Mem. Fac. Sci., Shimane Univ.,
18, pp. 37–44 Dec. 20, 1984

The Construction of Dual Amplitude from the Duality Scheme with Quark-Model Hadron Spectrum and the General Veneziano-type Amplitude

Taketoshi INO
Department of Physics, Shimane University, Matsue 690
(Received September 3, 1984)

The way to construct $0^- - 0^-$ dual amplitudes from the duality scheme in Ref. 1) and the most general Veneziano-type amplitudes is discussed generally. A new condition to restrict terms for the unitary-symmetry breaking is proposed in accordance with the required symmetric treatment of the two dual channels.

§1. Introduction

Although the Veneziano-type models have no unique way to specify their pole structures, the combination of them and the duality relation based on the quark-model hadron spectrum proposed by the author[1)] is promising to construct a satisfactory Born term of the scattering amplitude. In fact, by the duality relation and a condition at $s\to\infty$, Munakata, Sakamoto and the author[2)] have been able to determine an infinite number of coefficients of the most general $\pi^- - \pi^+$ Veneziano-type amplitude except for a overall multiplying factor, and have found that the obtained amplitude has good properties; absence of negative-norm state, compatibility with available experiments of 2π decay, and the consistency with the PCAC hypothesis in the limit of $m_\pi\to 0$. It is also found that this amplitude takes a simple form when all the spin (and unitary-spin) dependent forces between quarks and anti-quarks are neglected.

The purpose of the present paper is to show that the method employed in Ref. 2) may be applicable to constract dual amplitudes for other $0^- - 0^-$ processes where terms for the unitary-symmetry breaking are needed, differently from the $\pi^- - \pi^+$ scattering.

Let us consider the $K^-K^0\to K^-K^0$ process, where the u-channel is exotic. The duality scheme (system of duality relations)[1)] with the harmonic-oscillator spectrum of $SU(6)\otimes O(3)_L$ multiplets is

$$\sum_{a\in N} R_a^{(s)}(t)|_{t=m_{N'}^2} = \sum_{b\in N'} R_b^{(t)}(s)|_{s=m_N^2},\ N,\ N'=0,\ 1,\ 2,\ldots, \tag{1.1}$$

where $N(N')$ is the s-(t-) channel resonance family. $R_a^{(s)}(t)$ ($R_b^{(t)}(s)$) is the residue of the scattering amplitude at the s-(t-) channel resonance $a(b)$ in the narrow-width

Table I. Resonance families and their members in the $s-(t-)$ channel of the $K^- - K^0$ scattering. The double circles imply that there are two states with different quark-orbital angular momenta at their places.

Family J^P	0	1	2	3	4
5^-					○ ( )
4^+				○ ( )	
3^-			○ $g(\phi(1850))$		◎ ( )
2^+		○ $A_2(f')$		◎ ( )	
1^-	○ $\rho(\phi)$		ρ', ◎ ρ''( )		◎ ( )
0^+		○ δ( )		○ ( )	

approximation. The s- and t-channel resonance families and their members are shown in Table I.

Here, the oscillator model is adopted for the squared mass, and mass-splitting due to spin-orbit and tensor interactions are neglected. And the ideal mixing and the OZI-decoupling rule are also assumed. The resonances have a degenerate mass for each family, and there is no odd daughter in Table I because J takes alternate values for a fixed N in the harmonic oscillator model.

Now, we start with the most general Veneziano-type amplitude which has no ancester,

$$F(s,\,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=\max(m,n)}^{m+n}\lambda_k^{m,n}\frac{\Gamma(m-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}\,, \tag{1.2}$$

where $\lambda_k^{m,n}$ are arbitrary constants and

$$\alpha_s\equiv\alpha_{\rho-A_2}(s)=\alpha' s+\alpha_0,\quad \tilde{\alpha}_t\equiv\alpha_{\Phi-f'}(t)=\alpha' t+\tilde{\alpha}_0. \tag{1.3}$$

Here the universal slope is assumed for simplicity.

In §2, the coefficients $\lambda_k^{m,n}$ of (1.2) is restricted by the duality scheme (1.1) and a new condition proposed here in order to restrict terms for the symmetry breaking. Solutionsof them are also searched. The section 3 is devoted to the summary and the discussion of the condition at $s\to\infty$.

§2. Solutions of the scheme (1.1)

It is convenient to divide the constraints of the scheme (1.1) into two conditions as

$$\begin{cases}\{\lim_{\alpha_s\to J}(J-\alpha_s)F(s,\,t)\}|_{\tilde{\alpha}_t=J'}=\{\lim_{\tilde{\alpha}_t\to J'}(J'-\tilde{\alpha}_t)F(s,\,t)\}|_{\alpha_s=J}, & (2.1\text{a})\\ (\text{contribution of each odd daughter})=0. & (2.1\text{b})\end{cases}$$

Here, $J(J')$ denotes the spin of the parent belongint to the s-(t-) channel family $N(N')$. It is noted that the linearity between s and α_s (t and $\tilde{\alpha}_t$) is used to rewrite (1.1) as (2.1a, b).

In the subsection 2.1, particular solutions of (2.1a) are searched, considering the linear property of the condition (2.1a) and regarding the most general Veneziano-type amplitude (1.2) as the sum of

$$F^{m,n}(s,t)\equiv\sum_{k=\max(m,n)}^{m+n}\lambda_k^{m,n}\frac{\Gamma(m-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}\,. \tag{2.2}$$

In the subsection, we pay attention also to the symmetric treatment of the two dual channels, and find a sum of particular solutions which has just the degree of freedom to provide an arbitrary residue at each of parents and their daughters, including odd daughters. In the subsection 2.2, we try to determine remaining coefficients $\lambda_k^{m,n}$ by the condition (2.1b). Resultant solutions are found to be significant only in some approximations. In the subsection 2.3, we give an amplitude which explicitly satisfies the conditions (2.1a, b) in the first-order approximation of the symmetry breaking.

2.1 *Solutions under the condition* (2.1a)

First, it is examined that which of $F^{m,n}(s,t)$ (m, $n=1, 2,\ldots$) of (2.2) satisfy the condition (2.1a) non-trivially. In Table II, non-trivial solutions are listed.

Table II. The series of $F^{m,n}(s,t)$'s which satisfy the condition (2.1a) non-trivially.

$F^{1,1}$	$F^{1,2}$					
$F^{2,1}$	$F^{2,2}$	$F^{2,3}$				
	$F^{3,2}$	$F^{3,3}$	$F^{3,4}$			
		$F^{4,3}$	$F^{4,4}$	$F^{4,5}$		
			$F^{5,4}$	$F^{5,5}$	$F^{5,6}$	
				$F^{6,5}$	$F^{6,6}$	$F^{6,7}$
				.	.	.

Other $F^{m,n}(s,t)$ satisfy (2.1a) when and only when all the parameters $\lambda_k^{m,n}$ in them are zero,

$$\lambda_{m+n}^{m,n}=\lambda_{m+n-1}^{m,n}=\cdots=\lambda_{\max(m,n)}^{m,n}=0. \tag{2.3}$$

It is noted that some combination of $F^{1,3}(s,t)$ and $F^{3,1}(s,t)$ and etc. are also non-trivial solutions of (2.1a). The roles of such combinations are discussed later in the subsection 2.3.

The sum of the particualr solutions in Table II is

$$F(s,t)=\sum_{n=1}^{\infty}\sum_{k=n}^{2n}\lambda_k^{n,n}\frac{\Gamma(n-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}

$$+\sum_{n=1}^{\infty}\sum_{k=n+1}^{2n+1}\left\{\lambda_k^{n,n+1}\frac{\Gamma(n-\alpha_s)\,\Gamma(n+1-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}\right.$$

$$\left.+\lambda_k^{n+1,n}\frac{\Gamma(n+1-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}\right\}. \tag{2.4}$$

The first term of (2.4) is just the amplitude which is taken in Ref. 2). (This term is the most general Veneziano-type amplitude for the $\pi^- - \pi^+$ scattering, when $\tilde{\alpha}_0=\alpha_0$.) It is clear that the second term of (2.4) should be used to express the symmetry breaking. Therefore $\lambda_k^{n,n+1}$ and $\lambda_k^{n+1,n}$ are functions of the difference between α_0 and $\tilde{\alpha}_0$ and are odd functions of the difference

$$\lambda_k^{n,n+1}=\lambda_k^{n,n+1}(\varDelta)=-\lambda_k^{n,n+1}(-\varDelta),\quad \lambda_k^{n+1,n}=\lambda_k^{n+1,n}(\varDelta)=-\lambda_k^{n+1,n}(-\varDelta), \tag{2.5}$$

where

$$\varDelta\equiv\alpha_0-\tilde{\alpha}_0. \tag{2.6}$$

The two dual channels have to be treated symmetrically. We propose a way for such symmetrical treatment here. We require that the amplitude (2.4) satisfy the new condition

$$F(\alpha_s,\,\alpha_0;\,\tilde{\alpha}_t,\,\tilde{\alpha}_0)=F(\tilde{\alpha}_t,\,\tilde{\alpha}_0;\,\alpha_s,\,\alpha_0). \tag{2.7}$$

Because of the condition for the symmetric treatment, (2.7), $\lambda_k^{n,n+1}(\varDelta)$ and $\lambda_k^{n+1,n}(\varDelta)$ have to satisfy

$$\lambda_k^{n+1,n}(-\varDelta)=\lambda_k^{n,n+1}(\varDelta). \tag{2.8}$$

In concluding this subsection, we state that the amplitude (2.4) with the requirement (2.8) has just the degree of freedom to provide an arbitrary residue at each of the s- and t-channel parents and their daughters, including odd daughters. In the physical sense, this amplitude is a candidate which we should take as the starting point. The explicit form of (2.4), obtained after the removal of vanishing elements under the condition (2.1a), is

$$F(s,\,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{n+1}\lambda_k^{n,n}\frac{\Gamma(n-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(k-\alpha_s-\tilde{\alpha}_t)}$$

$$+\sum_{n=1}^{\infty}\left\{\lambda_{n+1}^{n,n+1}\frac{\Gamma(n-\alpha_s)\,\Gamma(n+1-\tilde{\alpha}_t)}{\Gamma(n+1-\alpha_s-\tilde{\alpha}_t)}\right.$$

$$\left.+\lambda_{n+1}^{n+1,n}\frac{\Gamma(n+1-\alpha_s)\,\Gamma(n-\tilde{\alpha}_t)}{\Gamma(n+1-\alpha_s-\tilde{\alpha}_t)}\right\}. \tag{2.9}$$

In the Appendix, it is shown that (2.9) satisfies (2.1a). It is noted that for the $\pi^- - \pi^+$ scattering, we have been able to accommodate the first term to the condition (2.1b), that is, absence of odd daughters.[2] As will be seen later, the case of $K^-K^0\to K^-K^0$

scattering is also successful with (2.9) in certain approximations. But, in order to avoid odd daughters explicitly, we must prepare the study in the subsection 2.3 using the combinations of $F^{1,3}(s, t)$ and $F^{3,1}(s, t)$ and etc. as already stated.

2.2 *Approximate solutions under the conditions* (2.1a) *and* (2.1b)

We impose the condition (2.1b) that odd daughters do not contribute to the scattering amplitude. For this purpose we introduce the values of s, t for the s- and t-channels in their respective centers of mass.
s-channel:

$$s=4(q_s^2+\mu^2),\quad t=-2q_s^2(1-\cos\theta_s),\quad s+t+u=4\mu^2. \tag{2.10}$$

t-channel:

$$t=4(q_t^2+\mu^2),\quad s=-2q_t^2(1-\cos\theta_t),\quad s+t+u=4\mu^2. \tag{2.10}$$

Here μ is the kaon mass. The condition of absence of odd daughter in the s-channel upon $R_J(x_s)$ ($x_s\equiv\cos\theta_s$), the residue of $F(s, t)$ at $\alpha_s=J$, is written as

$$R_J(-x_s)^J=(-1)^J R_J(x_s). \tag{2.12a}$$

Similarly, the condition in the t-channel upon $R_{J'}(x\equiv\cos\theta_t)$, the residue of $F(s, t)$ at $\tilde{\alpha}_t=J'$, is

$$R_{J'}(-x_t)=(-1)^{J'}R_{J'}(x_t). \tag{2.12b}$$

Starting with $J=1$, $J'=1$, we impose (2.12a, b) upon the amplitude (2.9) to determine $\lambda_n^{n,n}$, $\lambda_{n+1}^{n,n}$ and $\lambda_{n+1}^{n,n+1}$ ($\lambda_{n+1}^{n+1,n}$) successively with increasing J and J'. We find, in the first-order approximation of Δ,

$$\lambda_2^{1,1}=\frac{1-\beta_c}{2}\lambda_1^{1,1}+0\left(\left(\frac{\Delta}{2}\right)^2\right),$$

$$\lambda_2^{1,2}=-\lambda_2^{2,1}=-\frac{1}{1+\beta_c}\lambda_1^{1,1}\cdot\left(\frac{\Delta}{2}\right),$$

$$\lambda_3^{2,2}=-\frac{1-\beta_c}{2}\lambda_1^{1,1}-(1+\beta_c)\lambda_2^{2,2}+0\left(\left(\frac{\Delta}{2}\right)^2\right)$$

$$\lambda_3^{2,3}=-\lambda_3^{3,2}=-\frac{1}{(1+\beta_c)(3+\beta_c)}\lambda_1^{1,1}\cdot\left(\frac{\Delta}{2}\right)-\frac{1}{3+\beta_c}\lambda_2^{2,2}\cdot\left(\frac{\Delta}{2}\right),$$

$$\lambda_3^{3,3}=\frac{(1-\beta_c)(1+\beta_c)}{2(3+\beta_c)(5+\beta_c)}\lambda_1^{1,1}+\frac{1+\beta_c}{5+\beta_c}\lambda_2^{2,2}+0\left(\left(\frac{\Delta}{2}\right)^2\right), \tag{2.13}$$

$$\lambda_4^{3,3}=-\frac{3(1-\beta_c)(1+\beta_c)}{4(5+\beta_c)}\lambda_1^{1,1}-\frac{(1+\beta_c)(2+\beta_c)}{5+\beta_c}\lambda_2^{2,2}+0\left(\left(\frac{\Delta}{2}\right)^2\right),$$

$$\lambda_4^{3,4} = -\lambda_4^{4,3} = -\frac{1}{5+\beta_c}\lambda_3^{3,3}\cdot\left(\frac{\Delta}{2}\right) - \frac{1}{3(1+\beta_c)(3+\beta_c)}\lambda_1^{1,1}\cdot\left(\frac{\Delta}{2}\right)$$
$$-\frac{2}{3(3+\beta_c)(5+\beta_c)}\lambda_2^{2,2}\cdot\left(\frac{\Delta}{2}\right) + 0\left(\left(\frac{\Delta}{2}\right)^3\right),$$
$$\cdots\cdots\cdots\cdots\cdots .$$

Here

$$\beta_c = \frac{\beta+\tilde{\beta}}{2}, \quad \beta \equiv 2-\alpha_0-2\tilde{\alpha}_0-4\mu^2\alpha', \quad \tilde{\beta} \equiv 2-\tilde{\alpha}_0-2\alpha_0-4\mu^2\alpha'. \tag{2.14}$$

The linear combinations of the Regge alope times μ^2 and the zero intercepts, β and $\tilde{\beta}$, are introduced for (2.12a, b) respectively. We have used $\beta \equiv 2-3\alpha_0-4m_\pi^2\alpha'$ for the $\pi^- - \pi^+$ scattering in Ref. 2). It is natural that the coefficients of the main terms, $\lambda_n^{n,n}$ and $\lambda_{m+1}^{n,n}$, are expressed in terms of β_c, the average of β and $\tilde{\beta}$. From (2.13), we can infer the features of $\lambda_n^{n,n}$, $\lambda_{n+1}^{n,n}$ and $\lambda_{n+1}^{n,n+1}$ ($\lambda_{n+1}^{n+1,n}$).
(a) The coefficients of the main terms, $\lambda_n^{n,n}$ and $\lambda_{n+1}^{n,n}$, are

$$\lambda_n^{n,n} = \frac{1+\beta_c}{(n-1)!(2n-1+\beta_c)(2n-3+\beta_c)}\{(n-2)(1-\beta_c)\lambda_1^{1,1}$$
$$+(n-1)(3+\beta_c)\lambda_2^{2,2}\}, \tag{2.15a}$$

$$\lambda_{n+1}^{n,n} = -\frac{(1-\beta_c^2)\lambda_1^{1,1}}{2(n-1)!}\left\{\frac{n(n-2)}{2n-1+\beta_c} - \frac{(n-1)(n-3)}{2n-3+\beta_c}\right\}$$
$$-\frac{(1+\beta_c)(3+\beta_c)\lambda_2^{2,2}}{2(n-2)!}\left\{\frac{n}{2n-1+\beta_c} - \frac{n-2}{2n-3+\beta_c}\right\}. \tag{2.15b}$$

(b) It is expected that the terms expressing the symmetry breadking are not so important at high energies, as the effect of the breaking itself may be neglected at the energies. Therefore the terms with $\lambda_{n+1}^{n,n+1}$ (and $\lambda_{n+1}^{n+1,n}$) will converge more rapidly with $n\to\infty$ than those with $\lambda_n^{n,n}$ which dominate at high energies. In fact such feature are observed.

The amplitude (2.9) with (2.13) and (2.15a, b) avoid contributions from odd daugters in a good approximation. Strictly speaking, the amplitude does not have the odd daughters of the $J(J')=1, 2$ parents and the spin$=2$ odd daughters of the $J(J')=3$ parents in the first-order approximation of Δ, but it does not avoid the spin$=0$ odd daughters of the $J(J')=3$ parents explicitly even in the approximation. This is due to the fact that the property of the gamma function can not be accommodated to the expressions of both of the main terms and the symmetry-breaking terms in one rule.

2.3 *A solution under the conditions* (2.1a) *and* (2.1b) *in the first-order approximation of the symmetry breaking*

In this subsection, we propose a way to avoid odd daughters explicitly in the first-order approximation of the symmetry breaking, using the combinations of $F^{1,3}(s, t)$ and $F^{3,1}(s, t)$ and etc.

In order to avoid the 0 odd daughters of the 3 parents, we must prepare, for example,

$$\lambda_3^{1,3}\frac{\Gamma(1-\alpha_s)\,\Gamma(3-\tilde{\alpha}_t)}{\Gamma(3-\alpha_s-\tilde{\alpha}_t)}+\lambda_3^{3,1}\frac{\Gamma(3-\alpha_s)\,\Gamma(1-\tilde{\alpha}_t)}{\Gamma(3-\alpha_s-\tilde{\alpha}_t)}$$

$$=C_3^{1,3}|\Delta|\left\{\frac{\Gamma(1-\alpha_s)\,\Gamma(3-\tilde{\alpha}_t)}{\Gamma(3-\alpha_s-\tilde{\alpha}_t)}+\frac{\Gamma(3-\alpha_s)\,\Gamma(1-\tilde{\alpha}_t)}{\Gamma(3-\alpha_s-\tilde{\alpha}_t)}\right\}, \tag{2.16}$$

which also satisfies the condition (2.1a). We can eliminate the 3, 1 odd daughters of the 4 parents with the terms of $\lambda_4^{4,4}$, $\lambda_5^{4,4}$ and $\lambda_5^{4,5}(\lambda_5^{5,4})$ and

$$C_4^{1,4}|\Delta|\left\{\frac{\Gamma(1-\alpha_s)\,\Gamma(4-\tilde{\alpha}_t)}{\Gamma(4-\alpha_s-\tilde{\alpha}_t)}+\frac{\Gamma(4-\alpha_s)\,\Gamma(1-\tilde{\alpha}_t)}{\Gamma(4-\alpha_s-\tilde{\alpha}_t)}\right\}, \tag{2.17}$$

and so on. As we add the terms (2.16), (2.17) and etc. to the amplitude (2.4) with (2.8), we must subtract some terms of the amplitude so as to retain the total number of the freedoms. From the observation of (2.9), it is clear that what terms have to be subtracted.

In conclusion, one has to prepare also the explicitly-symmetric terms (2.16), (2.17) and etc. of the first order of Δ. The discrimination of these terms from the main ones is possible by the treatment of β_c and Δ as independent parameters. We suppose that the terms will converge more rapidly with $n\to\infty$ than the terms with $\lambda_n^{n,n}$. The resultant amplitude takes a form of (terms as similar as for the $\pi^- -\pi^+$ scattering) plus (ones for the symmetry breaking).

§3. Summary and convergence condition at $s\to\infty$

It is suggested that one can construct a $K^- - K^0$ amplitude with the form (terms as similar as for the $\pi^- -\pi^+$ scattering) plus (ones for the symmetry breaking) at least in the first-order approximation of Δ, starting with the most general Veneziano-type amplitude and the duality scheme proposed the author. The newly proposed condition (2.7) serves to restrict terms for the symmetry breaking. While, the needed explicitly-symmetric terms (2.16), (2.17) and etc. can be distinguished from the dependence on the symmetry breaking, Δ.

Last, we discuss the convergence condition at $s\to\infty$. As all the terms for the breaking will converge more rapidly than the terms with $\lambda_n^{n,n}$, the condition may be written as for the $\pi^- -\pi^+$ scattering,

$$\lambda_2^{2,2}=-\frac{1-\beta_c}{3+\beta_c}\lambda_1^{1,1}. \tag{3.1}$$

Appendix

Derivation of Equation (2.9)

In the Appendix, we show that the parameters $\lambda_k^{n,n}$ and $\lambda_k^{n,n+1}(\lambda_k^{n+1,n})$ in the amplitude (2.4) vanish for $2n \geqq k \geqq n+2$ and $2n+1 \geqq k \geqq n+2$ $(2n+1 \geqq k \geqq n+2)$ under the condition (2.1a), respectively. Let us begin with the $(J, J')=(1, 1)$ case. Under the condition of this case, we have

$$\lambda_3^{1,2} = \lambda_3^{2,1}. \tag{A.1}$$

As $\lambda_3^{1,2}$ and $\lambda_3^{2,1}$ are the odd functions of Δ, (A.1) and (2.8) lead to

$$\lambda_3^{1,2} = \lambda_3^{2,1} = 0. \tag{A.2}$$

We find, from the $(J, J')=(2, 1)$ case,

$$\lambda_4^{2,2} + \lambda_4^{2,3} + \lambda_5^{2,3} = 0, \tag{A.3}$$

and do, form the $(J, J')=(1, 2)$ case,

$$\lambda_4^{2,2} + \lambda_4^{3,2} + \lambda_5^{3,2} = 0. \tag{A.4}$$

The $(J, J')=(2, 2)$ case gives

$$\lambda_5^{2,3} = \lambda_5^{3,2}. \tag{A.5}$$

Because of the odd property of $\lambda_4^{2,3}$, $\lambda_4^{3,2}$, $\lambda_5^{2,3}$ and $\lambda_5^{3,2}$ with respect to Δ, (A.3), (A.4), (A.5) and (2.8) lead to

$$\lambda_4^{2,2} = 0,$$
$$\lambda_4^{2,3} = \lambda_4^{3,2} = \lambda_5^{2,3} = \lambda_5^{3,2} = 0. \tag{A.6}$$

The same manipulation for higher J and J' leads the consequence (2.9).

References

1) T. Ino, Prog. Theor. Phys. **62** (1979), 1177; **61** (1979), 1863.
T. Ino, Prog. Theor. Phys. **71** (1984), 864.
2) T. Ino, Y. Munakata and J. Sakamoto, to be published in Prog. Theor. Phys. **73** (1985), No. 1.