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Totally Geodesic Imbeddings of Homogeneous Systems into their Enveloping Lie Groups*

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The enveloping Lie group $A = G \times K_e$ of a connected analytic homogeneous system (G, η) contains a submanifold $G \times \{1\}$ which can be identified with G under the canonical imbedding. In this paper, we characterize the class of homogeneous systems imbedded totally geodesically into their enveloping Lie groups, carrying with their canonical connections. It is shown that the class of symmetric homogeneous systems and that of homogeneous systems of Lie groups are essentially the case, among K-semisimple homogeneous systems (Theorem 4).

Introduction

Let (G, η) be a connected analytic homogeneous system and $A = G \times K_e$ the enveloping Lie group of G at a base point $e \in G$, where K_e is the closure of the left inner mapping group A_e in the isotropy group at e of $\mathrm{Aut}(\eta)$ (cf. [3–I]). We denote by $\mathfrak{G} = T_e(G)$ the tangent Lie triple algebra of (G, η) at e, $\mathfrak{A}$ the Lie algebra of A and $\mathfrak{K}$ the Lie subalgebra of $\mathfrak{A}$ corresponding to the Lie subgroup $\{e\} \times K_e$ (which will be identified with K_e) of A. The pair $(\mathfrak{A}, \mathfrak{K})$ is a reductive pair of Lie algebras, that is, $\mathfrak{A} = \mathfrak{G} \oplus \mathfrak{K}$ with $[\mathfrak{K}, \mathfrak{G}] \subset \mathfrak{G}$. We consider the canonical imbedding j of G into A given by $j(x) = (x, 1)$ for $x \in G$. By the works of E. Cartan (e.g. [1]) it has been well-known that any totally geodesic submanifold of a Lie group is symmetric. In this paper, we shall investigate the homogeneous systems whose images under the canonical imbeddings are totally geodesic submanifolds in their enveloping Lie groups, and show that symmetric homogeneous systems and homogenous systems of Lie groups are typical ones with this property. The terminology in this paper will be referred to [3].

§ 1. Geodesic homogeneous systems

An analytic homogeneous system (G, η) is said to be *geodesic* if, for any two points x and y on any geodesic curve c of G (w. r. t. the canonical connection), the linear map induced from the displacement $\eta(x, y)$ coincedes with the parallel displacement of tangent vectors, from x to y along c. A geodesic homogeneous system is said to be

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regular if the linear representation of the group K_e on the tangent space $\mathfrak{G}=T_e(G)$ at e coincides with the restricted holonomy group at e. In Proposition 8 of [3–I] it has been shown that a connected geodesic homogeneous system (G, η) is regular if and only if $\mathfrak{K}=D(\mathfrak{G}, \mathfrak{G})$ holds, where $D(\mathfrak{G}, \mathfrak{G})$ denotes the inner derivation algebra of the tangent Lie triple algebra $\mathfrak{G}$.

Let $j: G\to A=G\times K_e$ be the canonical imbedding of G into the enveloping Lie group A which is given by $j(x)=(x, 1)$. Denote the group-multiplication of $j(x)$ and $j(y)$ in A by $j(x)j(y)=(\mu(x, y), \alpha(x, y))$. We can show that the multiplication $xy=\mu(x, y)$ in G is given by $xy=\eta(e, x, y)$ and the left inner mapping $L_{x,y}$ can be identified with the action ad $\alpha(x, y)$ of the element of K_e on H, by identifying G (resp. K_e) with the submanifold $j(G)$ (resp. Lie subgroup $\{e\}\times K_e$) of A (cf. [3–V]).

THEOREM 1. *A connected analytic homogeneous system (G, η) with a base point e is geodesic if and only if the submanifold (G, j) of $A=G\times K_e$ with the canonical connection of (G, η) is totally geodesic at the identity element $(e, 1)$ of A (w. r. t. the $(-)$-connection).*

PROOF. Assume that (G, η) is geodesic. Let $c: t\to x(t)$, $t\in I\subset \boldsymbol{R}$, be a geodesic of G through $e=x(0)$. Since the canonical connection of G is complete, we can assume that the domain I of c is the whole real numbers $\boldsymbol{R}$. The displacement $\eta(e, x(t))$ induces the parallel displacement $d\eta(e, x(t))$: $T_e(G)\to T_{x(t)}(G)$ from e to $x(t)$ along c. Since any displacement of (G, η) is an affine transformation of the canonical connection, it sends any geodesic into a geodesic and especially we have

$$\mu(x(t), x(s))=\eta(e, x(t), x(s))=x(t+s)=\eta(e, x(s), x(t)).$$

Also we get $\eta(x(s), x(t+s))\eta(e, x(s))=\eta(e, x(t+s))$, for $t, s\in \boldsymbol{R}$, by comparing the induced linear map of both sides. These equalities imply the relation $L_{x(t),x(s)}=1$. In fact, we can see it by the equalities $L_{x(s),x(t)}=\eta(e, x(s+t))^{-1}\eta(e, x(s))\eta(e, x(t))=\eta(x(s+t), e)\eta(x(s), x(s+t))\eta(e, x(s))=1$. Thus we can see that the image $j(x(t))=(x(t), 1)$ of the geodesic c under the canonical imbedding is a 1-parameter subgroup of A, that is, the submanifold (G, j) is totally geodesic at the identity $(e, 1)$ of A.

Conversely, suppose that the submanifold (G, j) with the canonical connection is totally geodesic at the identity $(e, 1)$ of A. Then, the 1-parameter subgroup $\tilde{x}(t)$ generated by any $X\in\mathfrak{G}\subset\mathfrak{A}=\mathfrak{G}\oplus\mathfrak{K}$ is contained in $j(G)=(G, 1)$. Set $\tilde{x}(t)=(x(t), 1)$ for $t\in\boldsymbol{R}$, and we get $L_{x(t),x(s)}=1$ by $(x(t+s), 1)=\tilde{x}(t+s)=\tilde{x}(t)\tilde{x}(s)=(\mu(x(t), x(s)), \alpha(x(t), x(s)))$. Now, for any $Y_0\in\mathfrak{G}=T_e(G)$, let $Y(t)=d\eta(e, x(t))Y_0$ be a vector field along the curve c given by $x(t)$, $t\in\boldsymbol{R}$. By the definition of the canonical connection of (G, η) we have

$$\nabla_{\dot{c}}Y(t)=\lim_{h\to 0}\frac{1}{h}[\tau(x(t), x(t+h))^{-1}Y(t+h)-Y(t)],$$

where $\tau(x(t), x(t+h))=d\eta(x(t), x(t+h))$. Since $\eta(e, x(t))^{-1}\eta(x(t+h), x(t))\eta(e, x(t+h))$

$=L_{x(t+h),x(-h)}=1$ holds, we can show that $\nabla_{\dot{c}}Y(t)=0$. Hence, it follows that $d\eta(e, x(t))$ is the parallel displacement of vectors from e to $x(t)$ along c. Especially, the curve c is a geodesic tangent to $X=\frac{d}{dt}\Big|_0 x(t)$. For any two points $x(t)$ and $x(s)$ on the geodesic c, $d\eta(x(t), x(s))=d\eta(e, x(s))d\eta(e, x(t))^{-1}$ also gives the parallel displacement from $x(t)$ to $x(s)$ along c. Any geodesic c': $t\to x'(t)$ through any point $x_0=x'(0)$ in G is obtained by $c'=\eta(e, x_0)c$ for some geodesic c through $e=x(0)$. Here, $\eta(e, x_0)$ is an affine transformation and we can see that $d\eta(x'(t), x'(s))=d\eta(e, x_0)d\eta(x(t), x(s))$ is the parallel displacement from $x'(t)$ to $x'(s)$ along c', for any $x'(t)$, $x'(s)$ on c'. Thus, we can see that the homogeneous system is geodesic. q.e.d.

§2. Totally geodesic imbeddings into the enveloping groups

Let (G, η) be a connected analytic homogeneous system with a base point e. In this section, we consider the situation where the affinely connected manifold G can be imbedded under the canonical imbedding j: $G\to A$ as a totally geodesic submanifold of the enveloping Lie group A.

PROPOSITION 1. *If the submanifold (G, j) with the canonical connection is a totally geodesic submanifold of A (w.r.t. the $(-)$-connection), then the subspace $\mathfrak{G}_0=\mathfrak{G}\mathfrak{G}$ of the tangent Lie triple algebra $\mathfrak{G}=T_e(G)$ of (G, η) is reduced to a Lie algebra, and it is an ideal of $\mathfrak{G}$. Moreover, $D(\mathfrak{G}_0, \mathfrak{G})=\{0\}$ holds.*

PROOF. It is well-known that a totally geodesic submanifold N of an affine symmetric space M is itself a symmetric space, and that the tangent space $\mathfrak{N}$ of N at a point e satisfies $[[\mathfrak{N}, \mathfrak{N}], \mathfrak{N}]\subset\mathfrak{N}$ in the enveloping Lie algebra of the tangent Lie triple system $\mathfrak{M}$ of M at e (cf. [9], [10]; also [1]). In our case, the enveloping Lie group A will be regarded as an affine symmetric space with respect to the (0)-connection. If the submanifold (G, j) is totally geodesic with respect to the $(-)$-connection, so is it with respect to the (0)-connection and we have $[[\mathfrak{G}, \mathfrak{G}], \mathfrak{G}]\subset\mathfrak{G}$ in the Lie algebra $\mathfrak{A}=\mathfrak{G}\oplus\mathfrak{K}$ of A. For any $X, Y, Z\in\mathfrak{G}$, we get $[X, Y]=XY+D(X, Y)$ and $[[X, Y], Z]=(XY)Z+D(X, Y)Z+D(XY, Z)$. The $\mathfrak{K}$-component of $[[X, Y], Z]$ must be zero and it follows that the subspace $\mathfrak{G}_0=\mathfrak{G}\mathfrak{G}$ satisfies $D(\mathfrak{G}_0, \mathfrak{G})=\{0\}$. Since $\mathfrak{G}\mathfrak{G}_0\subset\mathfrak{G}_0$ and $D(\mathfrak{G}, \mathfrak{G}_0)\mathfrak{G}=\{0\}\subset\mathfrak{G}_0$, the subspace $\mathfrak{G}_0$ is an ideal of $\mathfrak{G}$, which is reduced to a Lie algebra as $D(\mathfrak{G}_0, \mathfrak{G}_0)=\{0\}$. q.e.d.

Here, let us recall several results concerning symmetric homogeneous systems (cf. [3]). A homogeneous system (G, η) is *symmetric* if the map S_x sending y into $S_x(y)=\eta(y, x, x)$ is an automorphism of (G, η) for every x. If an analytic homogeneous system (G, η) is symmetric, then G is an affine symmetric space with the canonical connection of (G, η) and the tangent Lie triple algebra $\mathfrak{G}$ is reduced to a Lie triple system, i.e., $\mathfrak{G}\mathfrak{G}=\{0\}$.

PROPOSITION 2. *Let (G, η) be a symmetric homogeneous system with a base point e. The multipication $\mu(x, y)=\eta(e, x, y)$ at e and its left inner mappings $L_{x,y}$ satisfy the following relations*:

(1) $\mu(x, y)^{-1}=\mu(x^{-1}, y^{-1})$.

(2) $L_{x,y}=L_{x^{-1},y^{-1}}=L_{y,x^{-1}}=L_{y,x}^{-1}$.

PROOF. Since $x^{-1}=S_e x$, the relation (1) is obtained by $\mu(x^{-1}, y^{-1})=\eta(e, S_e x, S_e y) =S_e\eta(e, x, y)=\mu(x, y)^{-1}$. It was shown in [3–V] that the left ranslations $L_x y= \mu(x, y)$ have the same properties as those of homogeneous loops in §1 of [2]. Therefore, the relation (2) is proved in the same way as was shown in Lemma 1.8 and (4) of Proposition 1.13 of [2]. q. e. d.

We say that the homogeneous system (G, η) is *simple* (resp. *K-semisimple*) if the tangent Lie triple algebra $\mathfrak{G}$ of (G, η) is simple (resp. K-semisimple in the sense of [7]). The *homogeneous system of a Lie group G* is given by $\eta(x, y, z)=yx^{-1}z$ for $x, y, z \in G$ (cf. [3–II]). It has the enveloping Lie group $A=G\times\{1\}$ and, of course, this can be identified with G under the canonical imbedding $j: G\rightarrow A$. The tangent Lie triple algebra of this homogeneous system at the identity e is reduced to the Lie algebra of the Lie group G.

THEOREM 2. *Assume that the homogeneous system (G, η) is simple. If the submanifold (G, j) imbedded in the enveloping Lie group $A=G\times K_e$ is a totally geodesic submanifold of A (w. r. t. the $(-)$-connection), then G is a symmetric homogeneous system unless it is a homogeneous system of a Lie group.*

PROOF. In [3–II] we have proved that a geodesic homogeneous system is symmetric (resp. a homogeneous system of a Lie group) if and only if its tangent Lie triple algebra is reduced to a Lie triple system (resp. Lie algebra). Since G is geodesic by Theorem 1, it is sufficient to show that the tangent Lie triple algebra $\mathfrak{G}$ is reduced to a Lie triple system unless it is reduced to a Lie algebra. In Proposition 1 we have shown that $\mathfrak{G}_0=\mathfrak{G}\mathfrak{G}$ is an ideal of $\mathfrak{G}$. Since $\mathfrak{G}$ is assumed to be simple, $\mathfrak{G}_0=\{0\}$ or $\mathfrak{G}_0=\mathfrak{G}$. If $\mathfrak{G}_0=\mathfrak{G}$ holds, then $\mathfrak{G}$ should be reduced to a Lie algebra by Proposition 1. Hence we get $\mathfrak{G}_0=\{0\}$ and it follows that $\mathfrak{G}$ is reduced to a Lie triple system. q. e. d.

As the converse of this theorem we have

THEOREM 3. *If an analytic homogeneous system (G, η) is symmetric, then the submanifold (G, j) with the canonical connection imbedded by the canonical imbedding $j: G\rightarrow A$ is a totally geodesic submanifold of the enveloping Lie group A.*

PROOF. Assume that the homogeneous system (G, η) is symmetric. We regard the enveloping Lie group A as an affine symmetric space with respect to the (0)-connection. Then, to prove that (G, j) is a totally geodesic submanifold of A, we firstly show that $j(G)=G\times\{1\}$ is a symmetric subspace of the Lie group A, that is, $\tilde{x}\tilde{y}^{-1}\tilde{x}$ is contained in $j(G)$ for any $\tilde{x}=(x, 1)$ and $\tilde{y}=(y, 1)$ in $j(G)$ (cf. Ch. III of [9]). By the definition of the group-multiplication in $A=G\times K_e$, we have

$$\begin{aligned}\tilde{x}\tilde{y}^{-1}\tilde{x}&=(x, 1)(y^{-1}, 1)(x, 1)\\&=(xy^{-1}, L_{x,y^{-1}})(x, 1)\\&=(x(y^{-1}x), L_{xy^{-1},u}L_{x,y^{-1}})\\&=(x(y^{-1}x), L_{x,y^{-1}}L_{v,x}),\end{aligned}$$

where $u=L_{x,y^{-1}}x$ and $v=L^{-1}{}_{x,y^{-1}}(xy^{-1})$. On the other hand, we get

$$\begin{aligned}\tilde{x}\tilde{y}^{-1}\tilde{x}&=(x, 1)(y^{-1}x, L_{y^{-1},x})\\&=(x(y^{-1}x), L_{x,y^{-1}x}L_{y^{-1},x}).\end{aligned}$$

Comparing the K_e-components of both expressions of $\tilde{x}\tilde{y}^{-1}\tilde{x}$, we have

$$L_{x,y^{-1}}L_{v,x}=L_{x,y^{-1}x}L_{y^{-1},x},\quad v=L^{-1}_{x,y^{-1}}(xy^{-1}).$$

Since (G, η) is assumed to be symmetric, Proposition 2 implies

$$L^{-1}_{x,y^{-1}}=L_{y,x^{-1}}=L_{y^{-1},x}$$

and

$$v=L_{y^{-1},x}(xy^{-1})=L_{y,x^{-1}}(xy^{-1})=(yx^{-1})^{-1}=y^{-1}x.$$

Therefore, we have

$$L_{x,y^{-1}}L_{y^{-1}x,x}=L_{x,y^{-1}x}L_{y^{-1},x}.$$

However, Proposition 2 implies also that the right hand side of this equation is equal to the inverse of the left hand side. Set $L(x, y)=L_{x,y^{-1}}L_{y^{-1}x,x}$ for $x, y\in G$, then we have

$$\tilde{x}\tilde{y}^{-1}\tilde{x}=(x(y^{-1}x), L(x, y))$$

and

$$L(x, y)^2=1.$$

The automorphism $L(x, y)$ of (G, η) is an affine transformation leaving the base point e fixed. Suppose that there exists a pair x and y of elements of G such that $L(x, y)\neq 1$. Then, the linear transformation $dL(x, y)$ on $T_e(G)$ satisfies $dL(x, y)=-1$. This is a contradiction, since the set $\{dL(x, y)|x, y\in G\}\subset\{1, -1\}$ must be a connected subset of the restricted holonomy group at e. Thus, we can see that $L(x, y)=1$ and $\tilde{x}\tilde{y}^{-1}\tilde{x}$ belongs to $j(G)$ for any $\tilde{x}, \tilde{y}\in j(G)$. From this we can see that $j(G)$ is a totally geodesic submanifold of A with respect to the (0)-connection, and so with respect to the $(-)$-connection of A. The tangent Lie triple system of this symmetric space at $(e, 1)$ is the tangent space $\mathfrak{G}$ with the trilinear product $[[X, Y], Z]$ in the Lie algebra $\mathfrak{A}=\mathfrak{G}\oplus\mathfrak{K}$. However, the tangent Lie triple algebra of (G, η) at e is reduced to a Lie triple system with $D(X, Y)Z=[[X, Y], Z]$ and we can see that both symmetric spaces should be coincident. The proof is thus completed. q.e.d.

From this theorem combined with Theorem 1, we obtain

COROLLARY. *An analytic homogeneous system (G, η) is geodesic if it is symmetric.*

§3. The case of K-semisimple regular homogeneous systems

From now on we assume that the homogeneous system (G, η) is regular. Then, the Lie algebra $\mathfrak{K}$ of the Lie group $K_e=\bar{A}_e$ is just equal to the inner derivation algebra $D(\mathfrak{G}, \mathfrak{G})$ of the tangent Lie triple algebra $\mathfrak{G}$, and the Lie algebra $\mathfrak{A}$ of the enveloping Lie group A is equal to the standard enveloping Lie algebra $\mathfrak{G}\oplus D(\mathfrak{G}, \mathfrak{G})$ of $\mathfrak{G}$. By applying the results obtained in [5] and [7] to the tangent Lie triple algebra of the regular homogeneous system, we can show the following:

PROPOSITION 3. *Let (G, η) be a regular homogeneous system. Assume that the Killing form ϕ of the Lie algebra $\mathfrak{A}=\mathfrak{G}\oplus D(\mathfrak{G}, \mathfrak{G})$ satisfies $\phi(\mathfrak{G}, D(\mathfrak{G}, \mathfrak{G}))=0$. Then, the following* (1), (2) *and* (3) *are mutually equivalent*:

(1) *(G, η) is K-semisimple.*

(2) *The enveloping Lie group $A=G\times K_e$ is semisimple.*

(3) *The Killing-Ricci from $\beta=\phi|_{\mathfrak{G}\times\mathfrak{G}}$ of the tangent Lie triple algebra $\mathfrak{G}$ is nondegenerate.*

In [6] we have shown a decomposition theorem for simply connected regular homogeneous systems satisfying the conditions in the proposition above. Here, we will characterize a special class of those homogeneous systems in which they are decomposed into direct products of symmetric homogeneous systems and homogeneous systems of Lie groups.

THEOREM 4. *Let (G, η) be a simply connected regular analytic homogeneous system, $A=G\times K_e$ the enveloping Lie group of G at a base point e. Then, (G, η) is decomposed into a direct product $(G_0, \eta_0)\times(G_1, \eta_1)$ of a homogeneous system (G_0, η_0) of a semisimple Lie group G_0 and a K-semisimple symmetric homogeneous system (G_1, η_1) (one of them might be empty), if and only if the following three conditions are satisfied*:

(1) *The submanifold (G, j) of A imbedded under the canonical imbedding $j: G\to A$ is totally geodesic in A.*

(2) *The Killing form* ϕ *of the Lie algebra* $\mathfrak{A}=\mathfrak{G}\oplus D(\mathfrak{G}, \mathfrak{G})$ *of the enveloping Lie group satisfies* $\phi(\mathfrak{G}, D(\mathfrak{G}, \mathfrak{G}))=0$.

(3) (G, η) *is K-semisimple.*

PROOF. Assume that the conditions (1), (2) and (3) are satisfied. Then, the Killing-Ricci form β of the tangent Lie triple algebra $\mathfrak{G}$ of (G, η) is a nondegenerate invariant form (cf. [5]). Proposition 1 implies that $\mathfrak{G}_0=\mathfrak{G}\mathfrak{G}$ is an ideal of $\mathfrak{G}$ satisfying $D(\mathfrak{G}_0, \mathfrak{G})=\{0\}$, under the condition (1). Let $\mathfrak{G}_1$ be the orthogonal complement of $\mathfrak{G}_0$ with respect to the invariant form β. Then, $\mathfrak{G}_1$ is again an ideal of $\mathfrak{G}$ satisfying $\mathfrak{G}_1\mathfrak{G}_1=\{0\}$, and $\mathfrak{G}$ is decomposed into a direct sum $\mathfrak{G}=\mathfrak{G}_0\oplus\mathfrak{G}_1$. The restrictions β_0 and β_1 of β to $\mathfrak{G}_0$ and $\mathfrak{G}_1$ are nondegenerate Killing-Ricci forms of respective ideals. Now, by the theorem in [6], we can conclude that (G, η) is decomposed into a direct product of normal subsystems (G_0, η_0) and (G_1, η_1) of (G, η) whose tangent Lie triple algebras are $\mathfrak{G}_0$ and $\mathfrak{G}_1$, respectively. We can easily see that each of (G_i, η_i), $i=0, 1$, is geodesic. Since $\mathfrak{G}_0$ is reduced to a Lie algebra, β_0 is the Killing form of the Lie algebra $\mathfrak{G}_0$ and (G_0, η_0) is a homogeneous system of a semisimple Lie group. On the other hand, the ideal $\mathfrak{G}_1$ is reduced to a Lie triple system and (G_1, η_1) is a K-semisimple symmetric homogeneous system. In fact, by applying Proposition 3 to (G_1, η_1), we can see that it is K-semisimple.

Conversely, assume that $(G, \eta)=(G_0, \eta_0)\times(G_1, \eta_1)$ is a direct product decomposition of (G, η) into a homogeneous system (G_0, η_0) of a semisimple Lie group G_0 and a K-semisimple symmetric homogeneous system (G_1, η_1). The enveloping Lie group $A=G\times K_e$ can be expressed as the direct product $A_0\times A_1$, where A_0 and A_1 are the enveloping Lie groups of G_0 and G_1, respectively. Then, we can get the property (1) in the theorem. In fact, it follows from Theorem 3 that the canonical imbedding $j_1\colon G_1\to A_1$ is a totally geodesic imbedding into A_1 and so is $j\colon G\to A$, since $A_0=G_0\times\{1\}$. The tangent Lie triple system $\mathfrak{G}_1$ of the symmetric homogeneous system G_1 satisfies $\phi_1(\mathfrak{G}_1, D(\mathfrak{G}_1, \mathfrak{G}_1))=0$ for the Killing form ϕ_1 of the standard enveloping Lie algebra $\mathfrak{A}_1=\mathfrak{G}_1\oplus D(\mathfrak{G}_1, \mathfrak{G}_1)$. Hence, the Killing form $\phi=\phi_0+\phi_1$ of $\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1$ satisfies $\phi(\mathfrak{G}, D(\mathfrak{G}, \mathfrak{G}))=0$, that is, the condition (2) is satisfied. The Killing-Ricci forms β_i, $i=0, 1$, are nondegenerate and so is the Killing-Ricci form $\beta=\beta_0+\beta_1$ of $\mathfrak{G}=\mathfrak{G}_0\oplus\mathfrak{G}_1$. By Proposition 3, (G, η) must be K-semisimple. q.e.d.

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