and especially $x_\lambda=e_\lambda$ for each $\lambda\in E_\Gamma$ (where e_λ is the identity of S_λ). Since $S_{\gamma\gamma^*}x_\gamma=S_\gamma$ (see [2]), for any x_γ, x_δ there exists $C(\gamma, \delta)\in S_{\gamma\delta(\gamma\delta)^*}$ such that $x_\gamma x_\delta = C(\gamma, \delta)x_{\gamma\delta}$. Then, $ux_\gamma vx_\delta=uv^{\bar{\gamma}}C(\gamma, \delta)x_{\gamma\delta}$ for $u\in S_{\gamma\gamma^*}$, $v\in S_{\delta\delta^*}$, where $v^{\bar{\gamma}}=x_\gamma vx_\gamma^*$. Now, it follows from [2] that $\Delta=\{\bar{\gamma}, C(\eta, \delta)\}_{\gamma,\delta,\eta\in\Gamma}$ satisfies (C.3) of [2] and $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is $*$-isomorphic to S. Therefore, to complete the proof, it is need only to show that $\{C(\eta, \delta);\ \eta, \delta\in\Gamma\}$ satisfies (C.1). Let $\gamma\gamma^{-1}=\lambda$. Then, $x_\lambda x_\gamma=C(\lambda, \gamma)x_\gamma$. Now, $e_\lambda x_\gamma x_{\gamma^*}=C(\lambda, \gamma)x_\gamma x_{\gamma^*}$, and hence $e_\lambda e_{\gamma\gamma^*}C(\gamma, \gamma^*)=C(\lambda, \gamma)e_{\gamma\gamma^*}C(\gamma, \gamma^*)$. Therefore, $e_{\gamma\gamma^*}C(\gamma, \gamma^*)=C(\lambda, \gamma)C(\gamma, \gamma^*)$, and $e_{\gamma\gamma^*}=C(\lambda, \gamma)e_{\gamma\gamma^*}$. Hence, $e_{\gamma\gamma^*}=C(\lambda, \gamma)$. Similarly, we have $C(\gamma, \lambda)=e_{\gamma\gamma^*}$. Thus, $\{C(\eta, \delta): \eta, \delta\in\Gamma\}$ satisfies (C.1).

§3. Case II

In this section, we shall prove the following result:

THEOREM 2. *Let $(\Gamma, *)$*[2] *be an H-degenerate regular [otrhodox] $*$-semigroup, and $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ a partial groupoid [semigroup] which is a disjoint union [a band] of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ and satisfies (C.0) of [2]. Let F_Γ be the set of projections of $(\Gamma, *)$, and put $N_M(F_\Gamma)=\Sigma\{S_\lambda: \lambda\in F_\Gamma\}$. Let $\Delta=\{\bar{\gamma}, C(\delta, \eta)\}_{\gamma,\delta,\eta\in\Gamma}$ be a factor set belonging to $\{N_M(F_\Gamma), \Gamma\}$. Then, the $*$-regular product $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is an H-compatible [strongly H-compatible]*[3] *regular $*$-semigroup. Further, every H-compatible [strongly H-compatible] regular $*$-semigroup can be constructed in this fashion.*

PROOF. The first half: We first consider the part [ ]. It is need only to show that if $(\Gamma, *)$ is orthodox and if M is a band E_Γ of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ which satisfies (C.0) of [2] then the maximal subgroups of $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ form a band of groups. Now, the set of idempotents of $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is $E=\{(e_{\lambda\lambda^*}, \lambda): \lambda\in E_\Gamma\}$, where e_τ is the identity of S_τ for each $\tau\in E_\Gamma$ (see Theorem 2.1 of [2]). It is also easy to see that the H-class containing $(e_{\lambda\lambda^*}, \lambda)$ is $\{(a, \lambda): a\in S_{\lambda\lambda^*}\}=\underline{S}_\lambda$. Let $\lambda, \delta\in E_\Gamma$. Then, $(a, \lambda)(b, \delta)= (ab^{\bar{\lambda}}C(\lambda, \delta), \lambda\delta)$ and $\lambda\delta\in E_\Gamma$ since E_Γ is a band. Therefore, the maximal subgroups of $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ form a band E_Γ of the groups $\{\underline{S}_\lambda: \lambda\in E_\Gamma\}$.

Next consider the case where $(\Gamma, *)$ is an H-degenerate regular semigroup and M is a partial groupoid which is a disjoint union of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ and satisfies (C.0) of [2]. Now, $(a, \gamma)\mathscr{H}(b, \delta)$ implies $(a, \gamma)\mathscr{L}(b, \delta)$, where $\mathscr{H}$ and $\mathscr{L}$ are Green's H- and

2) A regular $*$-semigroup Γ with an involution $*$ is sometimes denoted by $(\Gamma, *)$.

3) A regular semigroup S is said to be H-compatible [strongly H-compatible] if the Green's H-relation on S is a congruence [if the maximal subgroups of S form a band of groups].

L-relations respectively. Hence, there exist (c, δ_1), (d, δ_2) such that $(c, \delta_1)(a, \gamma)=(b, \delta)$ and $(d, \delta_2)(b, \delta)=(a, \gamma)$. This implies $\delta_1\gamma=\delta$ and $\delta_2\delta=\gamma$, and hence $\gamma\mathscr{L}\delta$. Similarly, we have $\delta\mathscr{R}\gamma$, where $\mathscr{R}$ is Green's R-relation. That is, $r\mathscr{H}\delta$. Since Γ is H-degenerate, $\delta=\gamma$. Conversely assume that $\gamma=\delta$. Next, we shall show $(a, \gamma)\mathscr{H}(b, \delta)$. For any $c\in S_{\gamma^*\gamma}$, $(a, \gamma)(c, \gamma^*\gamma)=(ac^{\bar{\gamma}}C(\gamma, \gamma^*\gamma), \gamma\gamma^*\gamma)=(ac^{\bar{\gamma}}e_{\gamma\gamma^*}, \gamma)=(ac^{\bar{\gamma}}, \gamma)$. Since $\bar{\gamma}$ maps $S_{\gamma^*\gamma}$ onto $S_{\gamma\gamma^*}$, there exists c such that $c^{\bar{\gamma}}=a^{-1}b$, $a^{-1}\in S_{\gamma\gamma^*}$. Hence, $(ac^{\bar{\gamma}}, \gamma)=(b, \gamma)$. Therefore, $(a, \gamma)(c, \gamma^*\gamma)=(b, \gamma)$. Similarly, there exists $(t, \gamma^*\gamma)$ such that $(b, \gamma)(t, \gamma^*\gamma)=(a, \gamma)$. Hence, $(a, \gamma)\mathscr{R}(b, \gamma)$. Next, for $c\in S_{\gamma\gamma^*}$, $(c, \gamma\gamma^*)(a, \gamma)=(ca^{\overline{\gamma\gamma^*}}C(\gamma\gamma^*, \gamma), \gamma)=(ca^{\overline{\gamma\gamma^*}}e_{\gamma\gamma^*}, \gamma)=(ca^{\overline{\gamma\gamma^*}}, \gamma)=(ce_{\gamma\gamma^*}ae_{\gamma\gamma^*}, \gamma)=(ca, \gamma)$. The element ba^{-1} is contained in $S_{\gamma\gamma^*}$. If we take $c=ba^{-1}$, $(ca, \gamma)=(b, \gamma)$. Hence, $(c, \gamma\gamma^*)(a, \gamma)=(b, \gamma)$. Similarly, there exists $(d, \gamma\gamma^*)$ such that $(d, \gamma\gamma^*)(b, \gamma)=(a, \gamma)$. Hence, $(a, \gamma)\mathscr{L}(b, \gamma)$. Consequently, $(a, \gamma)\mathscr{H}(b, \gamma)$. Therefore, $\mathscr{H}$ is a congruence. Hence, $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ is H-compatible.

The latter half: Let $(S, \sharp)$ be an H-compatible regular $*$-semigroup. Since $S/\mathscr{H}=\Gamma$ is an H-degenerate regular $*$-semigroup with respect to the $*$-operation defined by $(x\mathscr{H})^*=x^\sharp\mathscr{H}$, it follows from Lemma 3.1 of [2] that S is $*$-isomorphic to $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$, where $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ ($S_\gamma=\gamma f^{-1}$, where f is the natural homomorphism of S onto $S/\mathscr{H}$), $N_M(F_\Gamma)=\Sigma\{S_\tau: \tau\in F_\Gamma\}$, and Δ a factor set belonging to $\{N_M(F_\Gamma), \Gamma\}$. In particular, if $(S, \sharp)$ above is strongly H-compatible, then it is easy to see that $S/\mathscr{H}=\Gamma$ is an H-degenerate orthodox $*$-semigroup and $S\cong N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$.

Remark. If $(\Gamma, *)$ is a fundamental orthodox $*$-semigroup, and M is a band E_Γ of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ satisfying (C.0) of [2], then $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ above is orthodox if and only if it satisfies the following condition:

(C.2) $$e_{\lambda\lambda^*}e_{\delta\delta^*}^{\bar{\lambda}}C(\lambda, \delta)=e_{\lambda\delta(\lambda\delta)^*} \quad \text{for any} \quad \lambda, \delta\in E_\Gamma.$$

In fact, let $(e_{\lambda\lambda^*}, \lambda)$, $(e_{\delta\delta^*}, \delta)$ be two idempotents of $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$. Then, it is easy to see that $(e_{\lambda\lambda^*}, \lambda)(e_{\delta\delta^*}, \delta)=(e_{\lambda\delta(\lambda\delta)^*}, \lambda\delta)$ if and only if $e_{\lambda\lambda^*}e_{\delta\delta^*}^{\bar{\lambda}}C(\lambda, \delta)=e_{\lambda\delta(\lambda\delta)^*}$. Hence, we have the following result:

$N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$, where $(\Gamma, *)$ is a fundamental orthodox $*$-semigroup and M is a band E_Γ of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ satisfying (C.0) of [2], is an orthodox $*$-semigroup if and only if the factor set Δ satisfies (C.2). Further, every orthodox $*$-semigroup can be obtained in this way.

References

[1] Yamada, M., On the structure of fundamental regular $*$-semigroups, Studia Sci. Math. Hungarica **16** (1981), 281–288.

[2] ———, Note on the construction of regular $*$-semigroups, Mem. Fac. Sci., Shimane Univ. **15** (1981), 17–22.