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EXAMPLES OF PSEUDO-HERMITIAN SYMMETRIC SPACES SATISFYING A CERTAIN SUPPOSITION

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Abstract. The main purpose of this paper is to give examples of effective semisimple pseudo-Hermitian symmetric spaces satisfying a certain supposition (S). If an effective semisimple pseudo-Hermitian symmetric space satisfies the supposition (S), then one can clarify several properties of the pseudo-Hermitian symmetric space—for example, any holomorphic function on the space is constant, the group of holomorphic automorphisms of the space is a (finite-dimensional) Lie group, and so on.

1. Introduction

For a complex manifold M we can set complex vector spaces, e.g., the complex vector space $\mathcal{O}(M)$ of holomorphic functions, the complex vector space $\mathcal{O}(T^{1,0}M)$ of holomorphic vector fields and the complex vector space $\Omega^r(M)$ of holomorphic r-forms, or more generally the complex vector space $\mathcal{V}_M$ of holomorphic cross-sections of a holomorphic vector bundle over M. These vector spaces sometimes play important roles in the study of complex manifold M. We think it is meaningful to judge whether the vector space $\mathcal{V}_M$ is finite-dimensional or not for a given connected complex manifold M.

This paper is a sequel to the paper [4]. In [4] we have dealt with the complex vector space $\mathcal{V}_{G/L}$ of holomorphic cross-sections of a homogeneous holomorphic vector bundle over a homogeneous pseudo-Käher manifold G/L of connected semisimple Lie group G and provided a sufficient condition (S) for the vector space $\mathcal{V}_{G/L}$ to be finite-dimensional in the case where G acts effectively on G/L. When the supposition (S) holds for G/L, it follows that $\dim_{\mathbb{C}} \mathcal{O}(G/L) < \infty$, $\dim_{\mathbb{C}} \mathcal{O}(T^{1,0}(G/L)) < \infty$ and $\dim_{\mathbb{C}} \Omega^r(G/L) < \infty$; and furthermore, one can assert that any holomorphic function on G/L is constant, the group $\mathrm{Hol}(G/L)$ of holomorphic automorphisms is a Lie group and so on. Then we want to give concrete examples of homogeneous

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pseudo-Käher manifolds G/L satisfying the supposition (S). Now, the open unit disk D in $\mathbb{C}$, the upper half-plane H in $\mathbb{C}$ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are effective semisimple Hermitian symmetric spaces. Any effective semisimple Hermitian symmetric space is one of the effective semisimple pseudo-Hermitian symmetric spaces. These imply that the set of effective semisimple pseudo-Hermitian symmetric spaces includes significant connected complex manifolds. Fortunately, an effective semisimple pseudo-Hermitian symmetric space is a homogeneous pseudo-Käher manifold G/L of connected semisimple Lie group G such that G acts effectively on G/L.

The main purpose of this paper is to give examples of effective semisimple pseudo-Hermitian symmetric spaces satisfying the supposition (S). See Theorem 3.25.

This paper consists of three sections. In Section 2 we recall fundamental facts about pseudo-Hermitian symmetric spaces and explain the supposition (S) more precisely (cf. Proposition 2.7). In Section 3 we devote ourselves to finding out pseudo-Hermitian symmetric spaces satisfying (S).

Notation. For a Lie group G, we denote its Lie algebra by the corresponding Fraktur small letter $\mathfrak{g}$ and utilize the following notation:

(n1) Ad, ad : the adjoint representations of G and $\mathfrak{g}$, respectively,
(n2) $C_G(T) := \{g \in G \mid \operatorname{Ad} g(T) = T\}$ for an element $T \in \mathfrak{g}$,
(n3) $Z(G)$: the center of G,
(n4) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$,
(n5) $i := \sqrt{-1}$,
(n6) $f|_A$: the restriction of a mapping f to a set A,
(n7) ϕ_* : the differential homomorphism of a Lie group homomorphism ϕ,
(n8) I_n : the unit matrix of degree n,
(n9) $E_{i,j}$: the matrix whose (i, j)-element is 1 and whose other elements are all 0.

2. Preliminaries

This section consists of two subsections. In Subsection 2.1 we recall that (a) an effective semisimple pseudo-Hermitian symmetric space G/L is an elliptic adjoint orbit,[1] (b) G/L can be embedded into a complex flag manifold $G_{\mathbb{C}}/Q^-$ via $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, and (c) its image $\iota(G/L)$ is a simply connected domain in $G_{\mathbb{C}}/Q^-$. In Subsection 2.2 we take the complex vector space $\mathcal{V}_{G/L}$ of holomorphic cross-sections of a holomorphic vector bundle $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and provide a sufficient condition for the vector space $\mathcal{V}_{G/L}$ to be finite-dimensional.

2.1. Pseudo-Hermitian symmetric spaces.

In this subsection we recall fundamental facts about pseudo-Hermitian symmetric spaces. A pseudo-Hermitian symmetric space is one of the affine symmetric spaces. First of all, let us recall the definition of affine symmetric space.

Definition 2.1 (cf. Nomizu [10, p.52, p.56]).

[1] We refer to Kobayashi [9] for the definitions of elliptic element and elliptic (adjoint) orbit.

(i) Let G be a connected (real) Lie group, and let L be a closed subgroup of G. Then the homogeneous space G/L is called an *affine symmetric space*, if there exists an involutive automorphism σ of G satisfying

$$(G^\sigma)_0 \subset L \subset G^\sigma,$$

where $(G^\sigma)_0$ stands for the identity component of $G^\sigma := \{x \in G \mid \sigma(x) = x\}$.

(ii) An affine symmetric space G/L is said to be *effective* (resp. *almost effective*), if G is effective (resp. almost effective) on G/L as a transformation group.

(iii) An affine symmetric space G/L is said to be *semisimple* (resp. *simple*), if the Lie algebra $\mathfrak{g}$ of G is semisimple (resp. simple).

(iv) An almost effective, semisimple affine symmetric space $(G/L, \sigma)$ is said to be *irreducible*, if $\operatorname{ad}\mathfrak{l}$ in $\mathfrak{u}$ is irreducible. Here, $\mathfrak{l} = \{X \in \mathfrak{g} \mid \sigma_*(X) = X\}$ and $\mathfrak{u} = \{Y \in \mathfrak{g} \mid \sigma_*(Y) = -Y\}$.

Here is the definition of pseudo-Hermitian symmetric space:

Definition 2.2 (cf. Berger [1, p.94])**.**

(1) An affine symmetric space G/L is said to be *pseudo-Hermitian*, if it admits a G-invariant complex structure J and a G-invariant pseudo-Hermitian metric g with respect to J.

(2) A symmetric Lie algebra $(\mathfrak{g}, \mathfrak{l}, \sigma)$ is said to be *pseudo-Hermitian*, if there exist an $\operatorname{ad}\mathfrak{l}$-invariant complex structure j on $\mathfrak{u}$ and an $\operatorname{ad}\mathfrak{l}$-invariant pseudo-Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle$ (with respect to j) on $\mathfrak{u}$. Here $\mathfrak{u} = \{Y \in \mathfrak{g} \mid \sigma(Y) = -Y\}$.

One knows the following fact:

Proposition 2.3 (cf. Shapiro [11, pp.533–534])**.** *Let $(G/L, \sigma, J, \mathsf{g})$ be any almost effective, semisimple pseudo-Hermitian symmetric space, and let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$ be the decomposition of $\mathfrak{g}$ with respect to σ_*. Then, there exists a unique $T \in \mathfrak{l}$ satisfying*

(i) $L = C_G(T) = (G^\sigma)_0$, (ii) $\sigma(g) = (\exp \pi T) g \exp(-\pi T)$ *for all* $g \in G$,
(iii) $J_o = \operatorname{ad} T$ *on* $T_o(G/L) = \mathfrak{u}$.

Here $\mathfrak{u}$ is identified with the tangent space $T_o(G/L)$ of G/L at the origin o.

Remark 2.4. Let us comment on the element T in Proposition 2.3.

(1) T is called the *canonical central element* of $\mathfrak{l}$. cf. Shapiro [11, p.533].

(2) T is a non-zero element of $\mathfrak{g}$ such that the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto [T, X]$, is semisimple and its eigenvalue is $\pm i$ or zero. Thus T is a non-zero elliptic element of $\mathfrak{g}$.

(3) Proposition 2.3-(i) tells us that the pseudo-Hermitian symmetric space G/L is the adjoint orbit of G through T, so that G/L is an elliptic adjoint orbit.

From now on, we are going to set the generalized Borel embedding by means of Shapiro [11] (see Proposition 2.5-(v) below). Let $G/L = (G/L, \sigma, J, \mathsf{g})$ be an effective semisimple pseudo-Hermitian symmetric space, and let T be the canonical central element of $\mathfrak{l}$. Proposition 2.3-(i) implies that $L = C_G(T)$ includes the center $Z(G)$ of G, and therefore $Z(G)$ is trivial because G acts effectively on G/L. That is to say, the connected semisimple Lie group G is isomorphic to the adjoint group of $\mathfrak{g}$. Consequently there exists a connected complex semisimple Lie group $G_{\mathbb{C}}$ so that

(1) $Z(G_{\mathbb{C}})$ is trivial,
(2) G is a connected closed subgroup of $G_{\mathbb{C}}$,
(3) $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$.

In this setting we put

$$\begin{aligned} &\mathfrak{g}^0 := \{Z \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(Z) = 0\}, \quad \mathfrak{g}^{-1} := \{W \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(W) = -iW\}, \\ &Q^- := \{q \in G_{\mathbb{C}} \mid \operatorname{Ad} q(\mathfrak{g}^0 \oplus \mathfrak{g}^{-1}) \subset \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}\}. \end{aligned} \tag{2.1}$$

In addition, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition such that $T \in \mathfrak{k}$, let G_u be the connected Lie subgroup of $G_{\mathbb{C}}$ corresponding to a (maximal compact) subalgebra $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$, and let $L_u := C_{G_u}(T)$. Define an inner automorphism σ_u of G_u by $\sigma_u(y) := (\exp \pi T) y \exp(-\pi T)$ for $y \in G_u$. Then, σ_u is involutive and $(G_u/L_u, \sigma_u)$ is an affine symmetric space. In view of the decomposition $\mathfrak{g}_u = \mathfrak{l}_u \oplus \mathfrak{u}_u$ of $\mathfrak{g}_u$ with respect to $(\sigma_u)_*$, one can construct a G_u-invariant complex structure J_u on G_u/L_u and a G_u-invariant Hermitian metric g_u on G_u/L_u from

$$(J_u)_o X := \operatorname{ad} T(X), \quad (\mathsf{g}_u)_o(Y, Z) := -B_{\mathfrak{g}_u}(Y, Z)$$

for $X, Y, Z \in \mathfrak{u}_u$, respectively, where we identify the vector space $\mathfrak{u}_u$ with the tangent space $T_o(G_u/L_u)$ at the origin $o \in G_u/L_u$ and denote by $B_{\mathfrak{g}_u}$ the Killing form of $\mathfrak{g}_u$.

Proposition 2.5 (cf. Shapiro [11][2]). *In the setting above*;

(i) $G_u/L_u = (G_u/L_u, \sigma_u, J_u, \mathsf{g}_u)$ *is an effective semisimple Hermitian symmetric space of the compact type,*
(ii) $L_u = G_u \cap Q^-$ *and* $L = G \cap Q^-$,
(iii) Q^- *is a connected, closed complex parabolic subgroup of* $G_{\mathbb{C}}$,
(iv) $\iota_u : G_u/L_u \to G_{\mathbb{C}}/Q^-$, $yL_u \mapsto yQ^-$, *is a* G_u*-equivariant biholomorphism of* G_u/L_u *onto* $G_{\mathbb{C}}/Q^-$,
(v) $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, *is a* G*-equivariant biholomorphism of* G/L *onto a simply connected domain in* $G_{\mathbb{C}}/Q^-$,
(vi) $G_uQ^- = G_{\mathbb{C}}$, *and* GQ^- *is a domain in* $G_{\mathbb{C}}$.

Remark 2.6. Here are comments on the mapping $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, in Proposition 2.5-(v).

(1) ι is called the *generalized Borel embedding.* cf. Shapiro [11, p.535].
(2) One can regard G/L as a simply connected domain in $G_{\mathbb{C}}/Q^-$ via ι.

2.2. **Homogeneous holomorphic vector bundles and a certain supposition.** In this subsection we take the complex vector space $\mathcal{V}_{G/L}$ of holomorphic cross-sections of a holomorphic vector bundle $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ and provide a sufficient condition for the vector space $\mathcal{V}_{G/L}$ to be finite-dimensional (see Proposition 2.7).

[2]We slightly modify Theorem 3.1 in Shapiro [11, p.535]. See Lemma 8.1.11-(1), Proposition 8.2.1-(ii), (iii), (v) and Lemma 11.1.2 in [5] if necessary.

Let $G/L = (G/L, \sigma, J, \mathsf{g})$ be an effective semisimple pseudo-Hermitian symmetric space, and let T be the canonical central element of $\mathfrak{l}$. We construct a complex flag manifold $G_{\mathbb{C}}/Q^-$ from (2.1), fix the generalized Borel embedding $\iota : G/L \to G_{\mathbb{C}}/Q^-$, $gL \mapsto gQ^-$, and identify G/L with $\iota(G/L)$.

$$\begin{array}{ccc} \iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V}) & & G_{\mathbb{C}} \times_\rho \mathsf{V} \\ \downarrow & & \downarrow \\ G/L & \xrightarrow{\ \iota\ } & G_{\mathbb{C}}/Q^- \end{array}$$

Take a finite-dimensional complex vector space V and a holomorphic homomorphism $\rho : Q^- \to GL(\mathsf{V})$, $q \mapsto \rho(q)$, where $GL(\mathsf{V})$ is the general linear group on V. Denote by $G_{\mathbb{C}} \times_\rho \mathsf{V}$ the homogeneous holomorphic vector bundle over $G_{\mathbb{C}}/Q^-$ associated with ρ, and by $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$ the restriction of $G_{\mathbb{C}} \times_\rho \mathsf{V}$ to $G/L \subset G_{\mathbb{C}}/Q^-$. Let

$$(2.2)\quad \begin{aligned} \mathcal{V}_{G_{\mathbb{C}}/Q^-} &:= \left\{ h : G_{\mathbb{C}} \to \mathsf{V} \,\middle|\, \begin{array}{l} \text{(i) } h \text{ is holomorphic,} \\ \text{(ii) } h(aq) = \rho(q)^{-1}\big(h(a)\big) \text{ for all } (a,q) \in G_{\mathbb{C}} \times Q^- \end{array} \right\}, \\ \mathcal{V}_{G/L} &:= \left\{ \psi : GQ^- \to \mathsf{V} \,\middle|\, \begin{array}{l} \text{(i) } \psi \text{ is holomorphic,} \\ \text{(ii) } \psi(xq) = \rho(q)^{-1}\big(\psi(x)\big) \text{ for all } (x,q) \in GQ^- \times Q^- \end{array} \right\}. \end{aligned}$$

Then one may assume that $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and $\mathcal{V}_{G/L}$ are the complex vector spaces of holomorphic cross-sections of the bundles $G_{\mathbb{C}} \times_\rho \mathsf{V}$ and $\iota^\sharp(G_{\mathbb{C}} \times_\rho \mathsf{V})$, respectively.

In general, the vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ is finite-dimensional (because $G_{\mathbb{C}}/Q^-$ is a connected compact complex manifold), but, in contrast, $\mathcal{V}_{G/L}$ is not always finite-dimensional. From now on, we are going to provide a sufficient condition for $\mathcal{V}_{G/L}$ to be finite-dimensional. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ with $T \in \mathfrak{k}$, and a maximal torus $i\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ containing T. Let $\mathfrak{h}_{\mathbb{C}}$ be the complex vector subspace of $\mathfrak{g}_{\mathbb{C}}$ generated by $i\mathfrak{h}_{\mathbb{R}}$, let $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$, let $\mathfrak{g}_\alpha$ be the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for $\alpha \in \triangle$, and let $\mathfrak{k}_{\mathbb{C}}$ be the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{k}$. Then one has

Proposition 2.7. *In the setting of Subsection* 2.2; *suppose that* (S) *there exists a fundamental root system* $\Pi_\triangle$ *of* $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ *satisfying two conditions*

(s1) $\alpha(-iT) \geq 0$ *for all* $\alpha \in \Pi_\triangle$, *and*

(s2) $\mathfrak{g}_\beta \subset \mathfrak{k}_{\mathbb{C}}$ *for every* $\beta \in \Pi_\triangle$ *with* $\beta(T) \neq 0$.

Then, the complex vector space $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ *is linear isomorphic to* $\mathcal{V}_{G/L}$ *via*

$$F : \mathcal{V}_{G_{\mathbb{C}}/Q^-} \to \mathcal{V}_{G/L},\ h \mapsto h|_{GQ^-};$$

and therefore $\dim_{\mathbb{C}} \mathcal{V}_{G/L} = \dim_{\mathbb{C}} \mathcal{V}_{G_{\mathbb{C}}/Q^-} < \infty$.

Proof. At this stage, our setting is as follows:

- $G_{\mathbb{C}}$ is a connected complex semisimple Lie group with the trivial center,
- G is a connected closed subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$,
- T is a non-zero elliptic element of $\mathfrak{g}$,
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$ with $T \in \mathfrak{k}$,
- $i\mathfrak{h}_{\mathbb{R}}$ is a maximal torus of $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ containing T,

- $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$, where $\mathfrak{h}_{\mathbb{C}}$ is the complex vector subspace of $\mathfrak{g}_{\mathbb{C}}$ generated by $i\mathfrak{h}_{\mathbb{R}}$,
- $\mathfrak{g}_\alpha$ is the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for $\alpha \in \triangle$,
- $L = C_G(T)$,
- Q^- is the closed complex subgroup of $G_{\mathbb{C}}$ defined by (2.1),
- $\mathfrak{k}_{\mathbb{C}}$ is the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{k}$,
- V is a finite-dimensional complex vector space,
- $\rho : Q^- \to GL(\mathsf{V})$, $q \mapsto \rho(q)$, is a holomorphic homomorphism,
- $\mathcal{V}_{G_{\mathbb{C}}/Q^-}$ and $\mathcal{V}_{G/L}$ are the complex vector spaces defined by (2.2).

Since we conform to the setting of Subsection 3.1 in [4], we can apply Theorem 3.1 in [4] to this proposition. Thus we can get the conclusion. □

Remark 2.8. Here are comments on Proposition 2.7.

(1) One can always take a fundamental root system $\Pi_\triangle \subset \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with (s1), by considering the lexicographic linear ordering on the dual space $(\mathfrak{h}_{\mathbb{R}})^*$ associated with an ordered real basis $-iT =: A_1, A_2, \dots, A_\ell$ of $\mathfrak{h}_{\mathbb{R}}$.
(2) If G is compact, then the pseudo-Hermitian symmetric space G/L always satisfies the supposition (S) because of $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$.
(3) If G/L is a symmetric bounded domain in $\mathbb{C}^n$, then it cannot satisfy the supposition (S) at all. cf. Example 4.2 in [4].

3. Examples of pseudo-Hermitian symmetric spaces satisfying (S)

Our aim is to find out effective semisimple pseudo-Hermitian symmetric spaces which satisfy the supposition (S) in Proposition 2.7.

3.1. **Reduction.** An effective semisimple pseudo-Hermitian symmetric space is biholomorphic to the direct product $G_1/L_1 \times G_2/L_2 \times \cdots \times G_r/L_r$, where all $G_1/L_1, \dots, G_r/L_r$ are effective simple pseudo-Hermitian symmetric spaces. There are four types of simple pseudo-Hermitian symmetric spaces:

Table A: four types of simple pseudo-Hermitian symmetric spaces	
(I)	an irreducible Hermitian symmetric space of the compact type
(II)	an irreducible Hermitian symmetric space of the non-compact type
(III)	a simple irreducible pseudo-Hermitian (non-Hermitian) symmetric space
(IV)	a simple reducible pseudo-Hermitian symmetric space

Here a simple pseudo-Hermitian symmetric space G/L is *reducible* if and only if the Lie algebra $\mathfrak{g}$ is complex (cf. Shapiro [11, p.532]). From the next subsection we will mainly deal with (III) effective simple irreducible pseudo-Hermitian (non-Hermitian) symmetric spaces; due to Remark 2.8-(2), (3) and

Proposition 3.1. (IV) *Any effective simple reducible pseudo-Hermitian symmetric space G/L cannot satisfy the supposition* (S) *in Proposition* 2.7 *at all.*

Proof. Let $\overline{\mathfrak{g}}$ be the complex conjugate Lie algebra to $\mathfrak{g}$. Then, the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of $\mathfrak{g}$ is complex Lie algebra isomorphic to the direct product $\mathfrak{g} \times \overline{\mathfrak{g}}$ via

$$\phi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g} \times \overline{\mathfrak{g}},\ X + iY \mapsto (X + iY, X - iY)$$

$(X, Y \in \mathfrak{g})$ since $\lambda(A, B) = (\lambda A, \overline{\lambda} B)$ for all $\lambda \in \mathbb{C}$ and $(A, B) \in \mathfrak{g} \times \overline{\mathfrak{g}}$. Moreover, the real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ corresponds to $\{(X, X) \,|\, X \in \mathfrak{g}\} = \phi(\mathfrak{g}) \subset \mathfrak{g} \times \overline{\mathfrak{g}}$. Identifying $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g} \times \overline{\mathfrak{g}}$ via ϕ, we will explain the reason why G/L cannot satisfy the condition (s2) $\mathfrak{g}_\beta \subset \mathfrak{k}_{\mathbb{C}}$ in Proposition 2.7.

Let T be the canonical central element of $\mathfrak{l}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with $T \in \mathfrak{k}$, and let $i\mathfrak{h}_{\mathbb{R}}$ be a maximal torus of $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ containing T. Besides, let $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ be the complex vector subspace and subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $i\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{k}$, respectively. Then $\mathfrak{k}_{\mathbb{C}}$ corresponds to

$$\{(K_1 + iK_2, K_1 - iK_2) \,|\, K_1, K_2 \in \mathfrak{k}\} = \phi(\mathfrak{k}_{\mathbb{C}}). \tag{3.1}$$

Since $\mathfrak{g}$ is complex (semi)simple, it follows that $\mathfrak{p} = i\mathfrak{k}$, so that $\mathfrak{g}_u$ corresponds to

$$\mathfrak{k} \times \mathfrak{k} = \phi(\mathfrak{g}_u).$$

Consequently there exist maximal tori $\mathfrak{t}_1$, $\mathfrak{t}_2 \subset \mathfrak{k}$ such that $T \in \mathfrak{t}_1 \cap \mathfrak{t}_2$ and $\mathfrak{t}_1 \times \mathfrak{t}_2 = \phi(i\mathfrak{h}_{\mathbb{R}})$. Letting $\mathfrak{c}_1$ and $\overline{\mathfrak{c}}_2$ be the complex vector subspaces of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ generated by $\mathfrak{t}_1$ and $\mathfrak{t}_2$, respectively, one can conclude that

$$\mathfrak{c}_1 \times \overline{\mathfrak{c}}_2 = \phi(\mathfrak{h}_{\mathbb{C}}).$$

From now on, we are going to confirm that G/L cannot satisfy the (s2). Let us use proof by contradiction. Suppose a root $\beta \in \triangle(\mathfrak{g} \times \overline{\mathfrak{g}}, \mathfrak{c}_1 \times \overline{\mathfrak{c}}_2)$ and a non-zero vector $E_\beta \in \mathfrak{g} \times \overline{\mathfrak{g}}$ to satisfy $[C, E_\beta] = \beta(C)E_\beta$ for all $C \in \mathfrak{c}_1 \times \overline{\mathfrak{c}}_2$ and $E_\beta \in \phi(\mathfrak{k}_{\mathbb{C}})$. Then, $\triangle(\mathfrak{g} \times \overline{\mathfrak{g}}, \mathfrak{c}_1 \times \overline{\mathfrak{c}}_2) \cong \triangle(\mathfrak{g}, \mathfrak{c}_1) \cup \triangle(\overline{\mathfrak{g}}, \overline{\mathfrak{c}}_2)$ implies that one of the following two cases only occurs:

(1) $\beta(C_1, C_2) = \beta(C_1, 0)$ for all $(C_1, C_2) \in \mathfrak{c}_1 \times \overline{\mathfrak{c}}_2$ and there exists a non-zero vector $E_1 \in \mathfrak{g}$ such that $E_\beta = (E_1, 0)$;

(2) $\beta(C_1, C_2) = \beta(0, C_2)$ for all $(C_1, C_2) \in \mathfrak{c}_1 \times \overline{\mathfrak{c}}_2$ and there exists a non-zero vector $E_2 \in \overline{\mathfrak{g}}$ such that $E_\beta = (0, E_2)$.

However, in any cases (1) $E_\beta = (E_1, 0)$ and (2) $E_\beta = (0, E_2)$ we obtain $E_\beta \notin \phi(\mathfrak{k}_{\mathbb{C}})$ from (3.1), which is a contradiction to $E_\beta \in \phi(\mathfrak{k}_{\mathbb{C}})$. For this reason G/L cannot satisfy (s2) at all. □

Table B	
type	the supposition (S) in Proposition 2.7
(I)	O.K.
(II)	N.G.
(III)	?
(IV)	N.G.

(Here (I), (II), (III) and (IV) correspond to those in Table A (p.32), respectively).

3.2. **Type (III).** The main purpose of this subsection is to give examples of simple irreducible pseudo-Hermitian (non-Hermitian) symmetric Lie algebras $(\mathfrak{g}, \mathfrak{l})$ satisfying the supposition (S) in Proposition 2.7.

Remark 3.2. From a simple irreducible pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{l})$, one can easily construct an effective simple irreducible pseudo-Hermitian symmetric space G/L. Indeed; for a given simple irreducible pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{l})$, let us take the canonical central element $T \in \mathfrak{l}$ and a connected Lie group G whose center $Z(G)$ is trivial and whose Lie algebra is $\mathfrak{g}$. Then $(G, C_G(T))$ is an effective simple irreducible pseudo-Hermitian symmetric space.

3.2.1. *AII.* Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{sl}(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{C}) \mid \operatorname{tr} A = 0\}$,

$$\mathfrak{g} := \mathfrak{su}^*(2n) = \left\{ \begin{pmatrix} X & Y \\ -\overline{Y} & \overline{X} \end{pmatrix} \middle| \begin{array}{l} X, Y \in \mathfrak{gl}(n, \mathbb{C}), \\ \operatorname{tr} X + \operatorname{tr} \overline{X} = 0 \end{array} \right\},$$

$$\mathfrak{k} := \left\{ \begin{pmatrix} K_1 & K_2 \\ -\overline{K}_2 & \overline{K}_1 \end{pmatrix} \middle| \begin{array}{l} K_1, K_2 \in \mathfrak{gl}(n, \mathbb{C}), \\ {}^t\overline{K}_1 = -K_1,\ {}^tK_2 = K_2 \end{array} \right\},$$

$$\mathfrak{p} := \left\{ \begin{pmatrix} P_1 & P_2 \\ -\overline{P}_2 & \overline{P}_1 \end{pmatrix} \middle| \begin{array}{l} P_1, P_2 \in \mathfrak{gl}(n, \mathbb{C}), \\ {}^t\overline{P}_1 = P_1,\ \operatorname{tr} P_1 + \operatorname{tr} \overline{P}_1 = 0,\ {}^tP_2 = -P_2 \end{array} \right\},$$

$$\mathfrak{h}_{\mathbb{R}} := \left\{ \begin{pmatrix} x_1 & & O \\ & \ddots & \\ O & & x_{2n} \end{pmatrix} \middle| \begin{array}{l} x_1, x_2, \dots, x_{2n} \in \mathbb{R}, \\ \sum_{i=1}^{2n} x_i = 0 \end{array} \right\},$$

where $n \geq 2$. Then it follows that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, that $i\mathfrak{h}_{\mathbb{R}}$ is a maximal torus of $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$, and that

$$\text{(aii.1)} \qquad \mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| \begin{array}{l} A, B, C \in \mathfrak{gl}(n, \mathbb{C}), \\ {}^tB = B,\ {}^tC = C \end{array} \right\}.$$

By setting a linear mapping $\alpha_j : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ as

$$\alpha_j \begin{pmatrix} z_1 & & O \\ & \ddots & \\ O & & z_{2n} \end{pmatrix} := z_j - z_{j+1} \text{ for } 1 \leq j \leq 2n-1,$$

one can get a fundamental root system $\Pi_\triangle := \{\alpha_j\}_{j=1}^{2n-1}$ of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

$$\Pi_\triangle\colon \quad \overset{1}{\underset{\alpha_1}{\circ}} - \overset{1}{\underset{\alpha_2}{\circ}} - \cdots - \overset{1}{\underset{\alpha_{2n-1}}{\circ}}.$$

Now, let us put

$$\text{(aii.2)} \qquad T := \frac{i}{2} \begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix}.$$

In this setting, we have $T \in (\mathfrak{k} \cap i\mathfrak{h}_{\mathbb{R}}) \subset \mathfrak{g}$ and $\alpha_j(-iT) = \delta_{j,n}$ for all $1 \leq j \leq 2n-1$. Hence the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ is semisimple and its eigenvalue is $\pm i$ or zero, so $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$ is a pseudo-Hermitian (non-Hermitian) symmetric Lie algebra and T is the canonical central element of $\mathfrak{c}_{\mathfrak{g}}(T)$. cf. Lemma 3.1.1 in [3, pp.22–23]. By a direct computation we obtain $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{t}$. Furthermore,

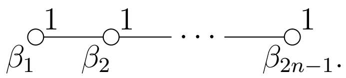
Proposition 3.3 (AII)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{su}^*(2n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7*. Here* $n \geq 2$ *and we refer to* (aii.2) *for* T*.*

Remark 3.4 (AII). If $n = 1$, then $(\mathfrak{su}^*(2n), \mathfrak{sl}(n,\mathbb{C}) \oplus \mathfrak{t})$ is an irreducible Hermitian symmetric Lie algebra of the compact type.

Proof of Proposition 3.3*.* Let

$$\begin{aligned} &\beta_k := \alpha_k \text{ for } 1 \leq k \leq n-1, \\ &\beta_n := \textstyle\sum_{p=n}^{2n-1} \alpha_p, \\ &\beta_{n+k} := -\alpha_{2n-k} \text{ for } 1 \leq k \leq n-1. \end{aligned}$$

Then $\Pi' := \{\beta_j\}_{j=1}^{2n-1}$ is a fundamental root system of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ whose Dynkin diagram is

Π': 

From (aii.2) we obtain $\beta_j(-iT) = \delta_{j,n} \geq 0$ for all $1 \leq j \leq 2n-1$, and the (s1) in Proposition 2.7 holds for this Π'. Moreover, (aii.1) implies that

$$\mathfrak{g}_{\beta_n} = \mathfrak{g}_{\alpha_n + \cdots + \alpha_{2n-1}} = \mathrm{span}_{\mathbb{C}}\{E_{n,2n}\} \subset \mathfrak{k}_{\mathbb{C}},$$

so that the (s2) in Proposition 2.7 also holds for Π'. □

3.2.2. *AIII.* Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{sl}(p+q,\mathbb{C}) = \{A \in \mathfrak{gl}(p+q,\mathbb{C}) \mid \mathrm{tr}\, A = 0\}$,

$$\mathfrak{g} := \mathfrak{su}(p,q) = \left\{ \begin{pmatrix} K_1 & Z \\ {}^t\overline{Z} & K_2 \end{pmatrix} \middle| \begin{array}{l} K_1 \in \mathfrak{u}(p),\ Z : p \times q \text{ complex matrix}, \\ K_2 \in \mathfrak{u}(q),\ \mathrm{tr}\, K_1 + \mathrm{tr}\, K_2 = 0 \end{array} \right\},$$

$$\mathfrak{k} := \left\{ \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix} \middle| \begin{array}{l} K_1 \in \mathfrak{u}(p),\ K_2 \in \mathfrak{u}(q), \\ \mathrm{tr}\, K_1 + \mathrm{tr}\, K_2 = 0 \end{array} \right\},$$

$$\mathfrak{p} := \left\{ \begin{pmatrix} O & Z \\ {}^t\overline{Z} & O \end{pmatrix} \middle| Z : p \times q \text{ complex matrix} \right\},$$

$$\mathfrak{h}_{\mathbb{R}} := \left\{ \begin{pmatrix} x_1 & & O \\ & \ddots & \\ O & & x_{p+q} \end{pmatrix} \middle| \begin{array}{l} x_1, x_2, \ldots, x_{p+q} \in \mathbb{R}, \\ \sum_{i=1}^{p+q} x_i = 0 \end{array} \right\},$$

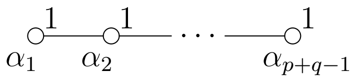
where $p, q \geq 1$ and $\mathfrak{u}(n) = \{K \in \mathfrak{gl}(n,\mathbb{C}) \mid {}^t\overline{K} = -K\}$. Then it turns out that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, $i\mathfrak{h}_{\mathbb{R}}$ is a maximal torus of $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$ and $i\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{k}$; besides,

(aiii.1) $$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} \middle| \begin{array}{l} B_1 \in \mathfrak{gl}(p,\mathbb{C}),\ B_2 \in \mathfrak{gl}(q,\mathbb{C}), \\ \mathrm{tr}\, B_1 + \mathrm{tr}\, B_2 = 0 \end{array} \right\}.$$

We define a linear mapping $\alpha_j : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ by

$$\alpha_j \begin{pmatrix} z_1 & & O \\ & \ddots & \\ O & & z_{p+q} \end{pmatrix} := z_j - z_{j+1} \text{ for } 1 \leq j \leq p+q-1$$

and obtain a fundamental root system $\Pi_{\triangle} := \{\alpha_j\}_{j=1}^{p+q-1}$ of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

$\Pi_\triangle$: 

Here, the dual basis $\{Z_j\}_{j=1}^{p+q-1}$ $(\subset \mathfrak{h}_\mathbb{R})$ of $\Pi_\triangle = \{\alpha_j\}_{j=1}^{p+q-1}$ is as follows:

$$Z_j = \frac{1}{p+q}\begin{pmatrix} (p+q-j)I_j & O \\ O & -jI_{p+q-j} \end{pmatrix} \quad (1 \leq j \leq p+q-1). \tag{aiii.2}$$

From now on, we are going to investigate the following three cases individually:

$$(1)\ T = iZ_a, \quad (2)\ T = iZ_{p+b}, \quad (3)\ T = i(Z_a - Z_p + Z_{p+b}),$$

where $1 \leq a \leq p-1$, $1 \leq b \leq q-1$. Remark that for each of the elements T above, we obtain $T \in i\mathfrak{h}_\mathbb{R} \subset \mathfrak{k} = (\mathfrak{g}_u \cap \mathfrak{g})$, the linear transformation $\operatorname{ad} T : \mathfrak{g} \to \mathfrak{g}$ is semisimple and its eigenvalue is $\pm i$ or zero; consequently, $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T))$ is a simple irreducible pseudo-Hermitian symmetric Lie algebra and T is the canonical central element of $\mathfrak{c}_\mathfrak{g}(T)$, cf. [3, pp.22–23].

Case (1). Let $T := iZ_a$ $(1 \leq a \leq p-1)$. Then one has $\alpha_j(-iT) = \alpha_j(Z_a) = \delta_{j,a} \geq 0$ for all $1 \leq j \leq p+q-1$, and the (s1) in Proposition 2.7 holds for $\Pi_\triangle = \{\alpha_j\}_{j=1}^{p+q-1}$. Furthermore, it follows from $1 \leq a \leq p-1$ and (aiii.1) that

$$\mathfrak{g}_{\alpha_a} = \operatorname{span}_\mathbb{C}\{E_{a,a+1}\} \subset \mathfrak{k}_\mathbb{C},$$

so that the (s2) in Proposition 2.7 holds for $\Pi_\triangle = \{\alpha_j\}_{j=1}^{p+q-1}$, also. Hence

Lemma 3.5 (AIII)**.** $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T)) = (\mathfrak{su}(p,q), \mathfrak{su}(a) \oplus \mathfrak{su}(p-a,q) \oplus \mathfrak{t})$ *satisfies the* (S) *in Proposition* 2.7*. Here* $1 \leq a \leq p-1$, $1 \leq q$ *and* $T = iZ_a$.

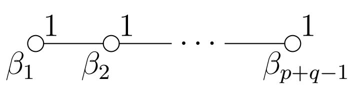
Case (2). In case of $T := iZ_{p+b}$ $(1 \leq b \leq q-1)$ one can demonstrate the following lemma by arguments similar to those in the case (1) above:

Lemma 3.6 (AIII)**.** $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T)) = (\mathfrak{su}(p,q), \mathfrak{su}(p,b) \oplus \mathfrak{su}(q-b) \oplus \mathfrak{t})$ *satisfies the* (S) *in Proposition* 2.7*. Here* $1 \leq p$, $1 \leq b \leq q-1$ *and* $T = iZ_{p+b}$.

Case (3). Now, let $T := i(Z_a - Z_p + Z_{p+b})$ $(1 \leq a \leq p-1,\ 1 \leq b \leq q-1)$, and set

$$\begin{aligned}
&\beta_k := \alpha_k \text{ for } 1 \leq k \leq a-1,\\
&\beta_a := \textstyle\sum_{n=a}^{p} \alpha_n,\\
&\beta_h := \alpha_{h-a+p} \text{ for } a+1 \leq h \leq a+q-1,\\
&\beta_{q+a} := -\textstyle\sum_{m=a+1}^{p+q-1} \alpha_m,\\
&\beta_\ell := \alpha_{\ell-q} \text{ for } q+a+1 \leq \ell \leq p+q-1.
\end{aligned}$$

Then we see that $\Pi' := \{\beta_j\}_{j=1}^{p+q-1}$ is a fundamental root system of $\triangle(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Π': 

Moreover, $\beta_j(-iT) = \delta_{j,a+b} \geq 0$ for all $1 \leq j \leq p+q-1$, and we deduce $\mathfrak{g}_{\beta_{a+b}} = \mathfrak{g}_{\alpha_{p+b}} = \operatorname{span}_\mathbb{C}\{E_{p+b,p+b+1}\} \subset \mathfrak{k}_\mathbb{C}$ from $1 \leq b \leq q-1$ and (aiii.1). Therefore the (s1) and (s2) in Proposition 2.7 hold for $\Pi' = \{\beta_j\}_{j=1}^{p+q-1}$.

Lemma 3.7 (AIII)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{su}(p,q), \mathfrak{su}(a,b) \oplus \mathfrak{su}(p-a, q-b) \oplus \mathfrak{t})$ *satisfies the* (S) *in Proposition* 2.7*. Here* $1 \leq a \leq p-1$, $1 \leq b \leq q-1$ *and* $T = i(Z_a - Z_p + Z_{p+b})$.

Three Lemmas 3.5, 3.6 and 3.7 provide us with

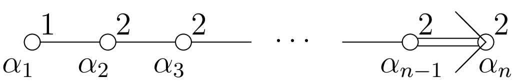
Proposition 3.8 (AIII)**.** *The supposition* (S) *in Proposition* 2.7 *holds for the following pseudo-Hermitian symmetric Lie algebras* $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$:

(1) $(\mathfrak{su}(p,q), \mathfrak{su}(a) \oplus \mathfrak{su}(p-a,q) \oplus \mathfrak{t})$, $1 \leq a \leq p-1$, $1 \leq q$ *and* $T = iZ_a$.
(2) $(\mathfrak{su}(p,q), \mathfrak{su}(p,b) \oplus \mathfrak{su}(q-b) \oplus \mathfrak{t})$, $1 \leq p$, $1 \leq b \leq q-1$ *and* $T = iZ_{p+b}$.
(3) $(\mathfrak{su}(p,q), \mathfrak{su}(a,b) \oplus \mathfrak{su}(p-a,q-b) \oplus \mathfrak{t})$, $1 \leq a \leq p-1$, $1 \leq b \leq q-1$ *and* $T = i(Z_a - Z_p + Z_{p+b})$.

Here we refer to (aiii.2) *for* Z_j $(1 \leq j \leq p+q-1)$.

3.2.3. *BI.* Let $\mathfrak{g}_{\mathbb{C}}$ be the classical complex simple Lie algebra of the type B_n $(n \geq 3)$. Assume that the Dynkin diagram of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is as follows:

(b.1) $\Pi_{\triangle}$:



(cf. Bourbaki [7, p.267]). Taking Chevalley's canonical basis $\{H^*_{\alpha_\ell}\}_{\ell=1}^n \cup \{E_\alpha \,|\, \alpha \in \triangle\}$ of $\mathfrak{g}_{\mathbb{C}}$, we construct a compact real form $\mathfrak{g}_u \subset \mathfrak{g}_{\mathbb{C}}$ from

$$\text{(b.2)} \qquad \begin{cases} \mathfrak{h}_{\mathbb{R}} := \operatorname{span}_{\mathbb{R}}\{H^*_{\alpha_\ell}\}_{\ell=1}^n, \\ \mathfrak{g}_u := i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \triangle} \operatorname{span}_{\mathbb{R}}\{E_\alpha - E_{-\alpha}\} \oplus \operatorname{span}_{\mathbb{R}}\{i(E_\alpha + E_{-\alpha})\}. \end{cases}$$

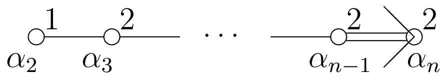
Denote by $\{Z_\ell\}_{\ell=1}^n$ $(\subset \mathfrak{h}_{\mathbb{R}})$ the dual basis of $\Pi_{\triangle} = \{\alpha_\ell\}_{\ell=1}^n$ and set an inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ as

$$\text{(bi.1)} \qquad \theta := \exp \pi \operatorname{ad} iZ_k,$$

where $1 \leq k \leq n$. Then θ is involutive, and (b.2) yields $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$, so we can consider the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_u$ with respect to θ and construct a noncompact real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. Here we remark that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{g}_u = \mathfrak{so}(2n+1)$, $\mathfrak{g} = \mathfrak{so}(2k, 2n-2k+1)$ and $i\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{k}$, and that

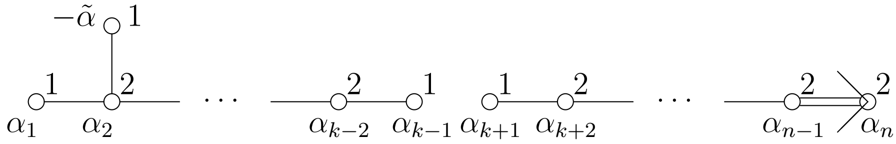
$\{\alpha_q\}_{q=2}^n$ $(k = 1)$,
$\{\alpha_a\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^n \cup \{-\tilde{\alpha}\}$ $(2 \leq k \leq n-1)$ and
$\{\alpha_p\}_{p=1}^{n-1} \cup \{-\tilde{\alpha}\}$ $(k = n)$

are fundamental root systems of $\triangle(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ whose Dynkin diagrams are



$(k = 1)$,

(bi.2)



$(2 \leq k \leq n-1)$ and

$(k = n)$,

respectively, where $\tilde{\alpha} := \alpha_1 + 2\sum_{q=2}^{n} \alpha_q$ and $\mathfrak{k}_{\mathbb{C}} = \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, \theta(X) = X\}$. In this setting we prove

Proposition 3.9 (BI)**.** *The supposition* (S) *in Proposition* 2.7 *holds for the following pseudo-Hermitian symmetric Lie algebras* $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$:

(1) $(\mathfrak{so}(2k, 2n-2k+1), \mathfrak{so}(2k-2, 2n-2k+1) \oplus \mathfrak{t})$, $n \geq 3$, $2 \leq k \leq n$ *and* $T = i(Z_{k-1} - Z_k)$.

(2) $(\mathfrak{so}(2k, 2n-2k+1), \mathfrak{so}(2k, 2n-2k-1) \oplus \mathfrak{t})$, $1 \leq k \leq n-2$ *and* $T = i(-Z_k + Z_{k+1})$.

Here $\{Z_\ell\}_{\ell=1}^{n}$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^{n}$ *in* (b.1) *and we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{so}(2k, 2n-2k+1)$ *from* (b.2) *and* (bi.1).

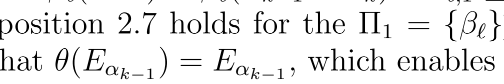
Remark 3.10 (BI). In case of (2) with $k = n-1$, we do not know whether $(\mathfrak{so}(2k, 2n-2k+1), \mathfrak{so}(2k, 2n-2k-1) \oplus \mathfrak{t}) = (\mathfrak{so}(2n-2, 3), \mathfrak{so}(2n-2, 1) \oplus \mathfrak{t})$ satisfies the supposition (S) or not.

Proof of Proposition 3.9. (1). Let

$$\beta_a := \alpha_{k-a} \text{ for } 1 \leq a \leq k-1,$$
$$\beta_k := -\textstyle\sum_{c=1}^{k} \alpha_c - 2\sum_{b=k+1}^{n} \alpha_b,$$
$$\beta_b := \alpha_b \text{ for } k+1 \leq b \leq n,$$

where the above implies $\beta_a = \alpha_{n-a}$ $(1 \leq a \leq n-1)$ and $\beta_n = -\sum_{c=1}^{n} \alpha_c$ in case of $k = n$. Then it turns out that $\Pi_1 := \{\beta_\ell\}_{\ell=1}^{n}$ is a fundamental root system of $\triangle$ whose Dynkin diagram is

(bi-1) Π_1:



Besides, $\alpha_a(Z_b) = \delta_{a,b}$ yields $\beta_\ell(-iT) = \beta_\ell(Z_{k-1} - Z_k) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq n$, and thus the (s1) in Proposition 2.7 holds for the $\Pi_1 = \{\beta_\ell\}_{\ell=1}^{n}$. It follows from (bi.1) and $\alpha_a(Z_b) = \delta_{a,b}$ that $\theta(E_{\alpha_{k-1}}) = E_{\alpha_{k-1}}$, which enables us to obtain

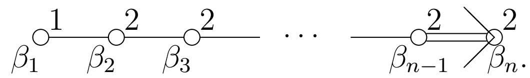
$$\mathfrak{g}_{\beta_1} = \mathfrak{g}_{\alpha_{k-1}} = \mathrm{span}_{\mathbb{C}}\{E_{\alpha_{k-1}}\} \subset \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, \theta(X) = X\} = \mathfrak{k}_{\mathbb{C}}.$$

This assures that the (s2) in Proposition 2.7 also holds for Π_1. Incidentally, (bi-1), (bi.2), $T = i(Z_{k-1} - Z_k)$ and $\alpha_a(Z_b) = \delta_{a,b}$ give rise to $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{so}(2n-1) \oplus \mathfrak{t}$, $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{so}(2k-2) \oplus \mathfrak{so}(2n-2k+1) \oplus \mathfrak{t}$, and $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(2k-2, 2n-2k+1) \oplus \mathfrak{t}$. cf. Corollary 3.6 in [2, p.1142].

(2). Setting

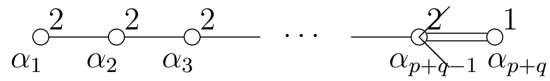
$$\begin{aligned} &\beta_s := \alpha_{k+s} \text{ for } 1 \leq s \leq n-k-1, \\ &\beta_{n-k} := -\textstyle\sum_{p=1}^{n-1} \alpha_p, \\ &\beta_t := \alpha_{t-n+k} \text{ for } n-k+1 \leq t \leq n-1, \\ &\beta_n := \textstyle\sum_{d=k}^{n} \alpha_d, \end{aligned}$$

we deduce that $\Pi_2 := \{\beta_\ell\}_{\ell=1}^n$ is a fundamental root system of $\triangle$ whose Dynkin diagram is

(bi-2) Π_2: 

By $\alpha_a(Z_b) = \delta_{a,b}$ and $T = i(-Z_k + Z_{k+1})$ we conclude $\beta_\ell(-iT) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq n$. One can complete the rest of proof, in a similar way to (1).[3] □

3.2.4. *CII.* Denote by $\mathfrak{g}_{\mathbb{C}}$ the classical complex simple Lie algebra of the type C_{p+q} $(p, q \geq 1)$, and assume that the Dynkin diagram of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is as follows:

(c.1) $\Pi_\triangle$: 

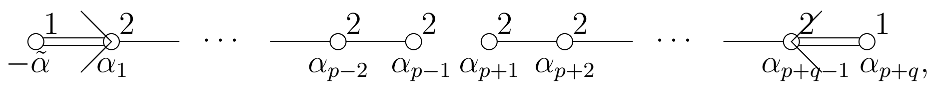
(cf. Bourbaki [7, p.269]). We take Chevalley's canonical basis $\{H^*_{\alpha_\ell}\}_{\ell=1}^{p+q} \cup \{E_\alpha \,|\, \alpha \in \triangle\}$ of $\mathfrak{g}_{\mathbb{C}}$ and construct a compact real form $\mathfrak{g}_u \subset \mathfrak{g}_{\mathbb{C}}$ from

(c.2)
$$\begin{cases} \mathfrak{h}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H^*_{\alpha_\ell}\}_{\ell=1}^{p+q}, \\ \mathfrak{g}_u := i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \triangle} \mathrm{span}_{\mathbb{R}}\{E_\alpha - E_{-\alpha}\} \oplus \mathrm{span}_{\mathbb{R}}\{i(E_\alpha + E_{-\alpha})\}. \end{cases}$$

In addition, let us take the dual basis $\{Z_\ell\}_{\ell=1}^{p+q}$ of $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^{p+q}$ and define an involutive inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ by

(cii.1)
$$\theta := \exp \pi \,\mathrm{ad}\, iZ_p.$$

Since $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$, one has the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_u$ with respect to θ and a non-compact real form $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$. Then it follows that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(p+q, \mathbb{C})$, $\mathfrak{g}_u = \mathfrak{sp}(p+q)$, $\mathfrak{g} = \mathfrak{sp}(p,q)$ and $i\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{k}$, and that $\{\alpha_a\}_{a=1}^{p-1} \cup \{\alpha_b\}_{b=p+1}^{p+q} \cup \{-\tilde{\alpha}\}$ is a fundamental root system of $\triangle(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and its Dynkin diagram is

(cii.2) 

where $\tilde{\alpha} := \alpha_{p+q} + 2\sum_{c=1}^{p+q-1} \alpha_c$. Now, we are in a position to demonstrate

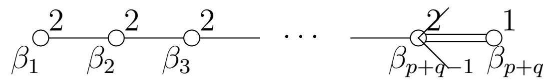
[3]Remark. If $k = n-1$, then the system Π_2 consists of $\beta_1 = -\sum_{p=1}^{n-1} \alpha_p$, $\beta_t = \alpha_{t-1}$ $(2 \leq t \leq n-1)$ and $\beta_n = \alpha_{n-1} + \alpha_n$; thus the (s2) in Proposition 2.7 cannot hold for Π_2.

Proposition 3.11 (CII)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{sp}(p,q), \mathfrak{su}(p,q) \oplus \mathfrak{t})$ *satisfies the* (S) *in Proposition* 2.7*. Here* $p, q \geq 1$*, we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{sp}(p,q)$ *from* (c.2) *and* (cii.1)*, and put* $T = i(-Z_p + Z_{p+q})$*, where* $\{Z_\ell\}_{\ell=1}^{p+q}$ *is the dual basis of* $\{\alpha_\ell\}_{\ell=1}^{p+q}$ *in* (c.1)*.*

Proof. Let

$$\begin{aligned}
&\beta_j := \alpha_{p+j} \text{ for } 1 \leq j \leq q-1,\\
&\beta_q := \textstyle\sum_{k=p}^{p+q} \alpha_k,\\
&\beta_h := \alpha_{p+q-h} \text{ for } q+1 \leq h \leq p+q-1,\\
&\beta_{p+q} := -\alpha_{p+q} - 2\textstyle\sum_{c=1}^{p+q-1} \alpha_c.
\end{aligned}$$

Then $\Pi' := \{\beta_\ell\}_{\ell=1}^{p+q}$ is a fundamental root system of $\triangle$,

(cii) Π': 

and it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta_\ell(-iT) = \beta_\ell(-Z_p + Z_{p+q}) = \delta_{\ell,p+q} \geq 0$ for all $1 \leq \ell \leq p+q$, so that the (s1) in Proposition 2.7 holds for $\Pi' = \{\beta_\ell\}_{\ell=1}^{p+q}$. Furthermore, it follows from (cii.1) and $1 \leq p \leq p+q-1$ that

$$\mathfrak{g}_{\beta_{p+q}} = \mathfrak{g}_{-2\alpha_1-\cdots-2\alpha_{p+q-1}-\alpha_{p+q}} = \mathrm{span}_{\mathbb{C}}\{E_{-2\alpha_1-\cdots-2\alpha_{p+q-1}-\alpha_{p+q}}\} \subset \mathfrak{k}_{\mathbb{C}}.$$

Thus the (s2) in Proposition 2.7 also holds for Π'. From (cii), (cii.2), $T = i(-Z_p + Z_{p+q})$ and $\alpha_a(Z_b) = \delta_{a,b}$ we obtain $\mathfrak{c}_{\mathfrak{g}_u}(T) = \mathfrak{su}(p+q) \oplus \mathfrak{t}$, $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{t}^2$, and $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{su}(p,q) \oplus \mathfrak{t}$. □

Remark 3.12 (CII). $(\mathfrak{sp}(1,1), \mathfrak{su}(1,1) \oplus \mathfrak{t}) = (\mathfrak{so}(4,1), \mathfrak{so}(2,1) \oplus \mathfrak{t})$ satisfies the supposition (S) in Proposition 2.7. cf. Proposition 3.9-(1).

3.2.5. *DI.* Let $\mathfrak{g}_{\mathbb{C}} := \mathfrak{so}(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{C}) \,|\, {}^tA = -A\}$,

$$\mathfrak{g} := \mathfrak{so}(p, 2n-p) = \left\{ \begin{pmatrix} K_1 & iD \\ -i{}^tD & K_2 \end{pmatrix} \middle| \begin{array}{l} K_1 \in \mathfrak{so}(p),\ D:\ p \times (2n-p) \text{ real matrix}, \\ K_2 \in \mathfrak{so}(2n-p) \end{array} \right\},$$

$$\mathfrak{k} := \left\{ \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix} \middle| K_1 \in \mathfrak{so}(p),\ K_2 \in \mathfrak{so}(2n-p) \right\},$$

$$\mathfrak{p} := \left\{ \begin{pmatrix} O & iD \\ -i{}^tD & O \end{pmatrix} \middle| D:\ p \times (2n-p) \text{ real matrix} \right\},$$

$$i\mathfrak{h}_{\mathbb{R}} := \left\{ \begin{pmatrix} \begin{matrix} 0 & x_1 \\ -x_1 & 0 \end{matrix} & & O \\ & \ddots & \\ O & & \begin{matrix} 0 & x_n \\ -x_n & 0 \end{matrix} \end{pmatrix} \middle| x_1, x_2, \ldots, x_n \in \mathbb{R} \right\},$$

where $n \geq 4$, $p \geq 1$ and $2n-p \geq 1$, and we note that the above notation $\mathfrak{so}(p, 2n-p)$ is different from Helgason's [8, p.446], but our Lie algebra $\mathfrak{so}(p, 2n-p)$ is isomorphic to Helgason's one. Here it follows that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, that $i\mathfrak{h}_{\mathbb{R}}$ is a maximal torus of $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$, and that

$$\text{(di.1)} \qquad \mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} \middle| \, B_1 \in \mathfrak{so}(p,\mathbb{C}),\ B_2 \in \mathfrak{so}(2n-p,\mathbb{C}) \right\}.$$

Following the notation in Helgason [8, p.187] we put

$$\text{(di.2)} \qquad \begin{cases} H_\ell := E_{2\ell-1,2\ell} - E_{2\ell,2\ell-1} \text{ for } 1 \le \ell \le n, \\ F_{a,b} := E_{a,b} - E_{b,a} \text{ for } 1 \le a \ne b \le 2n, \\ G^+_{k,j} := F_{2k-1,2j-1} + F_{2k,2j} + i(F_{2k-1,2j} - F_{2k,2j-1}) \text{ for } 1 \le k \ne j \le n, \\ G^-_{k,j} := F_{2k-1,2j-1} - F_{2k,2j} + i(F_{2k-1,2j} + F_{2k,2j-1}) \text{ for } 1 \le k < j \le n, \\ G^-_{j,k} := F_{2k-1,2j-1} - F_{2k,2j} - i(F_{2k-1,2j} + F_{2k,2j-1}) \text{ for } 1 \le k < j \le n. \end{cases}$$

Since $\mathfrak{h}_{\mathbb{C}} = \operatorname{span}_{\mathbb{C}}\{H_\ell\}_{\ell=1}^n$ one can define linear mappings $\alpha_r, \alpha_n : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ by

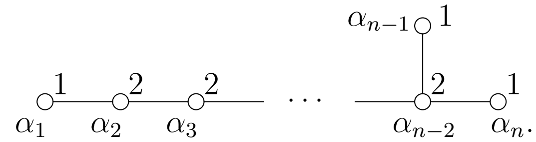
$$\alpha_r\big(\textstyle\sum_{\ell=1}^n z_\ell H_\ell\big) := -i(z_r - z_{r+1}) \text{ for } 1 \le r \le n-1,$$
$$\alpha_n\big(\textstyle\sum_{\ell=1}^n z_\ell H_\ell\big) := -i(z_{n-1} + z_n),$$

respectively. Then it turns out that $[H, G^+_{r,r+1}] = \alpha_r(H) G^+_{r,r+1}$ $(1 \le r \le n-1)$, $[H, G^-_{n,n-1}] = \alpha_n(H) G^-_{n,n-1}$ for all $H \in \mathfrak{h}_{\mathbb{C}}$, and that $\Pi_\triangle := \{\alpha_\ell\}_{\ell=1}^n$ is a fundamental root system of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and its dual basis $\{Z_\ell\}_{\ell=1}^n$ $(\subset \mathfrak{h}_{\mathbb{R}})$ is as follows:

$$\text{(di.3)} \qquad \begin{cases} Z_s = i(H_1 + H_2 + \cdots + H_s) \text{ for } 1 \le s \le n-2, \\ Z_{n-1} = (i/2)(-H_n + \sum_{r=1}^{n-1} H_r), \quad Z_n = (i/2)\sum_{\ell=1}^n H_\ell. \end{cases}$$

Remark that the Dynkin diagram of $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^n$ is

$\Pi_\triangle$:



In this setting we establish

Proposition 3.13 (DI-1)**.** *The supposition* (S) *in Proposition* 2.7 *holds for the following pseudo-Hermitian, non-Hermitian symmetric Lie algebras* $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$:

(1) $(\mathfrak{so}(p, 2n-p), \mathfrak{so}(p-2, 2n-p) \oplus \mathfrak{t})$*,* $n \ge 4$*,* $p \ge 4$*,* $2n - p \ge 1$ *and* $T = iZ_1 = -H_1$*.*

(2) $(\mathfrak{so}(p, 2n-p), \mathfrak{so}(p, 2n-p-2) \oplus \mathfrak{t})$*,* $n \ge 4$*,* $p \ge 1$*,* $2n - p \ge 4$ *and* $T = i(Z_{n-1} - Z_n) = H_n$*.*

Here we refer to (di.3)*,* (di.2) *for* Z_ℓ*,* H_ℓ $(1 \le \ell \le n)$*.*

Remark 3.14 (DI-1). We do not know whether the following two pseudo-Hermitian symmetric Lie algebras satisfy the supposition (S) or not:

- $(\mathfrak{so}(p, 2n-p), \mathfrak{so}(p-2, 2n-p) \oplus \mathfrak{t}) = (\mathfrak{so}(3, 2n-3), \mathfrak{so}(1, 2n-3) \oplus \mathfrak{t})$ in case of (1) with $p = 3$,
- $(\mathfrak{so}(p, 2n-p), \mathfrak{so}(p, 2n-p-2) \oplus \mathfrak{t}) = (\mathfrak{so}(2n-3, 3), \mathfrak{so}(2n-3, 1) \oplus \mathfrak{t})$ in case of (2) with $2n - p = 3$.

Poof of Proposition 3.13. (1). In view of $\alpha_a(Z_b) = \delta_{a,b}$ we see that $\alpha_\ell(-iT) = \alpha_\ell(Z_1) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq n$. Furthermore, (di.1) and $p \geq 4$ give rise to

$$\mathfrak{g}_{\alpha_1} = \mathrm{span}_{\mathbb{C}}\{G^+_{1,2}\} \subset \mathfrak{k}_{\mathbb{C}}.$$

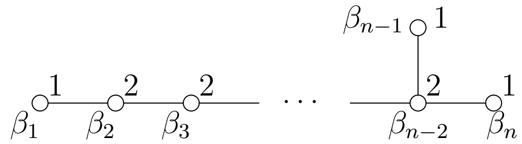
Therefore the (s1) and (s2) in Proposition 2.7 hold for the $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^n$. By a direct computation with $T = -H_1$, one obtains $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(p-2, 2n-p) \oplus \mathfrak{t}$, where we remark that $H_1 \in (\mathfrak{k} \cap i\mathfrak{h}_{\mathbb{R}}) \subset \mathfrak{g}$ comes from $p \geq 2$.

(2). Set

$$\beta_r := \alpha_{n-r} \text{ for } 1 \leq r \leq n-1,$$
$$\beta_n := -\alpha_1 - 2\sum_{c=2}^{n-2} \alpha_c - \alpha_{n-1} - \alpha_n.$$

Then, $\Pi' := \{\beta_\ell\}_{\ell=1}^n$ is a fundamental root system of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ whose Dynkin diagram is

Π':



and it follows from $T = i(Z_{n-1} - Z_n)$ and $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta_\ell(-iT) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq n$, so that the (s1) in Proposition 2.7 holds for the $\Pi' = \{\beta_\ell\}_{\ell=1}^n$. Moreover, (di.1) and $2n - p \geq 4$ yield $\mathfrak{g}_{\beta_1} = \mathfrak{g}_{\alpha_{n-1}} = \mathrm{span}_{\mathbb{C}}\{G^+_{n-1,n}\} \subset \mathfrak{k}_{\mathbb{C}}$. Thus the (s2) in Proposition 2.7 also holds for $\Pi' = \{\beta_\ell\}_{\ell=1}^n$. □

Now, let $p = 2m$. Then we have $\mathfrak{g} = \mathfrak{so}(2m, 2n-2m)$ and

Proposition 3.15 (DI-2)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{so}(2m, 2n-2m), \mathfrak{su}(m, n-m) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7*, where* $n \geq 4$*,* $m \geq 1$*,* $n - m \geq 2$ *and* $T = iZ_n = (-1/2)\sum_{\ell=1}^n H_\ell$*. Here we refer to* (di.3), (di.2) *for* Z_ℓ*,* H_ℓ $(1 \leq \ell \leq n)$*.*

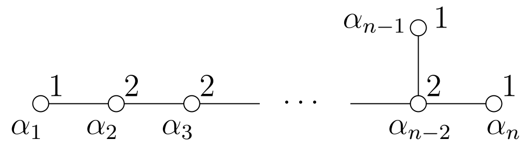
Proof. By a direct computation one obtains $\alpha_\ell(-iT) = \alpha_\ell(Z_n) = \delta_{\ell,n} \geq 0$ for all $1 \leq \ell \leq n$, and it follows from (di.2), $2n - 2m \geq 4$ and (di.1) with $p = 2m$ that

$$\mathfrak{g}_{\alpha_n} = \mathrm{span}_{\mathbb{C}}\{G^-_{n,n-1}\} \subset \mathfrak{k}_{\mathbb{C}}.$$

Therefore the (s1) and (s2) in Proposition 2.7 hold for $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^n$. □

3.2.6. *DIII.* Let $\mathfrak{g}_{\mathbb{C}}$ be the classical complex simple Lie algebra of the type D_n $(n \geq 3)$. Let us assume that the Dynkin diagram of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is

(diii.1) $\Pi_\triangle$:



(cf. Bourbaki [7, p.271]), take Chevalley's canonical basis $\{H^*_{\alpha_\ell}\}_{\ell=1}^n \cup \{E_\alpha \,|\, \alpha \in \triangle\}$ of $\mathfrak{g}_{\mathbb{C}}$ and define a compact real form $\mathfrak{g}_u \subset \mathfrak{g}$ by

(diii.2) $$\begin{cases} \mathfrak{h}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H^*_{\alpha_\ell}\}_{\ell=1}^n, \\ \mathfrak{g}_u := i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \triangle} \mathrm{span}_{\mathbb{R}}\{E_\alpha - E_{-\alpha}\} \oplus \mathrm{span}_{\mathbb{R}}\{i(E_\alpha + E_{-\alpha})\}; \end{cases}$$

in addition, denote by $\{Z_\ell\}_{\ell=1}^n$ ($\subset \mathfrak{h}_\mathbb{R}$) the dual basis of $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^n$. We are going to set a non-compact real form $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$. In order to do so, we first construct an involutive inner automorphism θ of $\mathfrak{g}$ from

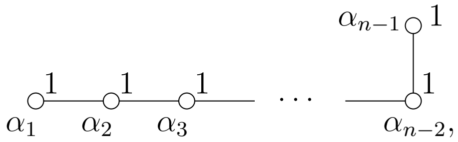
(diii.3) $$\theta := \exp \pi \operatorname{ad} iZ_n.$$

Then (diii.2) tells us that $\mathfrak{g}_u$ is stable under θ, so one can obtain the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_u$ with respect to θ and set a non-compact real form $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$ as follows: $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. Here it turns out that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2n,\mathbb{C}), \quad \mathfrak{g}_u = \mathfrak{so}(2n), \quad \mathfrak{k} = \mathfrak{u}(n), \quad \mathfrak{g} = \mathfrak{so}^*(2n), \quad i\mathfrak{h}_\mathbb{R} \subset \mathfrak{k} = (\mathfrak{g}_u \cap \mathfrak{g}),$$

and that $\{\alpha_j\}_{j=1}^{n-1}$ is a fundamental root system of $\triangle(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ whose Dynkin diagram is

(diii.4)



where $\mathfrak{k}_\mathbb{C} = \{X \in \mathfrak{g}_\mathbb{C} \,|\, \theta(X) = X\}$. Now, let us demonstrate

Proposition 3.16 (DIII)**.** *The supposition* (S) *in Proposition* 2.7 *holds for the following pseudo-Hermitian, non-Hermitian symmetric Lie algebras* $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T))$:

(1) $(\mathfrak{so}^*(2n), \mathfrak{su}(k, n-k) \oplus \mathfrak{t})$, $1 \le k \le n-2$ *and* $T = i(Z_k - Z_n)$.
(2) $(\mathfrak{so}^*(2n), \mathfrak{so}^*(2n-2) \oplus \mathfrak{t})$, $n \ge 3$ *and* $T = iZ_1$.

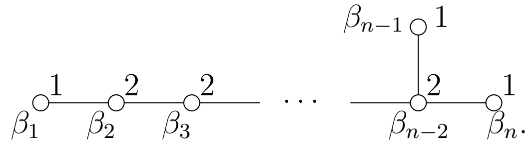
Here $\{Z_\ell\}_{\ell=1}^n$ *is the dual basis of* $\{\alpha_\ell\}_{\ell=1}^n$ *in* (diii.1) *and we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{so}^*(2n)$ *from* (diii.2) *and* (diii.3).

Proof. (1). We set

$$\begin{aligned}
&\beta_a := -\alpha_{k+1+a} \text{ for } 1 \le a \le n-k-2,\\
&\beta_{n-k-1} := -\textstyle\sum_{s=1}^{n-2} \alpha_s - \alpha_n,\\
&\beta_b := \alpha_{b-n+k+1} \text{ for } n-k \le b \le n-1,\\
&\beta_n := \alpha_k + 2\textstyle\sum_{c=k+1}^{n-2} \alpha_c + \alpha_{n-1} + \alpha_n,
\end{aligned}$$

where the above implies $\beta_1 = -\sum_{s=1}^{n-2} \alpha_s - \alpha_n$, $\beta_b = \alpha_{b-1}$ ($2 \le b \le n-1$), $\beta_n = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ in case of $k = n-2$. Then $\Pi_1 := \{\beta_\ell\}_{\ell=1}^n$ is a fundamental root system of $\triangle$ whose Dynkin diagram is

(diii-1) Π_1:



In view of $\alpha_a(Z_b) = \delta_{a,b}$ we see that $\beta_\ell(-iT) = \beta_\ell(Z_k - Z_n) = \delta_{\ell,n-1} \ge 0$ for all $1 \le \ell \le n$. Accordingly the (s1) in Proposition 2.7 holds for the $\Pi_1 = \{\beta_\ell\}_{\ell=1}^n$. From (diii.3) and $k \ne n$ we deduce that

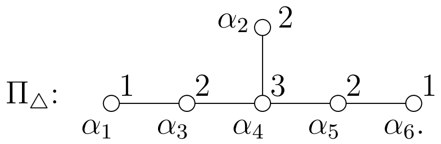
$$\mathfrak{g}_{\beta_{n-1}} = \mathfrak{g}_{\alpha_k} \subset \mathfrak{k}_\mathbb{C},$$

and therefore the (s2) in Proposition 2.7 also holds for $\Pi_1 = \{\beta_\ell\}_{\ell=1}^n$. It is easy to see that $\mathfrak{c}_\mathfrak{g}(T) = \mathfrak{su}(k, n-k) \oplus \mathfrak{t}$ by virtue of $\alpha_a(Z_b) = \delta_{a,b}$, $T = i(Z_k - Z_n)$, (diii-1) and (diii.4).

(2). One can show (2) by fixing the $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^n$ in (diii.1). □

3.2.7. *EII & EIII.* Let us denote by $\mathfrak{g}_\mathbb{C}$ the exceptional complex simple Lie algebra of the type E_6, and assume that the Dynkin diagram of $\triangle = \triangle(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is as follows (cf. Bourbaki [7, p.276]):

(e6.1)



Taking Chevalley's canonical basis $\{H^*_{\alpha_\ell}\}_{\ell=1}^6 \cup \{E_\alpha \,|\, \alpha \in \triangle\}$ of $\mathfrak{g}_\mathbb{C}$ into account, one can construct a compact real form $\mathfrak{g}_u \subset \mathfrak{g}_\mathbb{C}$ from

$$\text{(e6.2)} \qquad \begin{cases} \mathfrak{h}_\mathbb{R} := \operatorname{span}_\mathbb{R}\{H^*_{\alpha_\ell}\}_{\ell=1}^6, \\ \mathfrak{g}_u := i\mathfrak{h}_\mathbb{R} \oplus \bigoplus_{\alpha\in\triangle} \operatorname{span}_\mathbb{R}\{E_\alpha - E_{-\alpha}\} \oplus \operatorname{span}_\mathbb{R}\{i(E_\alpha + E_{-\alpha})\}; \end{cases}$$

and then we denote by $\{Z_\ell\}_{\ell=1}^6$ ($\subset \mathfrak{h}_\mathbb{R}$) the dual basis of $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^6$. In this setting, we are going to consider the following four simple irreducible pseudo-Hermitian (non-Hermitian) symmetric Lie algebras:

(EII-1) $(\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{t})$,
(EII-2) $(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$,
(EIII-1) $(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$,
(EIII-2) $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{t})$.

• Case (EII-1). Define an inner automorphism θ of $\mathfrak{g}_\mathbb{C}$ by

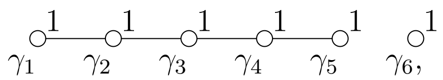
$$\text{(eii.1)} \qquad \theta := \exp \pi \operatorname{ad} i(Z_2 + Z_3 + Z_5).$$

Then it turns out that θ is involutive and $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$. Let $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ denote the decomposition of $\mathfrak{g}_u$ with respect to θ, and let $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. Setting

$$\begin{aligned} \gamma_1 &:= \alpha_2 + \alpha_3 + \alpha_4, \\ \gamma_2 &:= \alpha_1, \\ \gamma_3 &:= \alpha_3 + \alpha_4 + \alpha_5, \\ \gamma_4 &:= \alpha_6, \\ \gamma_5 &:= \alpha_2 + \alpha_4 + \alpha_5, \\ \gamma_6 &:= \alpha_4, \end{aligned}$$

we deduce that $\{\gamma_\ell\}_{\ell=1}^6$ is a fundamental root system of $\triangle(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and its Dynkin diagram is

(eii.2)



where $\mathfrak{k}_\mathbb{C} = \{X \in \mathfrak{g}_\mathbb{C} \,|\, \theta(X) = X\}$. Therefore it follows that $\mathfrak{k} = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$ and $\mathfrak{g} = \mathfrak{e}_{6(2)}$. Now, put

$$T := iZ_1.$$

Then $T \in i\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{k} = (\mathfrak{g}_u \cap \mathfrak{g})$, and $\alpha_\ell(-iT) = \alpha_\ell(Z_1) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq 6$. This assures that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T))$ is a pseudo-Hermitian symmetric Lie algebra, and that T is the canonical central element of $\mathfrak{c}_{\mathfrak{g}}(T)$. Moreover, the (s1) in Proposition 2.7 holds for $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^6$. From (eii.1) and $\alpha_a(Z_b) = \delta_{a,b}$ we see that $\theta(E_{\alpha_1}) = E_{\alpha_1}$, so that $\mathfrak{g}_{\alpha_1} \subset \mathfrak{k}_{\mathbb{C}}$. Hence, the (s2) in Proposition 2.7 also holds for $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^6$.

Proposition 3.17 (EII-1)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7. *Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{6(2)}$ *from* (e6.2) *and* (eii.1), *and put* $T = iZ_1$, *where* $\{Z_\ell\}_{\ell=1}^6$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^6$ *in* (e6.1).

Proof. The rest of proof is confirm that $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}(6,4) \oplus \mathfrak{t}$. However, that is immediate from (e6.1), (eii.2), $T = iZ_1$ and $\alpha_a(Z_b) = \delta_{a,b}$. □

• Case (EII-2). By arguments similar to those in Case (EII-1), one can assert

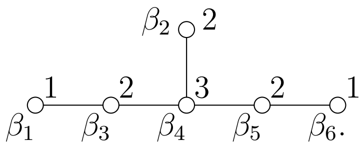
Proposition 3.18 (EII-2)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7. *Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{6(2)}$ *from* (e6.2) *and* (eii.1), *and put* $T = i(-Z_4 + Z_5)$, *where* $\{Z_\ell\}_{\ell=1}^6$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^6$ *in* (e6.1).

Proof. Setting

$$
\begin{aligned}
\beta_1 &:= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6,\\
\beta_2 &:= \alpha_6,\\
\beta_3 &:= \alpha_2,\\
\beta_4 &:= \alpha_4 + \alpha_5,\\
\beta_5 &:= \alpha_3,\\
\beta_6 &:= \alpha_1
\end{aligned}
$$

and $\Pi' := \{\beta_\ell\}_{\ell=1}^6$, one has

(eii-2) Π':



Then, it follows that $\Pi' = \{\beta_\ell\}_{\ell=1}^6$ is a fundamental root system of $\triangle$, and that $\beta_\ell(-iT) = \beta_\ell(-Z_4 + Z_5) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq 6$. Furthermore, we see that

$$\mathfrak{g}_{\beta_1} = \mathfrak{g}_{-\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6} \subset \mathfrak{k}_{\mathbb{C}}$$

in view of (eii.1) and $\alpha_a(Z_b) = \delta_{a,b}$. Consequently the (s1) and (s2) in Proposition 2.7 hold for the $\Pi' = \{\beta_\ell\}_{\ell=1}^6$. We obtain $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{so}^*(10) \oplus \mathfrak{t}$ from (eii-2), (eii.2), $T = i(-Z_4 + Z_5)$ and $\alpha_a(Z_b) = \delta_{a,b}$. □

• Cases (EIII-1) & (EIII-2). Let us define an involutive inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ by

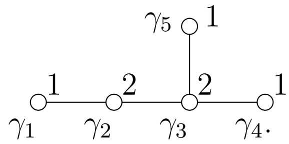
(eiii.1) $$\theta := \exp \pi \operatorname{ad} i(Z_1 - Z_6),$$

denote by $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ the decomposition of $\mathfrak{g}_u$ with respect to θ, and construct a non-compact real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$, where we note that $\mathfrak{g}_u$ is stable under θ (recall (e6.2) for $\mathfrak{g}_u$). Setting

$$\begin{aligned}
\gamma_1 &:= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6, \\
\gamma_2 &:= \alpha_2, \\
\gamma_3 &:= \alpha_4, \\
\gamma_4 &:= \alpha_3, \\
\gamma_5 &:= \alpha_5,
\end{aligned}$$

we deduce that $\{\gamma_k\}_{k=1}^5$ is a fundamental root system of $\triangle(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and its Dynkin diagram is

(eiii.2)



This shows that $\mathfrak{k} = \mathfrak{so}(10) \oplus \mathfrak{t}$, and $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. In this setting we demonstrate two propositions.

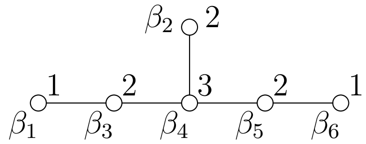
Proposition 3.19 (EIII-1)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7. *Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *from* (e6.2) *and* (eiii.1)*, and put* $T = i(Z_1 - Z_3)$*, where* $\{Z_\ell\}_{\ell=1}^6$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^6$ *in* (e6.1)*.*

Proof. Set β_ℓ $(1 \leq \ell \leq 6)$ as follows:

$$\begin{aligned}
\beta_1 &:= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6, \\
\beta_2 &:= \alpha_1 + \alpha_3, \\
\beta_3 &:= \alpha_2, \\
\beta_4 &:= \alpha_4, \\
\beta_5 &:= \alpha_5, \\
\beta_6 &:= \alpha_6.
\end{aligned}$$

Then we see that $\Pi_1 := \{\beta_\ell\}_{\ell=1}^6$ is a fundamental root system of $\triangle$, its Dynkin diagram is

(eiii-1) Π_1:



and $\beta_\ell(-iT) = \beta_\ell(Z_1 - Z_3) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq 6$. Moreover, we conclude

$$\mathfrak{g}_{\beta_1} = \mathfrak{g}_{-\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6} \subset \{X \in \mathfrak{g}_{\mathbb{C}} \mid \theta(X) = X\} = \mathfrak{k}_{\mathbb{C}}$$

by (eiii.1) and $\alpha_a(Z_b) = \delta_{a,b}$. Therefore the (s1) and (s2) in Proposition 2.7 hold for this $\Pi_1 = \{\beta_\ell\}_{\ell=1}^6$. □

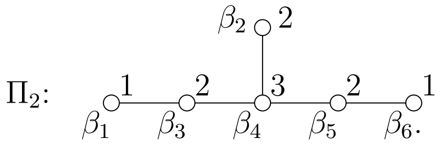
Proposition 3.20 (EIII-2)**.** $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7*. Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *from* (e6.2) *and* (eiii.1)*, and put* $T = i(-Z_3 + Z_5)$*, where* $\{Z_\ell\}_{\ell=1}^6$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^6$ *in* (e6.1)*.*

Proof. Setting

$$\begin{aligned}
\beta_1 &:= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,\\
\beta_2 &:= \alpha_4,\\
\beta_3 &:= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6,\\
\beta_4 &:= \alpha_2,\\
\beta_5 &:= \alpha_3 + \alpha_4 + \alpha_5,\\
\beta_6 &:= \alpha_6,
\end{aligned}$$

we deduce that $\Pi_2 := \{\beta_\ell\}_{\ell=1}^6$ is a fundamental root system of $\triangle$ and $\beta_\ell(-iT) = \beta_\ell(-Z_3 + Z_5) = \delta_{\ell,1} \geq 0$ for all $1 \leq \ell \leq 6$.

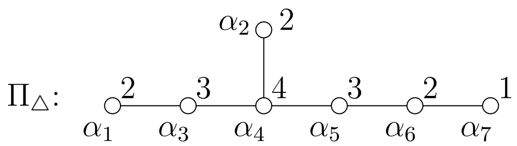
(eiii-2)



In addition, $\mathfrak{g}_{\beta_1} = \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} \subset \mathfrak{k}_{\mathbb{C}}$ due to (eiii.1). Therefore the (s1) and (s2) in Proposition 2.7 hold for $\Pi_2 = \{\beta_\ell\}_{\ell=1}^6$. □

3.2.8. *EV, EVI* & *EVII.* Denote by $\mathfrak{g}_{\mathbb{C}}$ the exceptional complex simple Lie algebra of the type E_7, and assume that the Dynkin diagram of $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is

(e7.1)



(cf. Bourbaki [7, p.280]). Taking Chevalley's canonical basis $\{H^*_{\alpha_\ell}\}_{\ell=1}^7 \cup \{E_\alpha \,|\, \alpha \in \triangle\}$ of $\mathfrak{g}_{\mathbb{C}}$, we construct a compact real form $\mathfrak{g}_u \subset \mathfrak{g}_{\mathbb{C}}$ from

(e7.2)
$$\begin{cases} \mathfrak{h}_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\{H^*_{\alpha_\ell}\}_{\ell=1}^7, \\ \mathfrak{g}_u := i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\triangle} \mathrm{span}_{\mathbb{R}}\{E_\alpha - E_{-\alpha}\} \oplus \mathrm{span}_{\mathbb{R}}\{i(E_\alpha + E_{-\alpha})\}. \end{cases}$$

Let $\{Z_\ell\}_{\ell=1}^7$ $(\subset \mathfrak{h}_{\mathbb{R}})$ be the dual basis of $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^7$. From now on, we are going to consider the following four simple irreducible pseudo-Hermitian (non-Hermitian) symmetric Lie algebras:

(EV) $(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$,
(EVI-1) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$,
(EVI-2) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$,
(EVII) $(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$.

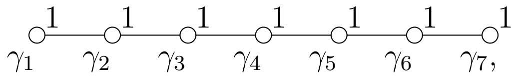
- Case (EV). Define an involutive inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ by

(ev.1)
$$\theta := \exp \pi \,\mathrm{ad}\, iZ_2.$$

Then it follows from (e7.2) that $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$, which enables us to obtain the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_u$ with respect to θ and construct a non-compact real form $\mathfrak{g}$ of $\mathfrak{g}_\mathbb{C}$ from $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. Setting

$$\begin{aligned}
&\gamma_1 := -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7,\\
&\gamma_2 := \alpha_1,\\
&\gamma_p := \alpha_p \text{ for } 3 \leq p \leq 7,
\end{aligned}$$

we have a fundamental root system $\{\gamma_\ell\}_{\ell=1}^7$ of $\triangle(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, and see that its Dynkin diagram is

(ev.2) 

where $\mathfrak{k}_\mathbb{C} = \{X \in \mathfrak{g}_\mathbb{C} \,|\, \theta(X) = X\}$. This implies $\mathfrak{k} = \mathfrak{su}(8)$ and $\mathfrak{g} = \mathfrak{e}_{7(7)}$. Now, let

$$T := iZ_7.$$

Then one has $T \in i\mathfrak{h}_\mathbb{R} \subset \mathfrak{k} \subset \mathfrak{g}$ and $\alpha_\ell(-iT) = \alpha_\ell(Z_7) = \delta_{\ell,7} \geq 0$ for all $1 \leq \ell \leq 7$. Moreover, (ev.1) yields $\mathfrak{g}_{\alpha_7} = \mathrm{span}_\mathbb{C}\{E_{\alpha_7}\} \subset \mathfrak{k}_\mathbb{C}$. Consequently the (s1) and (s2) in Proposition 2.7 hold for $\Pi_\triangle = \{\alpha_\ell\}_{\ell=1}^7$. For this reason we establish

Proposition 3.21 (EV)**.** $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T)) = (\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7*. Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{7(7)}$ *from* (e7.2) *and* (ev.1)*, and put* $T = iZ_7$*, where* $\{Z_\ell\}_{\ell=1}^7$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^7$ *in* (e7.1)*.*

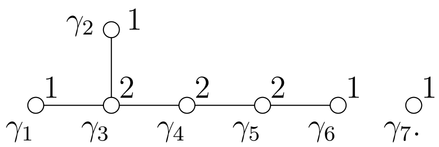
• Cases (EVI-1) & (EVI-2). Define an involutive inner automorphism θ of $\mathfrak{g}_\mathbb{C}$ by

(evi.1) $$\theta := \exp \pi \,\mathrm{ad}\, i(Z_2 + Z_7).$$

Let $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ be the decomposition of $\mathfrak{g}_u$ with respect to θ, where we remark that $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$ comes from (e7.2) and (evi.1). Let us set $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ and

$$\begin{aligned}
&\gamma_1 := \alpha_1,\\
&\gamma_2 := \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,\\
&\gamma_q := \alpha_q \text{ for } 3 \leq q \leq 6,\\
&\gamma_7 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.
\end{aligned}$$

Then it turns out that $\{\gamma_\ell\}_{\ell=1}^7$ is a fundamental root system of $\triangle(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and its Dynkin diagram is

(evi.2) 

So it follows that $\mathfrak{k} = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$ and $\mathfrak{g} = \mathfrak{e}_{7(-5)}$. We are in a position to verify

Proposition 3.22 (EVI)**.** *The supposition* (S) *in Proposition* 2.7 *holds for the following two pseudo-Hermitian symmetric Lie algebras* $(\mathfrak{g}, \mathfrak{c}_\mathfrak{g}(T))$:

(1) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$ *and* $T = iZ_7$.
(2) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$ *and* $T = i(Z_6 - Z_7)$.

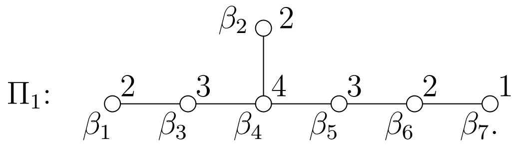
Here $\{Z_\ell\}_{\ell=1}^7$ *is the dual basis of* $\{\alpha_\ell\}_{\ell=1}^7$ *in* (e7.1) *and we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ *from* (e7.2) *and* (evi.1)*.*

Proof. (1). Let $T := iZ_7$, and let

$$\begin{aligned}
\beta_1 &:= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6,\\
\beta_2 &:= \alpha_6,\\
\beta_3 &:= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\\
\beta_4 &:= \alpha_5,\\
\beta_5 &:= \alpha_4,\\
\beta_6 &:= \alpha_3,\\
\beta_7 &:= \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.
\end{aligned}$$

Then $\Pi_1 := \{\beta_\ell\}_{\ell=1}^7$ is a fundamental root system of $\triangle$ whose Dynkin diagram is

(evi-1)



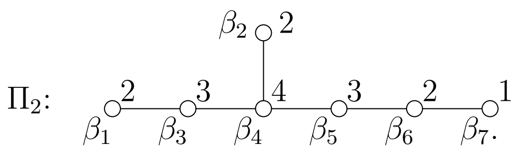
Moreover, $\alpha_a(Z_b) = \delta_{a,b}$ and (evi.1) imply that $\beta_\ell(-iT) = \beta_\ell(Z_7) = \delta_{\ell,7} \geq 0$ for all $1 \leq \ell \leq 7$ and $\mathfrak{g}_{\beta_7} = \mathfrak{g}_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \subset \mathfrak{k}_{\mathbb{C}}$, so that the (s1) and (s2) in Proposition 2.7 hold for this $\Pi_1 = \{\beta_\ell\}_{\ell=1}^7$. In addition, $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{e}_{6(2)} \oplus \mathfrak{t}$ follows from (evi-1), (evi.2) and $T = iZ_7$.

(2). One can conclude (2) by arguments similar to those above, and by setting $T := i(Z_6 - Z_7)$ and

$$\begin{aligned}
\beta_1 &:= -\alpha_2 - \alpha_3 - \alpha_4,\\
\beta_2 &:= -\alpha_1 - \alpha_3 - \alpha_4,\\
\beta_3 &:= \alpha_3,\\
\beta_4 &:= \alpha_4,\\
\beta_5 &:= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,\\
\beta_6 &:= -\alpha_6 - \alpha_7,\\
\beta_7 &:= \alpha_6,
\end{aligned}$$

where we remark that $\Pi_2 := \{\beta_\ell\}_{\ell=1}^7$ is a fundamental root system of $\triangle$ and its Dynkin diagram is

(evi-2)



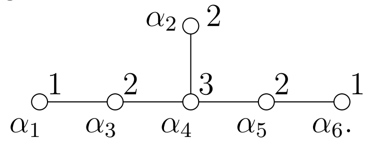
□

• Case (EVII). First, let us realize the exceptional real simple Lie algebra $\mathfrak{e}_{7(-25)}$. Define an involutive inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ by

(evii.1) $$\theta := \exp \pi \operatorname{ad} iZ_7.$$

Since (e7.2) we have $\theta(\mathfrak{g}_u) \subset \mathfrak{g}_u$. So, one can consider the decomposition $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_u$ with respect to θ, and set a non-compact real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ as $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$. It follows from (evii.1) and (e7.1) that $\{\alpha_k\}_{k=1}^{6}$ is a fundamental root system of $\triangle(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ whose Dynkin diagram is

(evii.2)



Therefore we show that $\mathfrak{k} = \mathfrak{e}_6 \oplus \mathfrak{t}$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$. Now, let us prove

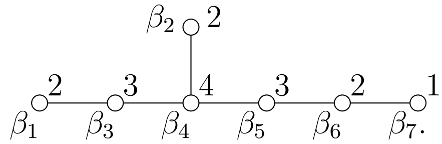
Proposition 3.23 (EVII). $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(T)) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$ *satisfies the supposition* (S) *in Proposition* 2.7. *Here we construct a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *of* $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ *from* (e7.2) *and* (evii.1), *and put* $T = i(Z_6 - Z_7)$, *where* $\{Z_\ell\}_{\ell=1}^{7}$ *stands for the dual basis of* $\{\alpha_\ell\}_{\ell=1}^{7}$ *in* (e7.1).

Proof. Setting

$$\begin{aligned}
\beta_1 &:= -\alpha_2 - \alpha_3 - \alpha_4, \\
\beta_2 &:= -\alpha_1 - \alpha_3 - \alpha_4, \\
\beta_3 &:= \alpha_3, \\
\beta_4 &:= \alpha_4, \\
\beta_5 &:= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
\beta_6 &:= -\alpha_6 - \alpha_7, \\
\beta_7 &:= \alpha_6,
\end{aligned}$$

we deduce that $\Pi' := \{\beta_\ell\}_{\ell=1}^{7}$ is a fundamental root system of $\triangle$ whose Dynkin diagram is

(evii) Π':



Then it turns out that $\beta_\ell(-iT) = \beta_\ell(Z_6 - Z_7) = \delta_{\ell,7} \geq 0$ for all $1 \leq \ell \leq 7$, and that $\mathfrak{g}_{\beta_7} = \mathfrak{g}_{\alpha_6} \subset \{X \in \mathfrak{g}_{\mathbb{C}} \,|\, \theta(X) = X\} = \mathfrak{k}_{\mathbb{C}}$ due to (evii.1) and $\alpha_a(Z_b) = \delta_{a,b}$. Accordingly the (s1) and (s2) in Proposition 2.7 hold for $\Pi' = \{\beta_\ell\}_{\ell=1}^{7}$. □

Remark 3.24. Seven Propositions 3.17 through 3.23 tell us that the supposition (S) in Proposition 2.7 holds for every simple irreducible pseudo-Hermitian, non-Hermitian symmetric Lie algebra of the exceptional type.

(EII-1) $(\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4) \oplus \mathfrak{t})$,
(EII-2) $(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$,
(EIII-1) $(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{t})$,
(EIII-2) $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{t})$,
(EV) $(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$,
(EVI-1) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t})$,
(EVI-2) $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$,
(EVII) $(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t})$.

Remark 2.8-(2), Remark 3.2 and the propositions in Subsection 3.2 lead to

Theorem 3.25. *The supposition* (S) *in Proposition* 2.7 *holds for every effective semisimple Hermitian symmetric space of the compact type and each effective simple irreducible pseudo-Hermitian* (*non-Hermitian*) *symmetric space* G/L *given in Table* C.

Table C
G/L, where we assume the center $Z(G)$ to be trivial for the G/L below.
(1) $SU^*(2n)/(SL(n,\mathbb{C})\cdot T)$, $n \geq 2$. cf. Proposition 3.3 (AII)
(2) $SU(p,q)/S(U(a)\times U(p-a,q))$, $1\leq a\leq p-1$, $1\leq q$. (3) $SU(p,q)/S(U(p,b)\times U(q-b))$, $1\leq p$, $1\leq b\leq q-1$. (4) $SU(p,q)/S(U(a,b)\times U(p-a,q-b))$, $1\leq a\leq p-1$, $1\leq b\leq q-1$. cf. Proposition 3.8 (AIII)
(5) $SO_0(2k,2n-2k+1)/(SO_0(2k-2,2n-2k+1)\cdot SO(2))$, $n\geq 3$, $2\leq k\leq n$. (6) $SO_0(2k,2n-2k+1)/(SO_0(2k,2n-2k-1)\cdot SO(2))$, $1\leq k\leq n-2$. cf. Proposition 3.9 (BI)
(7) $Sp(p,q)/U(p,q)$, $p,q\geq 1$. cf. Proposition 3.11 (CII)
(8) $SO_0(p,2n-p)/(SO_0(p-2,2n-p)\cdot SO(2))$, $n\geq 4$, $p\geq 4$, $2n-p\geq 1$. (9) $SO_0(p,2n-p)/(SO_0(p,2n-p-2)\cdot SO(2))$, $n\geq 4$, $p\geq 1$, $2n-p\geq 4$. cf. Proposition 3.13 (DI-1)
(10) $SO_0(2m,2n-2m)/U(m,n-m)$, $n\geq 4$, $m\geq 1$, $n-m\geq 2$. cf. Proposition 3.15 (DI-2)
(11) $SO^*(2n)/U(k,n-k)$, $1\leq k\leq n-2$. (12) $SO^*(2n)/(SO^*(2n-2)\cdot SO^*(2))$, $n\geq 3$. cf. Proposition 3.16 (DIII)
(13) $E_{6(2)}/(SO_0(6,4)\cdot SO(2))$. cf. Proposition 3.17 (EII-1)
(14) $E_{6(2)}/(SO^*(10)\cdot SO^*(2))$. cf. Proposition 3.18 (EII-2)
(15) $E_{6(-14)}/(SO^*(10)\cdot SO^*(2))$. cf. Proposition 3.19 (EIII-1)
(16) $E_{6(-14)}/(SO_0(8,2)\cdot SO(2))$. cf. Proposition 3.20 (EIII-2)
(17) $E_{7(7)}/(E_{6(2)}\cdot T)$. cf. Proposition 3.21 (EV)
(18) $E_{7(-5)}/(E_{6(2)}\cdot T)$. (19) $E_{7(-5)}/(E_{6(-14)}\cdot T)$. cf. Proposition 3.22 (EVI)
(20) $E_{7(-25)}/(E_{6(-14)}\cdot T)$. cf. Proposition 3.23 (EVII)

Remark 3.26. The pseudo-Hermitian symmetric spaces G/L in Table C, together with

(i) $SL(2n,\mathbb{R})/(SL(n,\mathbb{C})\cdot T)$, $n\geq 2$,
(ii) $SO_0(2n-2,3)/(SO_0(2n-2,1)\cdot SO(2))$, $n\geq 3$,
(iii) $SO_0(3,2n-3)/(SO_0(1,2n-3)\cdot SO(2))$, $n\geq 4$,
(iv) $SO_0(2n-3,3)/(SO_0(2n-3,1)\cdot SO(2))$, $n\geq 4$,
(v) $Sp(n,\mathbb{R})/U(k,n-k)$, $1\leq k\leq n-1$,

exhaust all the simple irreducible pseudo-Hermitian, non-Hermitian symmetric spaces in Tableau II of Berger [1, pp.157–161]. Unfortunately, we do not know whether the above pseudo-Hermitian symmetric spaces (i) through (v) satisfy the supposition (S) or not.

3.3. **An appendix.** It is known that for any effective irreducible Hermitian symmetric space G_u/L_u of the compact type, the complex vector space $\mathcal{O}(T^{1,0}(G_u/L_u))$ is linear isomorphic to $\mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}_u$ (e.g. Corollary 4.1-(ii) in [6, p.145]). So, Theorem 3.25, and two Propositions 2.7 and 2.5-(iv) lead to

Corollary 3.27. *For each effective simple irreducible pseudo-Hermitian symmetric space G/L in Table* C*, the complex vector space $\mathcal{O}(T^{1,0}(G/L))$ is linear isomorphic to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$.*

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