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POTENTIAL THEORY OF THE DISCRETE EQUATION $\Delta u = qu$

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Abstract. We develop a discrete potential theory for the equation $\Delta u = qu$ on an infinite network similar to the classical potential theory on Riemannian surfaces. The q-Green function for the Schrödinger operator $-\Delta + q$ plays the role of the Green function for the Laplace operator. We study some properties of q-Green potential whose kernel is the q-Green function. As an application, we give a classification of infinite networks by the classes of q-harmonic functions. We also focus on the role of the q-elliptic measure of the ideal boundary of the network.

1. Introduction

Many fruitful results in the theory of potentials related to Laplace operator had published in Constantinescu and Cornea [2]. Related to discrete Laplacian, some results were obtained by Soardi [12], Yamasaki [13], [15], and Kurata and Yamasaki [5], [6], etc. There are some papers related to Schrödinger operator $\Delta u - qu$, for instance Ozawa [10], Maeda [8] and Sario, Nakai, Wang, and Chung [11]. The discrete equation $\Delta_q u := \Delta u - qu = 0$ has been studied by Yamasaki [17], Kurata and Yamasaki [7], Anandam [1], and Fischer and Keller [3]. Their research methods are different. Anandam used the theory of axiomatic potentials. Our research method depends on the theory of Dirichlet space and reasoning in [2]. Fischer and Keller used semigroups of a self-adjoint realization of the Schrödinger operator. The aim of this paper is to study the discrete equation $\Delta_q u = 0$ on an infinite network along the same line in [17]. We always assume that q is a non-zero non-negative function and $q \not\equiv 0$. We show the fundamental results relating the spaces $\mathbf{E}$ and $\mathbf{E}_0$ and the norm $E(\cdot)^{1/2}$ in Section 3, and properties of q-superharmonic functions in Section 4. We define the q-Green function of $\mathcal{N}$ in Section 5. Most of these results were obtained in [17]. We give their proofs for completeness. The discrete analogues of Royden's decomposition of a function in $\mathbf{E}$ and Riesz's decomposition of a non-negative q-superharmonic function play the fundamental roles in our study.

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In Sections 6–9, we study potential-theoretic properties of q-Green potentials; for example, domination principle, equilibrium principle, etc. As the discrete analogy of q-elliptic measure in [11, Page 286], we introduce the q-elliptic measure ω of the ideal boundary of the network and study it in Section 10 more detail than in [17]. In case $\mathcal{N}$ is parabolic, we give some supplementary results in Section 11. We shall give a classification of infinite networks by using the classes of q-harmonic functions in Section 12. Analogous to the classification theory in Sario, Nakai, Wang, and Chung [11], we give some results of q-quasiharmonic classification of the networks by using q-elliptic measure ω in Section 13 which is similar to Yamasaki [16].

2. Fundamental Notion

Let $\mathcal{G} = \langle X, Y, K \rangle$ be an infinite graph which is connected and locally finite without self-loops (cf. [13]). Here we denote X by the countable set of nodes, Y by the countable set of arcs, and K by the node-arc incidence matrix. Namely, K is a function on $X \times Y$ and $K(x,y) = -1$ if x is the initial node of y, $K(x,y) = 1$ if x is the terminal node of y, and $K(x,y) = 0$ otherwise. Now we introduce several fundamental notation used in this paper. Let $L(X)$ be the set of all real functions on X, $L_0(X)$ the subset of $L(X)$ with finite support, and $L^+(X)$ the set of all non-negative functions on X. We define $L(Y)$, $L_0(Y)$, and $L^+(Y)$ similarly. Let $r \in L^+(Y)$ be a resistance, which is a strictly positive function, and let $q \in L^+(X)$ and $q \not\equiv 0$. In this paper, we call the triple $\mathcal{N} = \langle \mathcal{G}, r, q \rangle$ an infinite network. For $x \in X$, let $Y(x) = \{y \in Y\,;\, K(x,y) \neq 0\}$, which is the set of arcs incidence to x. We say that a sequence of finite networks $\{\mathcal{N}_n = \langle \mathcal{G}_n, r_n, q_n \rangle\}_n$ is an *exhaustion* of $\mathcal{N}$ if the sequence $\{\mathcal{G}_n = \langle X_n, Y_n, K_n \rangle\}_n$ of connected graphs satisfies $X_n \subset X_{n+1}$, $Y_n \subset Y_{n+1}$, $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$, and $Y(x) \subset Y_{n+1}$ for all $x \in X_n$. Here denote by K_n the restriction of K onto $X_n \times Y_n$ and by r_n and q_n the restrictions of r and q onto Y_n and X_n respectively. Hereafter we write $\mathcal{N}_n = \langle X_n, Y_n \rangle$ for short. For $u \in L(X)$, let

$$\begin{aligned}
du(y) &= -r(y)^{-1} \sum_{x \in X} K(x,y) u(x) \quad \text{(discrete derivative)},\\
D(u) &= \sum_{y \in Y} r(y)[du(y)]^2 \quad \text{(Dirichlet sum)},\\
E(u) &= D(u) + \sum_{x \in X} q(x) u(x)^2 \quad (q\text{-energy}),\\
\Delta u(x) &= \sum_{y \in Y} K(x,y)[du(y)] \quad \text{(discrete Laplacian)},\\
\Delta_q u(x) &= \Delta u(x) - q(x) u(x) \quad (\text{discrete } q\text{-Laplacian}).
\end{aligned}$$

We say that $u \in L(X)$ is *q-harmonic* on a subset A of X if $\Delta_q u(x) = 0$ on A. For $a \in X$, denote by $\varepsilon_a \in L(X)$ the characteristic function of $\{a\}$, i.e., $\varepsilon_a(a) = 1$ and $\varepsilon_a(x) = 0$ for $x \neq a$. Also for a set $A \subset X$ denote by $\varepsilon_A \in L(X)$ the characteristic function of A.

3. The Spaces $\mathbf{E}$ and $\mathbf{E}_0$

Let us put

$$\begin{aligned}\mathbf{D} &= \{u \in L(X)\,;\, D(u) < \infty\},\\ \mathbf{E} &= \{u \in L(X)\,;\, E(u) < \infty\},\\ \mathbf{H} &= \{u \in L(X)\,;\, \Delta_q u = 0\} \quad \text{(the set of q-harmonic functions)},\\ \mathbf{HE} &= \mathbf{H} \cap \mathbf{E}, \qquad \mathbf{HD} = \mathbf{H} \cap \mathbf{D}.\end{aligned}$$

For simplicity, we set for $u, v \in L(X)$

$$\begin{aligned}\langle u, v\rangle &= \sum_{x\in X} q(x)u(x)v(x),\\ \|u\|^2 &= \langle u, u\rangle,\\ D(u, v) &= \sum_{y\in Y} r(y)[du(y)][dv(y)],\\ E(u, v) &= D(u, v) + \langle u, v\rangle.\end{aligned}$$

Then $D(u) = D(u, u)$ and $E(u) = D(u) + \|u\|^2 = E(u, u)$.

Lemma 3.1. *For $a \in X$ there exists a constant $M_a > 0$ such that $|u(a)| \le M_a E(u)^{1/2}$ for $u \in \mathbf{E}$.*

Proof. Let $a \in X$. Let $b \in X$ be such that $q(b) > 0$. For $u \in \mathbf{E}$ we have $q(b)u(b)^2 \le E(u)$, or $|u(b)| \le q(b)^{-1/2}E(u)^{1/2}$. Let P be a path between a and b. Then

$$\begin{aligned}|u(a)| &\le |u(b)| + \sum_{y\in Y(P)} r(y)|du(y)|\\ &\le |u(b)| + \left(\sum_{y\in Y(P)} r(y)\right)^{1/2}\left(\sum_{y\in Y(P)} r(y)du(y)^2\right)^{1/2}\\ &\le q(b)^{-1/2}E(u)^{1/2} + \left(\sum_{y\in Y(P)} r(y)\right)^{1/2} E(u)^{1/2},\end{aligned}$$

where $Y(P)$ is the set of arcs belonging to P. □

It is easily seen that $\mathbf{E}$ is a Hilbert space with respect to the inner product $E(\cdot, \cdot)$. Note that if $u_n, u \in \mathbf{E}$ and $E(u_n - u) \to 0$ as $n \to \infty$, then $\{u_n\}_n$ converges pointwise to u. Denote by $\mathbf{E}_0$ the closure of $L_0(X)$ with respect to the norm $[E(\cdot)]^{1/2}$. Recall that $\mathbf{D}_0$ is the closure of $L_0(X)$ with respect to the norm $[D(\cdot) + u(x_0)^2]^{1/2}$, where x_0 is a fixed node of X (see [15, Theorem 1.1]). We say that $\mathcal{N}$ is hyperbolic (parabolic resp.) if the network $\langle \mathcal{G}, r\rangle$ is hyperbolic (parabolic resp.), i.e., $\mathbf{D} \ne \mathbf{D}_0$ ($\mathbf{D} = \mathbf{D}_0$ resp.) (cf. [14]).

Theorem 3.2. $\mathbf{E}_0 = \mathbf{D}_0 \cap \mathbf{E}$.

Proof. From $\mathbf{E}_0 \subset \mathbf{D}_0$ and $\mathbf{E}_0 \subset \mathbf{E}$, it follows that $\mathbf{E}_0 \subset \mathbf{D}_0 \cap \mathbf{E}$. To prove the converse relation, let $u \in \mathbf{D}_0 \cap \mathbf{E}$. There exists a sequence $\{f_n\}_n$ in $L_0(X)$ such that $D(u - f_n) \to 0$ as $n \to \infty$, $\{f_n\}_n$ converges pointwise to u, and $|f_n(x)| \leq |u(x)|$ on X. It suffices to show that $\|u - f_n\| \to 0$ as $n \to \infty$. Note that $L_2(X\,;\, q) = \{u \in L(X)\,;\, \|u\| < \infty\}$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$. Since $\|f_n\|^2 \leq \|u\|^2$ and $\{f_n\}_n$ converges pointwise to u, we see that $\{f_n\}_n$ converges weakly to u. We have $\|f_n\|^2 \to \|u\|^2$, so that $\|u - f_n\|^2 \to 0$ as $n \to \infty$. □

Lemma 3.3. *$E(u, f) = -\sum_{x \in X} [\Delta_q u(x)] f(x)$ for $u \in \mathbf{E}$ and $f \in L_0(X)$.*

Proof. Using [13, Lemma 3] we have

$$\begin{aligned} E(u, f) &= D(u, f) + \sum_{x \in X} q(x) u(x) f(x) \\ &= -\sum_{x \in X} [\Delta u(x)] f(x) + \sum_{x \in X} q(x) u(x) f(x) \\ &= -\sum_{x \in X} [\Delta_q u(x)] f(x). \end{aligned}$$ □

Lemma 3.4. **HE** *is the orthogonal complement of* $\mathbf{E}_0$ *in* $\mathbf{E}$.

Proof. Let $h \in \mathbf{HE}$. Then $E(h, f) = 0$ for every $f \in L_0(X)$ by Lemma 3.3, so that $E(h, v) = 0$ for every $v \in \mathbf{E}_0$. Conversely, suppose that $h \in \mathbf{E}$ satisfies $E(h, v) = 0$ for all $v \in \mathbf{E}_0$. Since $E(h, \varepsilon_x) = -\Delta_q h(x)$ by Lemma 3.3 for every $x \in X$, we see that $h \in \mathbf{HE}$. □

By a standard argument, we obtain

Theorem 3.5 (Royden's Decomposition)**.** *Every $u \in \mathbf{E}$ is decomposed uniquely in the form $u = v + h$ with $v \in \mathbf{E}_0$ and $h \in \mathbf{HE}$.*

Corollary 3.6. $\mathbf{HE} = \{0\}$ *if and only if* $\mathbf{E} = \mathbf{E}_0$.

We have by Theorem 3.2 and Corollary 3.6

Theorem 3.7. *If $\mathcal{N}$ is parabolic, then $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{HE} = \{0\}$.*

Theorem 3.8. *Assume that $\sum_{x \in X} q(x) < \infty$. Then $\mathcal{N}$ is parabolic if and only if $\mathbf{HE} = \{0\}$.*

Proof. Assume that $\mathbf{HE} = \{0\}$, or $\mathbf{E} = \mathbf{E}_0$. Since $E(1) = \sum_{x \in X} q(x) < \infty$, we have $1 \in \mathbf{E} = \mathbf{E}_0 \subset \mathbf{D}_0$. Therefore $\mathcal{N}$ is parabolic by [14, Theorem 3.2]. The converse follows from Theorem 3.7. □

We say that T is a *normal contraction of the real line* if $T0 = 0$ and $|Ts_1 - Ts_2| \leq |s_1 - s_2|$ for every real numbers s_1, s_2. We define $Tu \in L(X)$ for $u \in L(X)$ by $(Tu)(x) = Tu(x)$ for $x \in X$.

Lemma 3.9. *Let T be a normal contraction of the real line. Then $E(Tu) \leq E(u)$. If $u \in \mathbf{E}_0$, then $Tu \in \mathbf{E}_0$.*

Proof. For $u \in L(X)$, we have $D(Tu) \leq D(u)$ by [13, Lemma 2] and $\|Tu\| \leq \|u\|$, so that

$$E(Tu) = D(Tu) + \|Tu\|^2 \leq D(u) + \|u\|^2 = E(u).$$

Let $u \in \mathbf{E}_0$. Then $Tu \in \mathbf{E}$ by the above. We see by [15, Theorem 4.2] that $Tu \in \mathbf{D}_0$. Therefore, $Tu \in \mathbf{E}_0$ by Theorem 3.2. □

Corollary 3.10. *If $u \in \mathbf{E}$ ($\mathbf{E}_0$ resp.) and c is a positive constant, then* $\max(u, 0), \min(u, c), |u| \in \mathbf{E}$ *($\mathbf{E}_0$ resp.). In this case,*

$$E(\max(u, 0)) \leq E(u), \quad E(\min(u, c)) \leq E(u), \quad E(|u|) \leq E(u).$$

Proposition 3.11. *If $u, v \in \mathbf{E}_0$, then* $\min(u, v) \in \mathbf{E}_0$.

Proof. Since $u+v, |u-v| \in \mathbf{E}_0$, we see that $\min(u, v) = (u+v-|u-v|)/2 \in \mathbf{E}_0$. □

4. q-Superharmonic Functions

For $a \in X$, denote by $U(a)$ the set of neighboring nodes of a and a itself, i.e., $U(a) = \{x \in X\,;\, K(a, y)K(x, y) \neq 0 \text{ for some } y \in Y\}$. For a subset A of X, denote by $U(A)$ the union of $U(x)$ for $x \in A$. We say that $u \in L(X)$ is *q-superharmonic* on a subset A of X if $\Delta_q u(x) \leq 0$ on A. In order to express $\Delta_q u(x)$ in a more familiar form, let us put

$$t(x, z) = \sum_{y \in Y} |K(x, y)K(z, y)| r(y)^{-1} \quad \text{if } z \neq x, \quad t(x, x) = 0,$$

$$t(x) = \sum_{y \in Y} |K(x, y)| r(y)^{-1}.$$

Then $t(x, z) = t(z, x)$ for all $x, z \in X$ and $t(x) = \sum_{z \in X} t(x, z)$. Now we have

$$\Delta_q u(x) = -[t(x) + q(x)]u(x) + \sum_{z \in X} t(x, z)u(z).$$

Lemma 4.1. (1) *A non-negative harmonic function is q-superharmonic. Especially, a positive constant is q-superharmonic.*

(2) *If u and v are q-superharmonic on A, then both $u + v$ and* $\min(u, v)$ *are q-superharmonic on A.*

(3) *If u is q-harmonic on X, then* $-\max(u, 0)$ *is q-superharmonic on X.*

(4) *If $c > 0$ is a constant and u is q-superharmonic (q-harmonic resp.) on X, then cu is q-superharmonic (q-harmonic resp.) on X.*

Proof. (1) Let h be non-negative and harmonic. Then $\Delta_q h(x) = \Delta h(x) - q(x)h(x) = -q(x)h(x) \leq 0$ on X.

(2) If u and v are q-superharmonic, then $\Delta_q(u + v)(x) = \Delta_q u(x) + \Delta_q v(x) \leq 0$. Let $f = \min(u, v)$ and $a \in A$. We may assume that $f(a) = u(a)$. Since $f(x) \leq u(x)$, we have

$$\begin{aligned}\Delta_q f(a) &= \sum_{z \in X} t(z, a)f(z) - [t(a) + q(a)]f(a) \\ &\leq \sum_{z \in X} t(z, a)u(z) - [t(a) + q(a)]u(a) = \Delta_q u(a) \leq 0.\end{aligned}$$

(3) Let $f = \max(u, 0)$. Then $f \in L^+(X)$. If $f(a) = 0$, then $\Delta_q f(a) = \sum_{z\in X} t(a,z)f(z) \geq 0$. Let $f(a) > 0$, i.e., $f(a) = u(a)$. Since $f(x) \geq u(x)$ and u is q-harmonic, we have

$$\Delta_q f(a) = \Delta_q f(a) - \Delta_q u(a) = \sum_{z\in X} t(z,a)[f(z) - u(z)] \geq 0,$$

which means $\Delta_q(-f) \leq 0$.

(4) Our assertion follows from $\Delta_q(cu) = c\Delta_q u$. □

For $u \in L(X)$ and $a \in X$, we define *q-Poisson modification* $P_a u \in L(X)$ as

$$P_a u(a) = \frac{1}{t(a) + q(a)} \sum_{z\in X} t(z,a)u(z), \quad P_a u(x) = u(x) \quad \text{for } x \neq a.$$

Lemma 4.2. *If u is q-superharmonic on X, then $P_a u$ is q-superharmonic on X and q-harmonic at a and $P_a u \leq u$ on X.*

Proof. Since u is q-superharmonic at x, we have $P_a u(x) \leq u(x)$. In fact, in case $x \neq a$ our assertion is obvious. In case $x = a$, $\Delta_q u(a) \leq 0$ implies $\sum_{z\in X} t(z,a)u(z) \leq [q(a) + t(a)]u(a)$, so that $P_a u(a) \leq u(a)$. The proof is given in the following three cases: (1) $x \notin U(a)$, (2) $x = a$, and (3) $x \in U(a) \setminus \{a\}$.

(1). For $x \notin U(a)$, it is obvious that $\Delta_q P_a u(x) = \Delta_q u(x) \leq 0$.

(2). In case $x = a$, we have

$$\begin{aligned}\Delta_q P_a u(a) &= -[t(a) + q(a)]P_a u(a) + \sum_{z\in X} t(z,a)P_a u(z) \\ &= -\sum_{z\in X} t(z,a)u(z) + \sum_{z\in X} t(z,a)u(z) = 0.\end{aligned}$$

(3). In case $x \in U(a) \setminus \{a\}$, we have

$$\begin{aligned}\Delta_q P_a u(x) &= -[t(x) + q(x)]P_a u(x) + \sum_{z\in X} t(x,z)P_a u(z) \\ &\leq -[t(x) + q(x)]u(x) + \sum_{z\in X} t(x,z)u(z) = \Delta_q u(x) \leq 0.\end{aligned}$$

□

Lemma 4.3 (Local Minimum Principle)**.** *Let $u \in L(X)$ and $a \in X$. Assume that u is q-superharmonic at a and $u(z) \geq 0$ for all $z \in U(a) \setminus \{a\}$. Then $u(a) \geq 0$. Moreover, $u(a) = 0$ occurs only when $u(z) = 0$ for all $z \in U(a) \setminus \{a\}$.*

Proof. Since $\Delta_q u(a) \leq 0$ and $u(z) \geq 0$ for $z \in U(a) \setminus \{a\}$, we have

$$[q(a) + t(a)]u(a) \geq \sum_{z\in U(a)} t(a,z)u(z) \geq 0,$$

so that $u(a) \geq 0$. If $u(a) = 0$, then $u(z) = 0$ for $z \in U(a) \setminus \{a\}$ by the above inequality. □

Corollary 4.4. *Let u be q-superharmonic on X. If $u(x) \geq 0$ on X and $u(a) = 0$ for some $a \in X$, then $u(x) = 0$ on X.*

We have the following minimum principle:

Theorem 4.5 (Minimum Principle). *Let A be a finite subset of X and let $u \in L(X)$ be q-superharmonic on A. If $u(x) \geq 0$ on $X \setminus A$, then $u(x) \geq 0$ on X.*

Proof. Suppose that $c := \min\{u(x)\,;\, x \in A\} < 0$ and put $B = \{x \in X\,;\, u(x) = c\}$. Lemma 4.1 implies that $u - c$ is q-superharmonic on A. Since $u - c \geq 0$ on X and $u - c = 0$ on B, the local minimum principle implies $U(x) \subset B$ for all $x \in B$, so that $U(B) \subset B$. Since X is connected, we have $B = X$, which is a contradiction. □

Corollary 4.6. *Let A be a finite subset of X. If u is q-superharmonic on A and v is q-harmonic on A and if $u(x) \geq v(x)$ on $X \setminus A$, then $u(x) \geq v(x)$ on X.*

Proposition 4.7 (Harnack's Inequality). *Let $a, b \in X$. There exists a positive constant α depending only on a and b such that $\alpha^{-1}u(b) \leq u(a) \leq \alpha u(b)$ for all non-negative q-superharmonic function u on X.*

Proof. Let $x_0 \in X$ and $x_1 \in U(x_0) \setminus \{x_0\}$. Since $u(x) \geq 0$ and $\Delta_q u(x_0) \leq 0$, we have

$$t(x_1, x_0)u(x_1) \leq \sum_{x \in X} t(x, x_0)u(x) \leq [t(x_0) + q(x_0)]u(x_0),$$

or

$$u(x_1) \leq \frac{t(x_0) + q(x_0)}{t(x_1, x_0)} u(x_0).$$

If $x_2 \in U(x_1) \setminus \{x_1\}$, then

$$u(x_2) \leq \frac{t(x_1) + q(x_1)}{t(x_2, x_1)} u(x_1).$$

Repeat this argument to obtain the result. □

The following result was proved in Anandam [1, Theorem 2.4.9] in case $\mathcal{N}$ is a finite network.

Lemma 4.8. *Let $\mathcal{P}$ be a Perron family. Namely $\mathcal{P}$ is a non-empty family of q-superharmonic functions on X such that*

(1) *$\{u(x)\,;\, u \in \mathcal{P}\}$ is bounded from below for each $x \in X$,*
(2) *$\min(u, v) \in \mathcal{P}$ whenever $u, v \in \mathcal{P}$,*
(3) *$P_a u \in \mathcal{P}$ for any $u \in \mathcal{P}$ and $a \in X$.*

Then $u^*(x) = \inf\{u(x)\,;\, u \in \mathcal{P}\}$ is q-harmonic on X.

Proof. By (1), $u^* \in L(X)$. Let $a \in X$. Since $U(a)$ is a finite set, in view of (2), we can choose $u_n \in \mathcal{P}$ such that $u_n(z)$ converges decreasingly to $u^*(z)$ for all $z \in U(a)$. Put $v_n = P_a u_n$. Then $v_n \in \mathcal{P}$ and $u^* \leq v_n \leq u_n$. Hence $v_n(z) \to u^*(z)$ for all $z \in U(a)$. Since v_n is q-harmonic at a, so is u^*. □

Denote by **SH** the set of all q-superharmonic functions on X and let

$$\mathbf{H}^+ = \mathbf{H} \cap L^+(X), \quad \mathbf{HB} = \{u \in \mathbf{H}\,;\, \sup\{|u(x)|\,;\, x \in X\} < \infty\}.$$

Theorem 4.9. $\mathbf{H}^+ = \{0\}$ *implies* $\mathbf{HB} = \{0\}$.

Proof. Let $u \in \mathbf{HB}$ and consider $\mathcal{P} = \{v \in \mathbf{SH}\,;\, v \geq u^+ := \max(u,0)\}$. Since u is bounded, there exists $c > 0$ such that $|u| \leq c$. Note that $c \in \mathcal{P} \neq \emptyset$. Lemma 4.8 implies $\min(v_1, v_2) \in \mathcal{P}$ for $v_1, v_2 \in \mathcal{P}$. If $v \in \mathcal{P}$ and $a \in X$, then $P_a v - u^+ = v - u^+ \geq 0$ on $X \setminus \{a\}$ and $P_a v - u^+$ is q-superharmonic at a by Lemmas 4.1 and 4.2. By the local minimum principle, $P_a v(a) - u^+(a) \geq 0$, which implies $P_a v \in \mathcal{P}$. Lemma 4.8 shows that $h^+(x) := \inf\{v(x)\,;\, v \in \mathcal{P}\}$ is q-harmonic on X and $h^+ \geq u^+ \geq 0$, so that $h^+ \in \mathbf{H}^+ = \{0\}$, hence $u^+ = 0$. Similarly, $u^- := \max(-u, 0) = 0$, so that $u = 0$. □

This result was shown in [9] for a non-linear case.

5. The q-Green Function

Lemma 3.1 shows that $u \mapsto u(a)$ is a continuous linear mapping on $\mathbf{E}$ for each $a \in X$. By F. Riesz's theorem, there exists a reproducing kernel φ_a of $\mathbf{E}$, i.e., $\varphi_a \in \mathbf{E}$ and $E(\varphi_a, u) = u(a)$ for every $u \in \mathbf{E}$. Let $\varphi_a = g_a + \theta_a$ be Royden's decomposition, i.e., $g_a \in \mathbf{E}_0$ and $\theta_a \in \mathbf{HE}$. We call g_a the *q-Green function* of $\mathcal{N}$ with pole at a. By the uniqueness of the reproducing kernel and its Royden's decomposition, the q-Green function g_a exists uniquely. Note that in case $\mathbf{E} = \mathbf{E}_0$, $g_a = \varphi_a$ is the q-Green function of $\mathcal{N}$ with pole at a.

Theorem 5.1. *$E(g_a, u) = u(a)$ for all $u \in \mathbf{E}_0$ and $\Delta_q g_a(x) = -\varepsilon_a(x)$ on X.*

Proof. Let $u \in \mathbf{E}_0$. Then $E(\theta_a, u) = 0$ by Lemma 3.4, so that

$$E(g_a, u) = E(g_a + \theta_a, u) = E(\varphi_a, u) = u(a).$$

Since $\varepsilon_x \in L_0(X)$ for every $x \in X$, we see by Lemma 3.3 that

$$\varepsilon_x(a) = E(g_a, \varepsilon_x) = -\Delta_q g_a(x).$$ □

We do not use the notation $\tilde{g}_a$ used in [17]. In what follows, every statement related to the pair $(g_a, \mathbf{E}_0)$ remains true even in case $\mathbf{E}_0 = \mathbf{E}$. Since the reasoning related to $(g_a, \mathbf{E}_0)$ in case $\mathbf{E} \neq \mathbf{E}_0$ holds in the case $\mathbf{E} = \mathbf{E}_0$, we do not discern these cases.

Corollary 5.2. $g_a(a) = E(g_a) > 0$.

Lemma 5.3. *The function $u^* = g_a/g_a(a)$ is the unique optimal solution to the extremum problem: Minimize $E(u)$ subject to $u \in \mathbf{E}_0$ and $u(a) = 1$.*

Proof. Clearly, u^* is a feasible solution to our extremum problem. For any $u \in \mathbf{E}_0$ with $u(a) = 1$, we have

$$E(u^*) = \frac{E(g_a)}{g_a(a)^2} = \frac{1}{g_a(a)}, \quad 1 = E(g_a, u) \leq E(g_a)^{1/2} E(u)^{1/2},$$

so that $E(u) \geq 1/E(g_a) = E(u^*)$. To show the uniqueness of the optimal solution, let u_1 and u_2 be optimal solutions to our extremum problem. Then

$$\begin{aligned} \alpha &:= E(u_1) = E(u_2) \leq E((u_1 + u_2)/2) \\ &\leq E((u_1 + u_2)/2) + E((u_1 - u_2)/2) = (E(u_1) + E(u_2))/2 = \alpha, \end{aligned}$$

so that $E(u_1 - u_2) = 0$. Hence $u_1 = u_2$. □

Theorem 5.4. (1) *$g_a(b) = g_b(a)$ for every $a, b \in X$.*
(2) *$0 < g_a(x) \le g_a(a)$ on X.*

Proof. (1) $g_a(b) = E(g_b, g_a) = E(g_a, g_b) = g_b(a)$.
(2) Let $u^* = g_a/g_a(a)$. Since $E(\max(u^*, 0)) \le E(u^*)$ and $E(\min(u^*, 1)) \le E(u^*)$ by Corollary 3.10, we have $u^* = \max(u^*, 0) = \min(u^*, 1)$ by Lemma 5.3, and hence $0 \le u^* \le 1$. We see $u^* > 0$ by Corollary 4.4. □

Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$. There exists a unique *q-Green function* $g_a^{(n)}$ of $\mathcal{N}_n$ with pole at $a \in X_n$. This function is defined as the reproducing kernel of the linear mapping $u \in \mathbf{E}(X_n) \mapsto u(a)$, i.e., $E(u, g_a^{(n)}) = u(a)$ for $u \in \mathbf{E}(X_n)$, where $\mathbf{E}(X_n) = \{u \in L(X)\,;\, u = 0 \text{ on } X \setminus X_n\}$ is a Hilbert space with respect to the inner product $E(\cdot, \cdot)$. Needless to say, $g_a^{(n)}$ is the unique function of linear equation $\Delta_q g_a^{(n)} = -\varepsilon_a$ on X_n with the boundary condition $g_a^{(n)} = 0$ on $X \setminus X_n$. We have

Theorem 5.5. (1) *$g_a^{(n)}(b) = g_b^{(n)}(a)$ for every $a, b \in X_n$.*
(2) *$0 < g_a^{(n)}(x) \le g_a^{(n)}(a)$ for $a, x \in X_n$.*
(3) *$g_a^{(n)} \le g_a^{(n+1)} \le g_a$ on X and $\{g_a^{(n)}\}$ converges pointwise to g_a for $a \in X_n$.*
(4) *$E(g_a^{(n)} - g_a) \to 0$ as $n \to \infty$ for $a \in X$.*

Proof. (1) and (2) are shown by arguments similar to those of Theorem 5.4. Put $u_n = g_a^{(n+1)} - g_a^{(n)}$ and $v_n = g_a - g_a^{(n)}$. Then both u_n and v_n are q-harmonic on X_n and are non-negative on $X \setminus X_n$. We see by Theorem 4.5 that u_n and v_n are non-negative on X. This shows the first half of (3).

For $m > n$ and for $a \in X_n$, we have

$$E(g_a^{(n)}, g_a^{(m)}) = g_a^{(m)}(a) = E(g_a^{(m)}) \le g_a(a),$$
$$E(g_a^{(m)} - g_a^{(n)}) = E(g_a^{(m)}) - 2E(g_a^{(m)}, g_a^{(n)}) + E(g_a^{(n)}) = E(g_a^{(n)}) - E(g_a^{(m)}).$$

It follows that $\{g_a^{(n)}\}_n$ is a Cauchy sequence in the Hilbert space $\mathbf{E}_0$. There exists $f \in \mathbf{E}_0$ such that $E(g_a^{(n)} - f) \to 0$ as $n \to \infty$. Since $\{g_a^{(n)}\}_n$ converges pointwise to f, we have $\Delta_q f(x) = -\varepsilon_a(x)$ on X. Thus $f = g_a$. This shows (4) and the last half of (3). □

Example 5.6. Let $\mathcal{G}$ be the linear graph, $X = \{x_n\,;\, n \ge 0\}$, $Y = \{y_n\,;\, n \ge 1\}$, $K(x_n, y_{n+1}) = 1$, $K(x_{n+1}, y_{n+1}) = -1$ for $n \ge 0$, and $K(x, y) = 0$ for any other pair (x, y). Let $r_n = r(y_n)$ and assume that $R_0 := \sum_{j=1}^{\infty} r_j < \infty$. Let $q(x) = \varepsilon_{x_0}(x)$ and $\mathcal{N} = \{\mathcal{G}, r, q\}$. The q-Green function of $\mathcal{N}$ with pole at x_m ($m \ge 0$) is given by

$$g_{x_m}(x_n) = \frac{(1 + \rho_n)R_m}{1 + R_0} \quad \text{if } 0 \le n \le m,$$
$$g_{x_m}(x_n) = \frac{(1 + \rho_m)R_n}{1 + R_0} \quad \text{if } n \ge m,$$

where $R_n = \sum_{j=n+1}^{\infty} r_j$ and $\rho_n = R_0 - R_n$.

Proof. We prove only the case $m \geq 1$; the case $m = 0$ can be shown by a similar argument. Let $u_n = g_{x_m}(x_n)$ and $w_n = r_n^{-1}(u_n - u_{n-1})$. Then $\Delta_q g_{x_m}(x) = -\varepsilon_{x_m}(x)$ on X implies

$$w_1 - u_0 = 0, \quad w_{n+1} - w_n = 0 \quad \text{for } n \neq m, \quad w_{m+1} - w_m = -1.$$

We see that $w_n = u_0$ for $1 \leq n \leq m$ and $w_n = u_0 - 1$ for $n \geq m+1$, so that

$$u_n = u_0 + \rho_n u_0 \quad \text{for } 0 \leq n \leq m,$$
$$u_n = (u_0 - 1)(\rho_n - \rho_m) + u_m \quad \text{for } n \geq m.$$

Since $\mathcal{N}$ is hyperbolic, Kayano and Yamasaki [4, Theorem 3.3] show that $u_n \to 0$ as $n \to \infty$, so that $(u_0 - 1)R_m + u_m = 0$. Therefore $u_0 = R_m/(1 + R_0)$. □

Example 5.7. Let $\mathcal{G}$ be the homogeneous tree of order 3. We assume that $r = 1$ on Y and $q = 1$ on X. Denote by $\rho(a, b)$ the geodesic metric between two nodes a and b, i.e., the number of arcs of the path between a and b. Let $C(a\,;n) = \{x \in X\,;\rho(a, x) = n\}$. Then the q-Green function of $\mathcal{N}$ with pole at a is given by

$$g_a(x) = \frac{\alpha^n}{4 - 3\alpha} \quad \text{for } x \in C(a\,;n), \quad \alpha = 1 - \frac{1}{\sqrt{2}}.$$

Proof. Fix a node $a \in X$. By the symmetry, $g_a(x)$ depends only on $\rho(a, x)$. Define $u_n = g_a(x)$ for $\rho(x, a) = n$. The equation $\Delta_q g_a(x) = -\varepsilon_a(x)$ on X can be written as follows:

$$3u_1 - 4u_0 = -1, \quad 2u_{n+1} - 4u_n + u_{n-1} = 0 \quad \text{for } n \geq 1.$$

The characteristic equation $2t^2 - 4t + 1 = 0$ gives $t = 1 \pm 1/\sqrt{2}$. Since $\mathcal{N}$ is hyperbolic, we have that $u_n \to 0$ as $n \to \infty$, so that $u_n = A\alpha^n$ with $\alpha = 1 - 1/\sqrt{2}$ for $n \geq 0$. The condition $3u_1 - 4u_0 = -1$ shows $A = 1/(4 - 3\alpha)$. □

6. A Fundamental Existence Theorem

The following theorem plays a fundamental role for the study of q-Green potentials in the succeeding sections.

Theorem 6.1. *Let $f \in \mathbf{E}_0$ be non-negative and A a nonempty proper subset of X. Then there exists $u^* \in \mathbf{E}_0$ such that*

(1) $\Delta_q u^*(x) \leq 0$ *on* X,
(2) $\Delta_q u^*(x) = 0$ *on* $X \setminus A$,
(3) $u^*(x) \geq f(x)$ *on* A,
(4) $u^*(x) = f(x)$ *if* $x \in A$ *and* $\Delta_q u^*(x) < 0$.

Proof. Let us consider the following extremum problem:

$$\alpha = \inf\{E(u) - 2E(u, f)\,;u \in \mathcal{F}\},$$

where $\mathcal{F} = \{u \in \mathbf{E}_0\,;\Delta_q u \leq 0, \Delta_q u(x) = 0 \text{ on } X \setminus A\}$. Note that $\alpha < \infty$, since $g_x \in \mathcal{F}$ for $x \in A$. We see that α is finite by the inequality

$$E(u) - 2E(u, f) = E(u - f) - E(f) \geq -E(f).$$

Let $\{u_n\}_n$ be a minimizing sequence. Then

$$\begin{aligned}\alpha &\le E((u_n+u_m)/2) - 2E((u_n+u_m)/2, f)\\ &\le E((u_n+u_m)/2) - 2E((u_n+u_m)/2, f) + E((u_n-u_m)/2)\\ &= [E(u_n) - 2E(u_n, f)]/2 + [E(u_m) - 2E(u_m, f)]/2 \to \alpha\end{aligned}$$

as $n, m \to \infty$, so that $E(u_n - u_m) \to 0$ as $n, m \to \infty$. Since $\mathbf{E}_0$ is a Hilbert space, we see that there exists $u^* \in \mathbf{E}_0$ such that $E(u_n - u^*) \to 0$ as $n \to \infty$. Since $\{u_n\}_n$ converges pointwise to u^*, we see that $u^* \in \mathcal{F}$, which shows (1) and (2). We prove (3). Noting that

$$|E(u_n, f) - E(u^*, f)| = |E(u_n - u^*, f)| \le E(u_n - u^*)^{1/2} E(f)^{1/2} \to 0$$

as $n \to \infty$, we have $\alpha = E(u^*) - 2E(u^*, f)$. For $v \in \mathcal{F}$ and $t > 0$, we have $u^* + tv \in \mathcal{F}$, so that

$$\begin{aligned}\alpha &\le E(u^* + tv) - 2E(u^* + tv, f)\\ &= E(u^*) - 2E(u^*, f) + 2t[E(u^*, v) - E(v, f)] + t^2 E(v)\\ &= \alpha + 2t[E(u^*, v) - E(v, f)] + t^2 E(v).\end{aligned}$$

Therefore $E(u^*, v) - E(v, f) \ge 0$. By taking $v = g_x$ for $x \in A$ in this inequality, we obtain $u^*(x) \ge f(x)$ on A.

To prove (4), assume $\Delta_q u^*(a) < 0$ for $a \in A$. For any $t > 0$ with $\Delta_q u^*(a) + t < 0$, we see that $u^* - tg_a \in \mathbf{E}_0$ and $\Delta_q(u^* - tg_a)(x) = \Delta_q u^*(x) + t\varepsilon_a(x) \le 0$, so that $u^* - tg_a \in \mathcal{F}$. We have

$$\alpha \le E(u^* - tg_a) - 2E(u^* - tg_a, f) = \alpha - 2t[E(u^*, g_a) - E(g_a, f)] + t^2 E(g_a),$$

so that $E(u^*, g_a) - E(f, g_a) \le 0$. Thus $u^*(a) \le f(a)$. Hence $u^*(a) = f(a)$ by (3). □

7. q-Green Potentials

We define the *q-Green potential* $G\mu$ of $\mu \in L^+(X)$ and the *mutual q-Green potential energy* $G(\mu, \nu)$ of $\mu, \nu \in L^+(X)$ by

$$G\mu(x) = \sum_{z \in X} g_z(x)\mu(z), \quad G(\mu, \nu) = \sum_{x \in X} [G\mu(x)]\nu(x).$$

We call $G(\mu, \mu)$ the *q-Green potential energy* of μ. Let us put

$$\mathcal{M} = \{\mu \in L^+(X)\,;\, G\mu \in L(X)\}, \quad \mathcal{E} = \{\mu \in L^+(X)\,;\, G(\mu, \mu) < \infty\}.$$

We see easily

Lemma 7.1. *$\Delta_q G\mu(x) = -\mu(x)$ on X for every $\mu \in \mathcal{M}$.*

By Harnack's inequality, we note that $G\mu(a) < \infty$ for some $a \in X$ implies $G\mu(x) < \infty$ for any $x \in X$, so that $\mathcal{E} \subset \mathcal{M}$. We shall prove a discrete analogy of the Riesz decomposition theorem.

Theorem 7.2 (Riesz's Decomposition)**.** *Every non-negative q-superharmonic function u can be decomposed uniquely in the form $u = G\mu + h$, where $\mu \in \mathcal{M}$ and h is non-negative and q-harmonic on X. In this decomposition, $\mu = -\Delta_q u$ and h is the greatest q-harmonic minorant of u.*

Proof. Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$ and let $g_a^{(n)}$ be the q-Green function of $\mathcal{N}_n$ with pole at a. Put $\mu = -\Delta_q u$,

$$u_n(x) = \sum_{z\in X_n} g_z^{(n)}(x)\mu(z), \quad h_n = u - u_n.$$

Then $\Delta_q u_n = -\mu$ on X_n and $u_n = 0$ on $X \setminus X_n$, so that h_n is q-harmonic on X_n and $h_n \geq 0$ on $X \setminus X_n$. Thus $h_n \geq 0$ on X by the minimum principle. Since $g_z^{(n)} \leq g_z^{(n+1)}$ on X by Theorem 5.5, we have $u_n \leq u_{n+1}$ and $h_n \geq h_{n+1}$ on X. Let h be the pointwise limit of $\{h_n\}_n$. Then $h \in \mathbf{H}$. Since $\{u_n\}_n$ converges pointwise to $G\mu$, we have $u = G\mu + h$. The uniqueness of the decomposition is clear by Lemma 7.1. To prove the last assertion, let $h' \in \mathbf{H}$ and $0 \leq h' \leq u$ on X. Since $h_n - h'$ is q-harmonic on X_n and $h_n - h' = u - h' \geq 0$ on $X \setminus X_n$, we see by the minimum principle that $h_n \geq h'$ on X, and hence $h \geq h'$ on X. □

By this theorem, we obtain the following.

Theorem 7.3. *A non-negative q-superharmonic function u is a q-Green potential if and only if the greatest q-harmonic minorant of u is equal to zero.*

Corollary 7.4. *Let u be non-negative and q-superharmonic. If there exists $\mu \in \mathcal{M}$ such that $u(x) \leq G\mu(x)$ on X, then u is a q-Green potential.*

8. q-Potentials with Finite Energy

We begin with the study of q-potentials with finite energy.

Lemma 8.1. *If $\mu \in \mathcal{E}$, then $G\mu \in \mathbf{E}_0$ and $E(G\mu) = G(\mu, \mu)$.*

Proof. First let $\mu, \nu \in L_0(X)$. Since $g_x \in \mathbf{E}_0$, we have $G\mu, G\nu \in \mathbf{E}_0$. We obtain

$$E(G\mu, G\nu) = \sum_{z\in X} E(g_z, G\nu)\mu(z) = \sum_{z\in X}[G\nu(z)]\mu(z) = G(\mu, \nu).$$

Let $\mu \in \mathcal{E}$. Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$ and put $\mu_n = \mu\varepsilon_{X_n}$ and $u_n = G\mu_n$. Then $u_n \in \mathbf{E}_0$. For $m > n$ we have

$$E(u_n, u_m) = G(\mu_n, \mu_m) \geq G(\mu_n, \mu_n) = E(u_n),$$
$$E(u_n - u_m) = E(u_n) - 2E(u_n, u_m) + E(u_m) \leq E(u_m) - E(u_n).$$

Since $E(u_n) = G(\mu_n, \mu_n) \leq G(\mu, \mu) < \infty$, we have that $\{u_n\}_n$ is a Cauchy sequence in $\mathbf{E}_0$. Thus there exists $v \in \mathbf{E}_0$ such that $E(u_n - v) \to 0$ as $n \to \infty$. We have

$$G(\mu, \mu) \leq \liminf_{n\to\infty} G(\mu_n, \mu_n) = \lim_{n\to\infty} E(u_n) \leq G(\mu, \mu).$$

Therefore, $E(v) = G(\mu, \mu)$. Since $\{u_n\}_n$ converges pointwise to $G\mu$, we conclude that $G\mu = v \in \mathbf{E}_0$. □

Lemma 8.2. *Let $\mu \in \mathcal{E}$. Then $E(G\mu, u) = \sum_{x\in X} u(x)\mu(x)$ for every $u \in \mathbf{E}_0 \cap L^+(X)$.*

Proof. Since $E(g_x, u) = u(x)$ for $u \in \mathbf{E}_0$, our assertion is clear in case $\mu \in L_0(X)$ by Lemma 3.3 and Lemma 7.1. Let μ_n be the same as in the proof of the above lemma. Then $E(G\mu_n, u) = \sum_{x\in X} u(x)\mu_n(x)$. Since $E(G\mu - G\mu_n) \to 0$ as $n \to \infty$, we have $E(G\mu_n, u) \to E(G\mu, u)$ as $n \to \infty$. Since $u \in L^+(X)$, we see that

$$E(G\mu, u) = \lim_{n\to\infty} E(G\mu_n, u) = \lim_{n\to\infty} \sum_{x\in X} u(x)\mu_n(x)$$
$$= \lim_{n\to\infty} \sum_{x\in X_n} u(x)\mu(x) = \sum_{x\in X} u(x)\mu(x). \qquad \square$$

Lemma 8.3. *If $u \in \mathbf{E}_0$ is q-superharmonic on X, then $u \in L^+(X)$.*

Proof. Let $a \in X$ and $g_a^{(n)}$ be the q-Green function of $\mathcal{N}_n$. We may assume that $a \in X_n$ for large n. Since $E(g_a - g_a^{(n)}) \to 0$ as $n \to \infty$ and $u \in \mathbf{E}_0$, we have $E(u, g_a^{(n)} - g_a) \to 0$ as $n \to \infty$. By Lemma 3.2,

$$E(u, g_a^{(n)}) = -\sum_{z\in X} [\Delta_q u(z)] g_a^{(n)}(z) \geq 0,$$

so that $u(a) = E(u, g_a) = \lim_{n\to\infty} E(u, g_a^{(n)}) \geq 0$. $\square$

Theorem 8.4. $\{G\mu\,;\, \mu \in \mathcal{E}\} = \{u \in \mathbf{E}_0\,;\, \Delta_q u(x) \leq 0\}$.

Proof. Lemmas 7.1 and 8.1 shows that $\Delta_q G\mu \leq 0$ and $G\mu \in \mathbf{E}_0$ for $\mu \in \mathcal{E}$. To show the converse, let $u \in \mathbf{E}_0$ satisfy $\Delta_q u(x) \leq 0$ on X. Lemma 8.3 shows $u \in L^+(X)$. By Riesz's decomposition, there exist $\mu \in \mathcal{M}$ and $h \in \mathbf{H}^+$ such that $u = G\mu + h$. Consider an exhaustion $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ of $\mathcal{N}$ and put $\mu_n = \mu\varepsilon_{X_n}$ and $u_n = G\mu_n$. Since $G\mu \leq u$, we have

$$E(u_n) = G(\mu_n, \mu_n) \leq G(\mu, \mu_n) \leq \sum_{x\in X} u(x)\mu_n(x)$$
$$= E(G\mu_n, u) \leq E(u_n)^{1/2} E(u)^{1/2}$$

by Lemma 8.2, so that $G(\mu_n, \mu_n) \leq E(u) < \infty$. Therefore

$$G(\mu, \mu) \leq \liminf_{n\to\infty} G(\mu_n, \mu_n) \leq E(u),$$

hence $\mu \in \mathcal{E}$ and $G\mu \in \mathbf{E}_0$ by Lemma 8.1. It follows from Royden's decomposition that $u = G\mu$. $\square$

Let $\mathbf{HE}^+ = \mathbf{HE} \cap L^+(X)$.

Theorem 8.5. *If $u \in \mathbf{E}$ is non-negative and q-superharmonic, then u is decomposed uniquely in the form $u = G\mu + h$ with $\mu \in \mathcal{E}$ and $h \in \mathbf{HE}^+$.*

Proof. Royden's decomposition shows $u = v + h$ with $v \in \mathbf{E}_0$ and $h \in \mathbf{HE}$. Using $\Delta_q v = \Delta_q u \leq 0$, we have $v = G\mu$ for some $\mu \in \mathcal{E}$ by Theorem 8.4. Riesz's decomposition shows $u = G\mu' + h'$ for some $\mu' \in \mathcal{M}$ and $h' \in \mathbf{H}^+$. Note that $\mu' = -\Delta_q u = -\Delta_q v = \mu$, so that $h = h' \geq 0$. $\square$

Lemma 8.6. *Let $\mu \in \mathcal{M}$ and $\nu \in \mathcal{E}$. If $G\mu \leq G\nu$ on X, then $\mu \in \mathcal{E}$.*

Proof. We have

$$G(\mu,\mu) = \sum_{x\in X}[G\mu(x)]\mu(x) \leq \sum_{x\in X}[G\nu(x)]\mu(x) = \sum_{z\in X}[G\mu(z)]\nu(z)$$
$$\leq \sum_{z\in X}[G\nu(z)]\nu(z) = G(\nu,\nu) < \infty. \qquad \square$$

Denote by $S\mu$ the support of $\mu \in L(X)$, i.e., $S\mu = \{x \in X\,;\, \mu(x) \neq 0\}$.

Proposition 8.7. *Let $\mu, \nu \in \mathcal{E}$. If $G\mu \leq G\nu$ on $S\mu$, then the same inequality holds on X.*

Proof. Let $u = \min(G\mu, G\nu)$. Since $G\mu$ and $G\nu$ are q-superharmonic, so is u by Lemma 4.1. Proposition 3.11 implies $u \in \mathbf{E}_0$, so that there exists $\lambda \in \mathcal{E}$ such that $u = G\lambda$ by Theorem 8.4. Note that $u(x) = G\mu(x)$ on $S\mu$ by our assumption. Lemma 8.2 shows

$$E(G\mu, G\mu - u) = \sum_{x\in X}(G\mu(x) - u(x))\mu(x) = 0.$$

Therefore

$$E(G\mu - u) = E(G\mu, G\mu - u) - E(G\lambda, G\mu - u)$$
$$= -\sum_{x\in X}(G\mu(x) - u(x))\lambda(x) \leq 0,$$

and hence $E(G\mu - u) = 0$. Thus $u = G\mu$ and $G\mu \leq G\nu$ on X. $\square$

9. Potential Theoretic Properties of q-Green Potentials

Now we show some fundamental properties of q-Green potential which are well-known as the domination principle, the equilibrium principle and the balayage principle.

Proposition 9.1. *Let $\mu_1, \mu_2 \in \mathcal{M}$. Then there exists $\nu \in \mathcal{M}$ such that $G\nu = \min(G\mu_1, G\mu_1)$.*

Proof. Let $u = \min(G\mu_1, G\mu_2)$. Then u is non-negative and q-superharmonic by Lemma 4.1. Our assertion follows from Corollary 7.4. $\square$

By Proposition 9.1 and Lemma 8.6, we have

Corollary 9.2. *Let $\mu \in \mathcal{M}$ and $\nu \in \mathcal{E}$. Then there exists $\lambda \in \mathcal{E}$ such that $G\lambda = \min(G\mu, G\nu)$.*

Proposition 9.3 (Domination Principle)**.** *Let $\nu \in \mathcal{E}$ and $\mu \in \mathcal{M}$. If $G\mu(x) \leq G\nu(x)$ on $S\mu$, then the same inequality holds on X.*

Proof. Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$ and let $\mu_n = \mu\varepsilon_{X_n}$. Then $S\mu_n \subset S\mu$ and $\mu_n \in \mathcal{E}$. We have $G\mu_n(x) \leq G\nu(x)$ on $S\mu_n$. By Proposition 8.7, the same inequality holds on X. Since $G\mu_n(x) \to G\mu(x)$ as $n \to \infty$, we conclude that $G\mu(x) \leq G\nu(x)$ on X. $\square$

Theorem 9.4. *Let u be non-negative and q-superharmonic on X and $\mu \in \mathcal{M}$. If $G\mu(x) \le u(x)$ on $S\mu$, then the same inequality holds on X.*

Proof. Let $\{\mathcal{N}_n = \langle X_n, Y_n \rangle\}_n$ be an exhaustion of $\mathcal{N}$ and let $\mu_n = \mu\varepsilon_{X_n}$. Then $S\mu_n \subset S\mu$ and $\mu_n \in \mathcal{E}$. Let $u_n = \min(G\mu_n, u)$. Since $u_n \le G\mu_n$, we see by Corollary 7.4 that there exists $\lambda_n \in \mathcal{E}$ such that $G\lambda_n = u_n$. For $x \in S\mu_n$, we have $G\lambda_n(x) = \min(G\mu_n(x), u(x)) = G\mu_n(x)$, so that $G\mu_n(x) \le G\lambda_n(x) \le u(x)$ on X by Proposition 8.7. Since $G\mu_n(x) \to G\mu(x)$ as $n \to \infty$, we conclude that $G\mu(x) \le u(x)$ on X. □

Proposition 9.5 (Equilibrium Principle)**.** *For a nonempty finite subset A of X, there exists $\xi_A \in L^+(X)$ such that $S\xi_A \subset A$, $G\xi_A(x) = 1$ on A, and $G\xi_A(x) \le 1$ on X.*

Proof. Take $f = \varepsilon_A \in \mathbf{E}_0$ and let u^* be the function obtained in Theorem 6.1. Theorem 8.4 shows $u^* = G\xi_A$ for some $\xi_A \in \mathcal{E}$. Note that $\Delta_q u^* = -\xi_A$ by Lemma 7.1. We see that $G\xi_A(x) \ge 1$ on A and $S\xi_A \subset A$. Since $G\xi_A(x) = 1$ on $S\xi_A$, Theorem 9.4 shows that $G\xi_A(x) \le 1$ on X. □

Proposition 9.6 (Balayage Principle 1)**.** *Let $\mu \in \mathcal{E}$ and A a nonempty proper subset of X. Then there exists $\mu_A \in L^+(X)$ such that $S\mu_A \subset A$, $G\mu_A(x) = G\mu(x)$ on A, and $G\mu_A(x) \le G\mu(x)$ on X.*

Proof. Since $G\mu \in \mathbf{E}_0$ by Lemma 8.1, we take $f = G\mu$ in Theorem 6.1 and obtain u^*. Theorem 8.4 shows $u^* = G\mu_A$ for some $\mu_A \in \mathcal{E}$. We see that $S\mu_A \subset A$, $G\mu_A(x) \ge G\mu(x)$ on A, and $G\mu_A(x) = G\mu(x)$ on $S\mu_A$. Proposition 9.3 shows that $G\mu_A(x) \le G\mu(x)$ on X. □

Proposition 9.7 (Balayage Principle 2)**.** *Let $\mu \in \mathcal{M}$ and A a finite subset of X. If $\mu(X) := \sum_{x\in X} \mu(x) < \infty$, then there exists $\mu_A \in \mathcal{M}$ such that $S\mu_A \subset A$, $G\mu_A(x) = G\mu(x)$ on A, and $G\mu_A(x) \le G\mu(x)$ on X.*

Proof. Let $\{\mathcal{N}_n = \langle X_n, Y_n \rangle\}_n$ be an exhaustion of $\mathcal{N}$ and let $\mu_n = \mu\varepsilon_{X_n}$. Then $\mu_n \in \mathcal{E}$, so that by Proposition 9.6 there exists $\mu_n^* \in L^+(X)$ such that $S\mu_n^* \subset A$, $G\mu_n^*(x) = G\mu_n(x)$ on A, and $G\mu_n^*(x) \le G\mu_n(x)$ on X. Let $\xi_A \in L^+(X)$ be the function in Proposition 9.5, i.e., $S\xi_A \subset A$, $G\xi_A(x) = 1$ on A, and $G\xi_A(x) \le 1$ on X. We have

$$\begin{aligned}
\mu_n^*(A) := \sum_{x\in A} \mu_n^*(x) &= \sum_{x\in A} [G\xi_A(x)]\mu_n^*(x) = G(\xi_A, \mu_n^*) \\
&= \sum_{x\in X} [G\mu_n^*(x)]\xi_A(x) \le \sum_{x\in X} [G\mu_n(x)]\xi_A(x) \\
&\le \sum_{x\in X} [G\mu(x)]\xi_A(x) = \sum_{x\in X} [G\xi_A(x)]\mu(x) \le \mu(X) < \infty.
\end{aligned}$$

Taking a subsequence if necessary, we may assume that $\{\mu_n^*\}_n$ converges pointwise to μ^*. Since $G\mu_n^*$ ($G\mu_n$ resp.) converges pointwise to $G\mu^*$ ($G\mu$ resp.), we see that $S\mu^* \subset A$, $G\mu^*(x) = G\mu(x)$ on A, and $G\mu^*(x) \le G\mu(x)$ on X. We may take $\mu_A = \mu^*$. □

10. The q-Elliptic Measure of the Ideal Boundary of $\mathcal{N}$

We introduce the discrete version of q-elliptic measure in [11, Page 286]. Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$ and let ω_n be the unique solution of the following boundary problem.

$$\Delta_q u = 0 \quad \text{on } X_n \quad \text{and} \quad u = 1 \quad \text{on } X \setminus X_n.$$

Remark 10.1. The existence and uniqueness follows from the fact that our problem is reduced to a system of linear equations in a form: $A\mathbf{u} = \mathbf{b}$, where A is $m \times m$-matrix and $\mathbf{u}, \mathbf{b} \in \mathbb{R}^m$ with m the number of nodes in X_n. Our assertion follows from $\det A \neq 0$.

Another way to prove our assertion is to consider the extremum problem: $\beta_n = \inf\{E(u); u \in L(X), u = 1 \text{ on } X \setminus X_n\}$. We can show by a standard technique that there exists $u^* \in L(X)$ such that $u^* = 1$ on $X \setminus X_n$ and $\beta_n = E(u^*)$. By the variational technique used in the proof of Proposition 12.3 below, we see that u^* is the desired solution. In this case, the uniqueness follows from the minimum principle.

By the minimum principle, $0 \leq \omega_{n+1} \leq \omega_n \leq 1$ on X. The limit function ω of $\{\omega_n\}_n$ exists. It is easily seen that ω does not depend on the choice of an exhaustion of $\mathcal{N}$ and that ω is q-harmonic on X and $0 \leq \omega \leq 1$ on X. We call ω the *q-elliptic measure of the ideal boundary* of $\mathcal{N}$, shortly, *q-elliptic measure.*

Proposition 10.2. *Assume that u vanishes at the ideal boundary, i.e., for any $\varepsilon > 0$, there exists a finite subset X' of X such that $|u(x)| \leq \varepsilon$ on $X \setminus X'$. If u is q-harmonic on X, then $u = 0$.*

Proof. For any $\varepsilon > 0$, there exists a finite subset X' of X such that $|u(x)| \leq \varepsilon$ on $X \setminus X'$. Since both $\varepsilon \pm u$ are q-superharmonic and non-negative on $X \setminus X'$, the minimum principle shows that $\varepsilon \pm u \geq 0$ on X, i.e., $|u(x)| \leq \varepsilon$ on X. By the arbitrariness of ε, we have $u = 0$. □

Proposition 10.3. *If $c := \inf\{q(x)\,;\, x \in X\} > 0$, then* $\mathbf{HE} = \{0\}$.

Proof. Let $u \in \mathbf{HE}$. We have

$$c \sum_{x \in X} u(x)^2 \leq \|u\|^2 \leq E(u) < \infty,$$

so that u vanishes at the ideal boundary. Thus $u = 0$ by Proposition 10.2. □

Lemma 10.4. *Let $\{\mathcal{N}_n = \langle X_n, Y_n\rangle\}_n$ be an exhaustion of $\mathcal{N}$ and $g_a^{(n)}$ be the q-Green function of $\mathcal{N}_n$ with pole at $a \in X_n$. Then $\omega_n(x) = 1 - \sum_{z \in X_n} q(z) g_z^{(n)}(x)$.*

Proof. Let $u(x) = 1 - \sum_{z \in X_n} q(z) g_z^{(n)}(x)$. Then u is q-harmonic on X_n. In fact, for $x \in X_n$, we have

$$\Delta_q u(x) = \Delta_q 1(x) - \sum_{z \in X_n} q(z) \Delta_q g_z^{(n)}(x) = -q(x) - \sum_{z \in X_n} q(z)[-\varepsilon_z(x)] = 0.$$

Since $g_z^{(n)}(x) = 0$ for $x \in X \setminus X_n$ and $z \in X_n$, we have $u = 1$ on $X \setminus X_n$. Hence $u = \omega_n$. □

Letting $n \to \infty$ in this lemma, we obtain

Theorem 10.5. *Let ω be the q-elliptic measure of the ideal boundary. Then $\omega(x) = 1 - \sum_{z\in X} q(z)g_z(x)$.*

Corollary 10.6. *$Gq(x) = \sum_{z\in X} q(z)g_z(x) \le 1$ on X.*

Another proof of this fact was given without using the q-elliptic measure (cf. [17, Theorem 4.5]).

Lemma 10.7. *Let c be a positive constant. If u is q-superharmonic and $u(x) \ge -c$ on X, then $u(x) \ge -c\omega(x)$ on X. If u is q-harmonic and $|u(x)| \le c$ on X, then $|u(x)| \le c\omega(x)$ on X.*

Proof. Let $\{\omega_n\}_n$ be the determining sequence of ω. If u is q-superharmonic such that $u(x) \ge -c$ on X, then $u + c\omega_n$ is q-superharmonic on X_n and is non-negative on $X \setminus X_n$. The minimum principle implies $u + c\omega_n \ge 0$ on X. Therefore $u + c\omega \ge 0$ on X. If u is q-harmonic such that $|u(x)| \le c$ on X, then $u \ge -c$ and $-u \ge -c$. We have $u \ge -c\omega$ and $-u \ge -c\omega$, so that $|u(x)| \le c\omega(x)$ on X. □

Corollary 4.4, Theorem 10.5, and Lemma 10.7 imply

Theorem 10.8. *The following three properties are equivalent:*

(1) $\omega = 0$.
(2) $\mathbf{HB} = \{0\}$.
(3) $Gq(x) = 1$ *for some* $x \in X$.

Example 10.9. Let $\mathcal{G}$ be the same as in Example 5.6 and take $r(y_n) = 2^{-n}$ for $n \ge 1$ and $q(x_n) = 2^{n+1}$ for $n \ge 0$. Then $\mathcal{N}$ is hyperbolic and $\mathbf{HB} = \{0\}$.

Proof. Let $u \in \mathbf{H}$ and $u_n = u(x_n)$. The equation $\Delta_q u(x) = 0$ implies

$$\frac{u_1 - u_0}{2^{-1}} = 2u_0, \quad \frac{u_{n-1} - u_n}{2^{-n}} + \frac{u_{n+1} - u_n}{2^{-n-1}} = 2^{n+1}u_n \quad \text{for } n \ge 1,$$

or

$$u_1 = 2u_0, \quad 2u_{n+1} - 5u_n + u_{n-1} = 0 \quad \text{for } n \ge 1.$$

The general solution is $u_n = A\alpha^n + B\beta^n$ for $n \ge 0$ with $\alpha = (5 - \sqrt{17})/4$, $\beta = (5 + \sqrt{17})/4$. Note that $u_n = A\alpha^n$ does not satisfy $u_1 = 2u_0$ unless $A = 0$, which implies $\mathbf{HB} = \{0\}$. By the condition $u_1 = 2u_0$, we have $B = (7 - 3\alpha)A$, so that

$$u_n = A\alpha^n + (7 - 3\alpha)A\beta^n \quad \text{for } n \ge 0.$$

Now let $v_n = g_{x_0}(x_n)$. Then the equation $\Delta_q g_{x_0} = -\varepsilon_{x_0}$ implies

$$v_1 = 2v_0 - \frac{1}{2}, \quad 2v_{n+1} - 5v_n + v_{n-1} = 0 \quad \text{for } n \ge 1.$$

Since $\mathcal{N}$ is hyperbolic, Kayano and Yamasaki [4, Theorem 3.3] show that $v_n \to 0$ as $n \to \infty$, so that $v_n = A\alpha^n$. By the initial condition, we have $A = 1/(4 - 2\alpha)$, and hence

$$g_{x_0}(x_n) = \frac{\alpha^n}{4 - 2\alpha} \quad \text{for } n \ge 0.$$

We have

$$Gq(x_0) = \sum_{n=0}^{\infty} q(x_n) g_{x_0}(x_n) = \sum_{n=0}^{\infty} \frac{2^{n+1}\alpha^n}{4-2\alpha} = \frac{1}{(2-\alpha)(1-2\alpha)} = 1.$$

This also follows from Theorem 10.8. □

Proposition 10.10. *If $\mathcal{N}$ is hyperbolic and $q \in L_0^+(X)$, then $\omega \neq 0$.*

Proof. Suppose that $\omega = 0$. Then $Gq(x) = 1$ on X. Since Sq is a finite set, Kayano and Yamasaki [4, Theorem 3.3] show that there exists a sequence $\{x_n\}_n$ such that $g_z(x_n) \to 0$ as $n \to \infty$ for all $z \in Sq$, so that

$$1 = \lim_{n\to\infty} Gq(x_n) = \lim_{n\to\infty} \sum_{z\in Sq} g_z(x_n) q(z) = 0,$$

which is a contradiction. □

Proposition 10.11. *Assume that $q \in L_0^+(X)$ and $\omega \neq 0$. Then there exists a constant c with $0 < c < 1$ such that $\omega(x) \geq 1 - c$ on X.*

Proof. Let $c = \max\{Gq(x)\,;\, x \in Sq\}$. We have $Gq(x) < 1$ on X by Theorem 10.8. Since Sq is a finite set, it follows that $c < 1$. Namely $Gq(x) \leq c$ on Sq. We have $Gq(x) \leq c$ on X by Theorem 9.4. Theorem 10.5 shows that $\omega(x) = 1 - Gq(x) \geq 1 - c$ on X. □

Corollary 10.12. *Assume that $q \in L_0^+(X)$ and $\omega \neq 0$. Then there exists a constant c with $0 < c < 1$ such that $(1-c)G1(x) \leq G\omega(x) \leq G1(x)$ on X.*

11. The Case Where $\mathcal{N}$ is Parabolic

In this section, we consider the case where $\mathcal{N}$ is parabolic, i.e., $\mathbf{E} = \mathbf{E}_0$. We have

Proposition 11.1. *Assume that $\mathcal{N}$ is parabolic. Then $Gq(x) = 1$ on X.*

Proof. By [14, Theorem 3.2], we have $1 \in \mathbf{D}_0$, so that there exists a sequence $\{f_n\}_n$ in $L_0(X)$ such that $0 \leq f_n(x) \leq 1$ on X, $D(1 - f_n) \to 0$ as $n \to \infty$, and $\{f_n\}_n$ converges pointwise to 1. Let $a \in X$. Since $\Delta_q g_a(x) = -\varepsilon_a(x)$, we have $q(x)g_a(x) = \Delta g_a(x) + \varepsilon_a(x)$ and

$$\sum_{z\in X} f_n(z)[q(z)g_a(z)] = \sum_{z\in X} f_n(z)[\Delta g_a(z) + \varepsilon_a(z)] = -D(f_n, g_a) + f_n(a).$$

Since $D(f_n) = D(1 - f_n)$, we have

$$\lim_{n\to\infty} |D(f_n, g_a)| \leq \lim_{n\to\infty} D(f_n)^{1/2} D(g_a)^{1/2} = 0.$$

Since $Gq(a) \leq 1$ by Theorem 10.5, we have by Lebesgue's dominated convergence theorem

$$Gq(a) = \sum_{z\in X} q(z) g_a(z) = \lim_{n\to\infty} \sum_{z\in X} f_n(z)[q(z)g_a(z)]$$
$$= \lim_{n\to\infty} [-D(f_n, k_a) + f_n(a)] = 1$$

By Theorem 10.5 and the minimum principle, we see that $Gq = 1$ on X. □

By Theorems 10.8 and Proposition 11.1, we have

Theorem 11.2. *Assume that $\mathcal{N}$ is parabolic. Then* $\mathbf{HB} = \{0\}$.

We show the effect of the condition $Gq = 1$ by examples.

Example 11.3. Let $\mathcal{G}$ be the linear graph as in Example 5.6, $q = \varepsilon_{x_0} + \varepsilon_{x_1} + \varepsilon_{x_2}$, and $r(y) = 1$ on Y. Then $\mathcal{N}$ is parabolic (cf. [15, Example 3.1]) and g_{x_0} is given by

$$g_{x_0}(x_0) = \frac{5}{8}, \quad g_{x_0}(x_1) = \frac{2}{8}, \quad g_{x_0}(x_n) = \frac{1}{8} \quad \text{for } n \geq 2.$$

The class $\mathbf{H}^+$ consists of $h \in L(X)$ defined by

$$h(x_0) = t > 0, \quad h(x_1) = 2t, \quad h(x_n) = (8n - 11)t \quad \text{for } n \geq 2.$$

Proof. Let $h \in \mathbf{H}^+$ and $h_n = h(x_n)$. Then

$$h_1 - 2h_0 = 0, \quad h_2 + h_0 - 3h_1 = 0,$$
$$h_3 + h_1 - 3h_2 = 0, \quad h_{n+1} - 2h_n + h_{n-1} = 0 \quad \text{for } n \geq 3,$$

which implies

$$h(x_0) = t > 0, \quad h(x_1) = 2t, \quad h(x_n) = (8n - 11)t \quad \text{for } n \geq 2.$$

This means $\mathbf{HB} = \{0\}$. Let $u_n = g_{x_0}(x_n)$. The equation $\Delta_q g_{x_0} = -\varepsilon_{x_0}$ implies

$$u_1 - 2u_0 = -1, \quad u_2 + u_0 - 3u_1 = 0,$$
$$u_3 + u_1 - 3u_2 = 0, \quad u_{n+1} - 2u_n + u_{n-1} = 0 \quad \text{for } n \geq 3.$$

Proposition 11.1 implies $Gq(x_0) = 1$, which means

$$u_0 + u_1 + u_2 = 1.$$

These equations lead to $u_0 = 5/8$, $u_1 = 2/8$, $u_n = 1/8$ for $n \geq 2$. □

Example 11.4. Let $\mathcal{G}$ be the linear graph and let $q(x) = 1$ on X and $r(y) = 1$ on Y. Then $\mathcal{N}$ is parabolic (cf. [15, Example 3.1]) and

$$g_{x_0}(x_n) = \frac{\alpha^n}{2 - \alpha} \quad \text{for } n \geq 0, \quad \alpha = \frac{3 - \sqrt{5}}{2}.$$

Proof. Let $h \in \mathbf{H}^+$ and $h_n = h(x_n)$. The equation $\Delta_q h(x) = 0$ implies $h_1 = 2h_0$ and $h_{n+1} - 3h_n + h_{n-1} = 0$ for $n \geq 1$. The general solution is $u_n = A\alpha^n + B\beta^n$ for $n \geq 0$, where $\alpha = (3 - \sqrt{5})/2$ and $\beta = (3 + \sqrt{5})/2$ are solutions of the characteristic equation $t^2 - 3t + 1 = 0$. The initial condition shows $B = (3 - \alpha)A$ and $\mathbf{H} = \{A\alpha^n + (3 - \alpha)A\beta^n\,;\, n \geq 0\}$. This means $\mathbf{HB} = \{0\}$. Let $u_n = g_{x_0}(x_n)$. The equation $\Delta_q g_{x_0} = -\varepsilon_{x_0}$ implies

$$u_1 - 2u_0 = -1, \quad u_{n+1} - 3u_n + u_{n-1} = 0 \quad \text{for } n \geq 1.$$

Since $Gq(x) = 1$, we obtain $u_n = A\alpha^n$. By the condition $u_1 = 2u_0 - 1$, we have $A = 1/(2 - \alpha)$. □

12. Classification of Infinite Networks

Recall $\mathbf{HE}^+ = \mathbf{HE} \cap L^+(X)$ and let

$$\mathbf{HP} = \mathbf{H}^+ - \mathbf{H}^+ = \{h = h_1 - h_2\,;\, h_1, h_2 \in \mathbf{H}^+\},$$
$$\mathbf{HEP} = \mathbf{HE}^+ - \mathbf{HE}^+.$$

For a class C of $L(X)$, denote by O_C the collection of those infinite networks $\mathcal{N}$ for which C consists only of 0. Since $\mathbf{HP} \subset \mathbf{H}$, we have $O_H \subset O_{HP}$.

Proposition 12.1. $\mathbf{HB} \subset \mathbf{HP}$.

Proof. Let $u \in \mathbf{HB}$. Then there exists a constant such that $|u(x)| \leq c$ on X. By Lemma 10.7, $|u(x)| \leq c\omega(x)$ on X. Let $u_1 = (c\omega + u)/2$ and $u_2 = (c\omega - u)/2$. By Lemma 4.1, u_1 and u_2 are non-negative and q-harmonic and $u_1 - u_2 = u$. □

Corollary 12.2. $O_{HP} \subset O_{HB}$.

Clearly $\mathbf{HEB} \subset \mathbf{HB}$, so that $O_{HB} \subset O_{HEB}$.

Proposition 12.3. $\mathbf{HE} = \mathbf{HEP} = \{u_1 - u_2\,;\, u_1, u_2 \in \mathbf{HE}^+\}$.

Proof. Since $\mathbf{HEP} \subset \mathbf{HE}$ is clear, we prove the converse inclusion. Let $u \in \mathbf{HE}$ and $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$. For our purpose, we may assume that both u^+ and u^- are non-zero. Let $\{\mathcal{N}_n = \langle X_n, Y_n \rangle\}_n$ be an exhaustion of $\mathcal{N}$ and consider the following extremum problems:

$$\alpha_n = \inf\{E(v)\,;\, v \in \mathbf{E}, v = u^+ \text{ on } X \setminus X_n\}.$$

Note that $\alpha_n \leq E(u^+) \leq E(u)$ by Corollary 3.10. By the same reasoning as in the proof of Theorem 6.1, we see that there exists a unique solution v_n^* such that $\alpha_n = E(v_n^*)$. Let $f \in L(X)$ satisfy $f = 0$ on $X \setminus X_n$. Since $v_n^* + tf (\in \mathbf{E})$ is equal to u^+ on $X \setminus X_n$ for any real number t, we have

$$E(v_n^*) \leq E(v_n^* + tf) = E(v^*) + 2tE(v_n^*, f) + t^2 E(f).$$

Letting $t \nearrow 0$ and $t \searrow 0$, we obtain $E(v_n^*, f) = 0$. For any $x \in X \setminus X_n$, Lemma 3.3 shows

$$0 = E(v_n^*, \varepsilon_x) = -\Delta_q v_n^*(x),$$

namely v_n^* is q-harmonic on X_n. Note that $-u^+$ is q-superharmonic on X by Lemma 4.1. Since $v_n^* - u^+$ is q-superharmonic on X_n and vanishes on $X \setminus X_n$, we have $v_n^* - u^+ \geq 0$ on X by the minimum principle. From $v_{n+1}^* \geq u^+$ on X and $v_n^* = u^+$ on $X \setminus X_n$, we see that $v_{n+1}^* - v_n^* \geq 0$ on $X \setminus X_n$. Since $v_{n+1}^* - v_n^*$ is q-harmonic on X_n, we obtain by the minimum principle $v_{n+1}^* \geq v_n^*$ on X. Lemma 3.1 implies that, for each $x \in X$, there exists $M_x > 0$ such that $u^+(x) \leq v_n^*(x) \leq M_x E(v_n^*)^{1/2} \leq M_x E(u)^{1/2}$. Therefore, the sequence $\{v_n^*\}_n$ converges pointwise to $v^* \in L^+(X)$. Then v^* is q-harmonic on X and $v^* \geq u^+$. Note that

$$E(v^*) \leq \liminf_{n\to\infty} E(v_n^*) \leq E(u) < \infty,$$

so that $v^* \in \mathbf{HE}^+$. Theorem 8.5 shows that $v^* - u^+ = G\mu_1 + h_1$ with $\mu_1 \in \mathcal{E}$ and $h_1 \in \mathbf{HE}^+$. Similarly we find $w^* \in \mathbf{HE}^+$, $\mu_2 \in \mathcal{E}$, and $h_2 \in \mathbf{HE}^+$ such that $w^* - u^- = G\mu_2 + h_2$. Let $\varphi = v^* - w^* \in \mathbf{HE}$. Then

$$0 = \Delta_q(\varphi - u) = \Delta_q G(\mu_1 - \mu_2) = -\mu_1 + \mu_2.$$

Let $u_1 = v^* + h_2$ and $u_2 = w^* + h_1$. Then $u_1, u_2 \in \mathbf{HE}^+$ and

$$u = \varphi - (h_1 - h_2) = u_1 - u_2.$$

This completes the proof. □

Next theorem gives a sufficient condition for $\mathbf{H}^+ \neq \{0\}$.

Theorem 12.4. *If $\mathcal{N}$ is hyperbolic and $\sum_{x \in X} q(x) < \infty$, then $\mathbf{H}^+ \neq \{0\}$.*

Proof. If $\mathbf{H}^+ = \{0\}$, then $\mathbf{HE}^+ = \{0\}$, so that $\mathbf{HEP} = \{0\}$. Hence $\mathbf{HE} = \{0\}$ by Proposition 12.3. This contradicts Theorem 3.8. □

Proposition 12.5. *For every $u \in \mathbf{HE}$, there exists a sequence $\{h_n\}_n$ in $\mathbf{HEB}$ such that $E(u - h_n) \to 0$ as $n \to \infty$.*

Proof. Let $u \in \mathbf{HE}$ and $u \geq 0$ and let $u_n(x) = \min(u(x), n)$. Then $u_n \in \mathbf{E}$ is non-negative and q-superharmonic. Theorem 8.5 shows that $u_n = G\mu_n + h_n$ with $\mu_n \in \mathcal{E}$ and $h_n \in \mathbf{HE}^+$. We have $0 \leq h_n \leq u_n \leq n$ and $h_n \in \mathbf{HEB}$. Lemma 3.3 shows $E(u - h_n, G\mu_n) = 0$, which leads to $E(u - u_n) = E(G\mu_n) + E(u - h_n) \geq E(u - h_n)$. Note that $D(u - u_n) \to 0$ as $n \to \infty$ by [14, Lemma 3.1]. Since $\|u_n\| \leq \|u\|$ and $\{u_n\}_n$ converges pointwise to u, we see that $\{\langle u_n, v\rangle\}_n$ converges to $\langle u, v\rangle$ for every $v \in \mathbf{E}$. Furthermore, we have $\|u_n\|^2 \to \|u\|^2$ as $n \to \infty$, and that $\|u - u_n\|^2 \to 0$ as $n \to \infty$. Thus $E(u - u_n) \to 0$ as $n \to \infty$, which shows $E(u - h_n) \to 0$ as $n \to \infty$.

Now we consider the case where $u \in \mathbf{HE}$ is of any sign. By Proposition 12.3, there exist $u', u'' \in \mathbf{HE}^+$ such that $u = u' - u''$. By the above observation, we can find sequences $\{h'_n\}$ and $\{h''_n\}$ in $\mathbf{HEB}$ such that $E(u' - h'_n) \to 0$ and $E(u'' - h''_n) \to 0$ as $n \to \infty$. Let $h_n = h'_n - h''_n$. Then $h_n \in \mathbf{HEB}$ and $E(u - h_n) \to 0$ as $n \to \infty$. □

Corollary 12.6. *$O_{HE} = O_{HEB}$.*

Thus we have the following classification of infinite networks by the classes of q-harmonic functions:

Theorem 12.7. *$O_H \subset O_{HP} \subset O_{HB} \subset O_{HEB} = O_{HE}$.*

Note that $\mathbf{HD} = \mathbf{HE}$ if $q \in L_0^+(X)$.

13. q-Quasiharmonic Classification

We say that a function $u \in L(X)$ is *q-quasiharmonic* on X if $\Delta_q u = c\omega$ on X, where ω is the q-elliptic measure and c is a constant. Denote by $\mathbf{Q}$ the set of q-quasiharmonic functions on X normalized by $\Delta_q u = -\omega$. In this section, we always assume that $\omega \neq 0$. We consider the following classes of q-quasiharmonic functions:

$$\mathbf{QB} = \{u \in \mathbf{Q}\,;\, \sup\{|u(x)|\,;\, x \in X\} < \infty\},$$

$$\mathbf{QE} = \mathbf{Q} \cap \mathbf{E}, \quad \mathbf{Q}^+ = \mathbf{Q} \cap L^+(X).$$

In addition to $\mathcal{M}$ and $\mathcal{E}$, we introduce

$$\mathcal{M}_b = \{\mu \in \mathcal{M}\,;\, \sup\{G\mu(x)\,;\, x \in X\} < \infty\}.$$

Theorem 13.1. *Assume $\omega \neq 0$. The classes O_C for $C = \mathbf{Q}^+, \mathbf{QB}, \mathbf{QE}$ are characterized as follows:*

(1) *$\mathcal{N} \in O_{Q^+}$ if and only if $\omega \notin \mathcal{M}$;*
(2) *$\mathcal{N} \in O_{QB}$ if and only if $\omega \notin \mathcal{M}_b$;*
(3) *$\mathcal{N} \in O_{QE}$ if and only if $\omega \notin \mathcal{E}$.*

Proof. Let $u = G\omega$. If $\omega \in \mathcal{M}$, then $\Delta_q u = -\omega$ on X and $u > 0$, and hence $u \in \mathbf{Q}^+$. If $\omega \in \mathcal{M}_b$, then $u \in \mathbf{QB}$. If $\omega \in \mathcal{E}$, then $u \in \mathbf{QE}$ by Theorem 8.4. Thus the only-if parts in (1)–(3) are proved.

(1) Assume that $\mathcal{N} \notin O_{Q^+}$ and let $u \in \mathbf{Q}^+$. Since u is non-negative and q-superharmonic, we see by Riesz's decomposition that there exist $\mu \in \mathcal{M}$ and $h \in \mathbf{H}^+$ such that $u = G\mu + h$ and $\mu = -\Delta_q u = \omega$. Thus $\omega \in \mathcal{M}$.

(2) Assume that $\mathcal{N} \notin O_{QB}$ and $u \in \mathbf{QB}$. Then there exists a positive constant c such that $|u(x)| \leq c$ on X. Lemma 10.7 shows that $u + c\omega$ is non-negative and q-superharmonic. Riesz's decomposition shows that there exist $\mu \in \mathcal{M}$ and $h \in \mathbf{H}^+$ such that $u + c\omega = G\mu + h$ and $\mu = -\Delta_q(u + c\omega) = \omega$. Thus $G\omega \leq u + c\omega \leq 2c$ on X and $\omega \in \mathcal{M}_b$.

(3) Assume that $\mathcal{N} \notin O_{QE}$ and $u \in \mathbf{QE}$. Royden's decomposition implies that there exist $v \in \mathbf{E}_0$ and $h \in \mathbf{HE}$ such that $u = v + h$. Since $\Delta_q v = \Delta_q u = -\omega$, Theorem 8.4 shows that there exists $\mu \in \mathcal{E}$ such that $v = G\mu$. We obtain $\omega = -\Delta_q G\mu = \mu \in \mathcal{E}$. □

This theorem implies

Proposition 13.2. *If $\omega \neq 0$, then $O_{Q^+} \subset O_{QB}$.*

We have by Proposition 10.11 and Corollary 10.12

Lemma 13.3. *Assume that $q \in L_0^+(X)$ and $\omega \neq 0$. Then*

(1) *$\omega \in \mathcal{M}$ if and only if $1 \in \mathcal{M}$;*
(2) *$\omega \in \mathcal{M}_b$ if and only if $1 \in \mathcal{M}_b$;*
(3) *$\omega \in \mathcal{E}$ if and only if $1 \in \mathcal{E}$.*

We show by an example that there exists $\mathcal{N} \notin O_{Q^+}$ such that $\mathcal{N} \in O_{QB}$.

Example 13.4. Let $\mathcal{G}$ be the ladder as in [16, Example 4.3]. Namely $X = \{x_n, x'_n\,;\, n \geq 0\}$, $Y_n = \{y_n, y'_n, y''_n\,;\, n \geq 1\} \cup \{y''_0\}$ and $K(x, y)$ is defined by

$$K(x_n, y_{n+1}) = K(x'_n, y'_{n+1}) = K(x_n, y''_n) = -1,$$
$$K(x_{n+1}, y_{n+1}) = K(x'_{n+1}, y'_{n+1}) = K(x'_n, y''_n) = 1$$

for $n \geq 0$ and $K(x, y) = 0$ for any other pair. Let $q(x) = \varepsilon_{x_0}(x)$ and α_0 a constant with $0 < \alpha_0 < 1$. We choose $r(y)$ as follows:

$$r_n = 1, \quad r'_n = \frac{2^{-n-1}\alpha_0}{2n + 1 - \alpha_0},$$

$$r_0'' = \frac{\alpha_0}{2(2-\alpha_0)}, \quad r_n'' = (1-2^{-n-1})\alpha_0 + n$$

for $n \geq 1$, where $r_n = r(y_n)$, $r_n' = r(y_n')$, and $r_n'' = r(y_n'')$. This network is in $O_{QB} \setminus O_{Q^+}$.

Proof. Let us consider the function $u \in L(X)$ defined by

$$u_n = \alpha_0 + n, \quad u_n' = 2^{-n-1}\alpha_0 \quad \text{for } n \geq 0,$$

where $u_n = u(x_n)$ and $u_n' = u(x_n')$. We show $\Delta_q u = -1$. We compute

$$\begin{aligned}
du(y_n) &= -\frac{K(x_{n-1}, y_n)u(x_{n-1}) + K(x_n, y_n)u(x_n)}{r(y_n)} = \frac{u_{n-1} - u_n}{r_n} = -1, \\
du(y_n') &= -\frac{K(x_{n-1}', y_n')u(x_{n-1}') + K(x_n', y_n')u(x_n')}{r(y_n')} = \frac{u_{n-1}' - u_n'}{r_n'} = 2n + 1 - \alpha_0, \\
du(y_0'') &= -\frac{K(x_0, y_0'')u(x_0) + K(x_0', y_0'')u(x_0')}{r(y_0'')} = \frac{u_0 - u_0'}{r_0''} = 2 - \alpha_0, \\
du(y_n'') &= -\frac{K(x_n, y_n'')u(x_n) + K(x_n', y_n'')u(x_n')}{r(y_n'')} = \frac{u_n - u_n'}{r_n''} = 1
\end{aligned}$$

for $n \geq 1$. We have

$$\begin{aligned}
\Delta_q u(x_0) &= K(x_0, y_1)du(y_1) + K(x_0, y_0'')du(y_0'') - u(x_0) \\
&= -du(y_1) - du(y_0'') - u_0 = -1, \\
\Delta_q u(x_0') &= K(x_0', y_1')du(y_1') + K(x_0', y_0'')du(y_0'') \\
&= -du(y_1') + du(y_0'') = -1, \\
\Delta_q u(x_n) &= K(x_n, y_n)du(y_n) + K(x_n, y_{n+1})du(y_{n+1}) + K(x_n, y_n'')du(y_n'') \\
&= du(y_n) - du(y_{n+1}) - du(y_n'') = -1, \\
\Delta_q u(x_n') &= K(x_n', y_n')du(y_n') + K(x_n', y_{n+1}')du(y_{n+1}') + K(x_n', y_n'')du(y_n'') \\
&= du(y_n') - du(y_{n+1}') + du(y_n'') = -1.
\end{aligned}$$

By Riesz's decomposition, we have $u = G\mu + h$ with $\mu \in \mathcal{M}$ and $h \in \mathbf{H}^+$. Note that $1 = -\Delta_q u = \mu \in \mathcal{M}$. Also note that $\mathcal{N}$ is hyperbolic because of $\sum_n r_n' < \infty$ and [14, Theorem 4.1 and Lemma 4.3]. Proposition 10.10 shows $\omega \neq 0$.

To show $\mathcal{N} \in O_{QB} \setminus O_{Q^+}$, it suffices to show that $1 \in \mathcal{M} \setminus \mathcal{M}_b$ by Theorem 13.3 and Lemma 13.3. We show that $v := G1$ is unbounded. Suppose that v is bounded, i.e., there exists a positive constant c such that $|v(x)| \leq c$ on X. Let $v_n = v(x_n)$, $v_n' = v(x_n')$, $w_n = dv(y_n)$, $w_n' = dv(y_n')$, and $w_n'' = dv(y_n'')$. Then $|w_n| \leq 2c$ for all n. For any ε with $0 < \varepsilon < 1$, there exists n_0 such that $r_n'' \geq 2c/\varepsilon$ for all $n \geq n_0$, so that

$$|w_n''| = \frac{1}{r_n''}|v_n' - v_n| \leq \varepsilon.$$

Since $1 \in \mathcal{M}$, Lemma 7.1 shows $\Delta_q v = -1$, which implies $w_n - w_{n+1} - w_n'' = -1$, and that

$$-1 - \varepsilon \leq w_n - w_{n+1} \leq -1 + \varepsilon$$

for all $n \geq n_0$. This contradicts the boundedness of $\{w_n\}_n$. Thus v is unbounded. □

Proposition 13.5. *If $q \in L_0^+(X)$ and $\omega \neq 0$, then $O_{QB} \subset O_{QE}$.*

Proof. Assume $\mathcal{N} \notin O_{QE}$. Theorem 13.1 shows $\omega \in \mathcal{E}$. There exists a constant c with $0 < c < 1$ such that $\omega(x) \geq 1 - c$ on X by Proposition 10.11, so that

$$G(\omega, \omega) = \sum_{z \in X} G\omega(z)\omega(z) \geq (1-c) \sum_{z \in X} G\omega(z) \geq (1-c)G\omega(x)$$

for each $x \in X$. This means $\omega \in \mathcal{M}_b$. □

Example 13.6. Let $\mathcal{G}$ and q be the same as in Example 5.6. Define $r(y)$ by $r(y_n) = n^{-2} - (n+1)^{-2}$ for $n \geq 1$. Then $\omega \in \mathcal{M}_b$ and $\omega \notin \mathcal{E}$. Equivalently $\mathbf{QB} \neq \{0\}$ and $\mathbf{QE} = \{0\}$.

Proof. Let R_n and ρ_n be defined as in Example 5.6. Then

$$R_0 = 1, \quad R_n = \frac{1}{(n+1)^2}, \quad \rho_n = 1 - \frac{1}{(n+1)^2} < 1.$$

We have by Theorem 10.5

$$\omega(x_n) = 1 - g_{x_0}(x_n) = 1 - \frac{(1+\rho_0)R_n}{1+R_0} = \frac{1+\rho_n}{1+R_0} \quad \text{for } n \geq 0.$$

We obtain

$$\begin{aligned} G\omega(x_0) &= \frac{1}{(1+R_0)^2} \sum_{n=0}^{\infty} R_n(1+\rho_n) \leq \frac{1}{2} \sum_{n=0}^{\infty} R_n < \infty, \\ G\omega(x_m) &= \sum_{n=0}^{\infty} g_{x_m}(x_n)\omega(x_n) \\ &= \frac{1}{(1+R_0)^2} \left[R_m \sum_{n=0}^{m} (1+\rho_n)^2 + (1+\rho_m) \sum_{n=m+1}^{\infty} R_n(1+\rho_n) \right] \\ &\leq (m+1)R_m + \sum_{n=m+1}^{\infty} R_n \leq \frac{1}{m+1} + \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

so that $G\omega$ is bounded and $\omega \in \mathcal{M}_b$.

We have $G(\omega, \omega) = S_1 + S_2$, where

$$\begin{aligned} S_1 &= \frac{1}{(1+R_0)^3} \sum_{m=0}^{\infty} R_m(1+\rho_m) \sum_{n=0}^{m} (1+\rho_n)^2, \\ S_2 &= \frac{1}{(1+R_0)^3} \sum_{m=0}^{\infty} (1+\rho_m)^2 \sum_{n=m+1}^{\infty} R_n(1+\rho_n). \end{aligned}$$

Therefore

$$c_m := \sum_{n=m+1}^{\infty} R_n(1+\rho_n) \geq \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^2} \geq \int_{m+2}^{\infty} \frac{1}{t^2}\, dt = \frac{1}{m+2},$$

thus

$$G(\omega,\omega) \geq S_2 \geq \frac{1}{8}\sum_{m=0}^{\infty} c_m = \infty. \qquad \square$$

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