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Gauge Conditions and Correlations in Stochastic Quantization of U(1) Gauge Field

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Abstract

Stochastic quantization method is applied to U(1) gauge field. The stabilization of random trajectories of gauge-dependent quantities is discussed. It is shown that such a stabilization is possible by modifying the correlation of the white noise. As an example the propagator of the axial gauge condition is derived.

§1. Introduction

It is known that the stochastic quantization proposed by Parisi and Wu[1] is equivalent to other conventional quantizations for non-gauge theories. For gauge theories the equivalence between the stochastic and the other quantizations is proved perturbatively by many authors.[2)~8)] The outstanding point of this method is that we can quantize the gauge fields without any gauge fixing.

Nevertheless, it is still inconvenient to quantize the gauge fields without the gauge fixing because gauge-dependent quantities contain divergent terms while the gauge-invariant ones are of course calculated unambiguously (see below).

The outline of this method would be easily understood by showing the application to U (1) gauge field.

The starting point of the stochastic quantization is the Langevin equation,

$$\frac{\partial A_\mu(x, t)}{\partial t} = -\frac{\delta S[A_\mu]}{\delta A_\mu(x, t)} + \eta_\mu(x, t), \tag{1.1}$$

with the introduction of a fictitious time t. $S[A_\mu]$ is the classical Euclidean action for the gauge field A_μ and is given by

$$S[A_\mu] = \frac{1}{4}\int d^4x F_{\mu\nu}(x) F_{\mu\nu}(x). \tag{1.2}$$

Thus eq. (1.1) is explicitely written as

$$\frac{\partial}{\partial t} A_\mu(x, t) = -(\delta_{\mu\nu}\Box - \partial_\mu\partial_\nu) A_\nu(x, t) + \eta_\mu(x, t). \tag{1.1$'$}$$

The white noise $\eta_\mu(x, t)$ is supposed to satisfy the stochastic average

$$\begin{aligned}
&\langle \eta_\mu(x, t)\rangle = 0,\\
&\langle \eta_\mu(x, t)\eta_\nu(x', t')\rangle = 2\delta_{\mu\nu}\delta(x-x')\delta(t-t'),\\
&\langle \eta_\mu(x_1, t_1)\eta_\nu(x_2, t_2)\eta_\rho(x_3, t_3)\eta_\sigma(x_4, t_4)\rangle\\
&\quad = \langle \eta_\mu(x_1, t_1)\eta_\nu(x_2, t_2)\rangle \langle \eta_\rho(x_3, t_3)\eta_\sigma(x_4, t_4)\rangle\\
&\qquad + \text{other combinations},\\
&\qquad \cdots\cdots\cdots\cdots
\end{aligned} \tag{1.3}$$

It can be shown that

$$\lim_{t\to\infty} \langle \Phi(A_\mu(t))\rangle = \int \prod_x dA_\mu(x)\Phi(A_\mu)\, e^{-S[A]} \Big/ \int \prod_x dA_\mu\, e^{-S[A]}, \tag{1.4}$$

where $\Phi(A_\mu(x, t))$ is a gauge-invariant function of $A_\mu(x, t)$ with the same time t. The right-hand side is the Green function in the Euclidean field theory. If $\Phi(A_\mu)$ is a gauge-dependent function, however, the left-hand side of (1.4) never converges. This can be easily seen from (1.1′), which gives

$$\frac{\partial}{\partial t}\partial_\mu A_\mu(x, t) = \partial_\mu \eta_\mu(x, t). \tag{1.5}$$

The longitudinal component of A_μ never relaxes on the equilibrium for lack of friction terms on the right-hand side of the above equation.

In order to stabilize the random trajectory the friction terms are introduced by several authors in some different ways.[2),3),6)~8)]

In this paper we discuss another possibility for such a stabilization. It is shown in the next section that in U (1) gauge field the modification of the correlation of the white noise (1.3) with an adequate choice of the initial condition of the Langevin equation (1.1′) makes the stochastic average of $\Phi(A_\mu)$ converge. As an example we derive the propagator for the axial gauge condition by such a modification.

§2. Modification of the stochastic correlation of the white noise

The equation (1.1′) can be written in the momentum space as

$$\frac{\partial}{\partial t}\tilde{A}_\mu(k, t) = -(\delta_{\mu\nu}k^2 - k_\mu k_\nu)\tilde{A}_\nu(k, t) + \tilde{\eta}_\mu(k, t), \tag{2.1}$$

where the quantity with tilde is the Fourier transform of its correspondent. The above equation is solved as

$$\tilde{A}_\mu(k, t) = \int_0^t dt' \left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\exp\left\{-k^2(t-t')\right\} + \frac{k_\mu k_\nu}{k^2}\right]\tilde{\eta}_\nu(k, t')$$

$$+ \left\{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\exp(-k^2 t) + \frac{k_\mu k_\nu}{k^2}\right\}\tilde{A}_\nu(k, 0). \tag{2.2}$$

Then the stochastic correlation of the product $\tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)$ is given by

$$\langle \tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle = \delta(k+p)\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\{1 - \exp(-2k^2 t)\}k^{-2} + \frac{2k_\mu k_\nu}{k^2}t\right]$$

$$+ \left\{\left(\delta_{\mu\lambda} - \frac{k_\mu k_\lambda}{k^2}\right)\exp(-k^2 t) + \frac{k_\mu k_\lambda}{k^2}\right\}\tilde{A}_\lambda(k, 0)$$

$$\times \left\{\left(\delta_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}\right)\exp(-p^2 t) + \frac{p_\nu p_\sigma}{p^2}\right\}\tilde{A}_\sigma(p, 0), \tag{2.3}$$

where we use

$$\langle \tilde{\eta}_\mu(k, t)\tilde{\eta}_\nu(p, t')\rangle = 2\delta_{\mu\nu}\delta(k+p)\delta(t-t'). \tag{2.4}$$

For large t, eq. (2.3) leads to

$$\langle \tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle \longrightarrow \delta(k+p)\left\{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)k^{-2} + \frac{2k_\mu k_\nu}{k^2}t\right\}$$

$$+ \frac{k_\mu k_\lambda p_\nu p_\sigma}{k^2 p^2}\tilde{A}_\lambda(k, 0)\tilde{A}_\sigma(p, 0) \tag{2.5}$$

Namiki et al.[5] have taken the average with respect to the initial value $\tilde{A}_\mu(k, 0)$ and have supposed

$$\{\tilde{A}_\mu(k, 0)\tilde{A}_\nu(p, 0)\}_{aver.} = \alpha\delta_{\mu\nu}\delta(k+p). \tag{2.6}$$

Then they have obtained

$$\langle \tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle_{aver.} \longrightarrow \delta(k+p)k^{-2}\left\{\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2} + 2k_\mu k_\nu t\right\}. \tag{2.7}$$

It was shown that the initial value of the field is related to the gauge parameter of the propagator in the conventional quantizations.

The divergent term in the right-hand side on (2.7) is due to the non-relaxation of the longitudinal component of the field as explained before. A direct way to cancel such a divergent term is to impose a constraint on the field A_μ. For example we suppose

$$\partial_\mu A_\mu(x, t) = 0. \tag{2.8}$$

In this case all the components of the white noise η_μ are no longer independent of each other; we have

$$\partial_\mu \eta_\mu(x, t) = 0, \tag{2.9}$$

from (1.5). The above equation implies that the correlation (1.3) of η_μ should be modified as

$$\langle \tilde{\eta}_\mu(k, t)\tilde{\eta}_\nu(p, t')\rangle = 2\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\delta(k+p)\delta(t-t'). \tag{2.10}$$

The stochastic correlation of the product $\tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)$ is

$$\begin{aligned}\langle \tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle &= \delta(k+p)\cdot\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\{1-\exp(-2k^2 t)\}k^{-2} \\ &\quad + \exp\{-(k^2+p^2)t\}\tilde{A}_\mu(k, 0)\tilde{A}_\nu(p, 0) \\ &\longrightarrow \delta(k+p)\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)k^{-2}\end{aligned} \tag{2.11}$$

Conversely the modification (2.10) of the correlation of η_μ means the constraint (2.8) by tracing back the above line of reasoning (more precisely we need additionally $\partial_\mu A_\mu(x, 0) = 0$ as the initial condition).

Now, we consider the modification of the correlation of η_μ which does not induce the constraint on A_μ. Here we search such a modification that the propagator of the axial gauge condition is derived. We put

$$\begin{aligned}\langle \tilde{\eta}_\mu(k, t)\tilde{\eta}_\nu(p, t')\rangle &= 2\delta(p+k)\delta(t-t') \\ &\quad \times \{\delta_{\mu\nu} + B\delta_{\mu 3}\delta_{\nu 3} + C(\delta_{\mu 3}k_\nu + \delta_{\nu 3}k_\mu) + Dk_\mu k_\nu\},\end{aligned} \tag{2.12}$$

where B, C and D are functions of k_μ. From (2.2) the correlation of the product $\tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)$ is given by

$$\begin{aligned}\langle \tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle &= 2\delta(k+p)\int_0^t dt'\Big[\Big(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\exp\{-2k^2(t-t')\} + \frac{k_\mu k_\nu}{k^2} \\ &\quad + B\Big\{\Big(\delta_{\mu 3} - \frac{k_\mu k_3}{k^2}\Big)\exp k^2(t'-t) + \frac{k_\mu k_3}{k^2}\Big\}\Big\{\Big(\delta_{\nu 3} - \frac{k_\nu k_3}{k^2}\Big)\exp k^2(t'-t) + \frac{k_\nu k_3}{k^2}\Big\} \\ &\quad + C\Big\{k_\mu\Big(\delta_{\nu 3} - \frac{k_\nu k_3}{k^2}\Big)\exp k^2(t'-t) + \frac{k_3}{k^2}k_\mu k_\nu + (\mu \longleftrightarrow \nu)\Big\} + Dk_\mu k_\nu\Big] \\ &\quad + \Big\{\Big(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\Big)\exp(-k^2 t) + \frac{k_\mu k_\rho}{k^2}\Big\}\Big\{\Big(\delta_{\nu\sigma} - \frac{p_\nu p_\rho}{p^2}\Big)\exp(-p^2 t) \\ &\quad + \frac{p_\nu p_\nu}{p^2}\Big\}\tilde{A}_\rho(k, 0)\tilde{A}_\sigma(p, 0).\end{aligned} \tag{2.13}$$

The condition that the terms of $O(t)$ in the above equation should vanish is

$$k^2+Bk_3^2+2Ck^2k_3+Dk^4=0. \tag{2.14}$$

Taking the average with respect to the initial value $\tilde{A}_\mu(k, 0)$ and the limit $t\to\infty$, we have

$$\begin{aligned}
&\lim_{t\to\infty}\langle\tilde{A}_\mu(k, t)\tilde{A}_\nu(p, t)\rangle_{aver.}\\
&=\delta(k+p)\Big\{\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\Big(1+\frac{3k_3^2}{k^2}B+4k_3C\Big)+B\delta_{\mu 3}\delta_{\nu 3}\\
&\quad+2(k_\mu\delta_{3\nu}+k_\nu\delta_{3\mu})\Big(\frac{k_3}{2k^2}B+C\Big)\Big\}k^{-2}\\
&\quad+\frac{k_\mu k_\rho p_\nu p_\sigma}{k^2p^2}\{\tilde{A}_\rho(k, 0)\tilde{A}_\sigma(p, 0)\}_{aver.}
\end{aligned} \tag{2.15}$$

This should be equal to the axial gauge propagator,

$$\delta(k+p)\Big\{\delta_{\mu\nu}-\frac{1}{k_3}(k_\mu\delta_{\nu 3}+k_\nu\delta_{\mu 3})+\frac{k_\mu k_\nu}{k_3^2}\Big\}k^{-2}. \tag{2.16}$$

The functions B, C and D and $\{\tilde{A}_\rho(k, 0)\tilde{A}_\sigma(p, 0)\}_{aver.}$ should satisfy

$$\begin{aligned}
&B=0,\\
&2C=-\frac{1}{k_3},\\
&\Big(\frac{1}{k^2}+\frac{4k_3}{k^2}C+\frac{1}{k_3^2}\Big)\delta(k+p)=\frac{k_\rho p_\sigma}{k^2p^2}\{\tilde{A}_\rho(k, 0)\tilde{A}_\sigma(p, 0\}_{aver.}.
\end{aligned}$$

The combination of the above equations with (2.14) gives

$$\begin{aligned}
&B=D=0,\\
&C=-\frac{1}{2k_3},\\
&\{\tilde{A}_\mu(k, 0)\tilde{A}_\nu(p, 0)\}_{aver.}=\delta_{\mu\nu}\Big(\frac{1}{k_3^2}-\frac{1}{k^2}\Big)\delta(k+p).
\end{aligned} \tag{2.17}$$

We write the modified correlation of $\tilde{\eta}_\mu$;

$$\langle\tilde{\eta}_\mu(k, t)\tilde{\eta}_\nu(p, t')\rangle=2\delta(k+p)\delta(t-t')\Big\{\delta_{\mu\nu}-\frac{1}{2k_3}(\delta_{\mu 3}k_\nu+\delta_{\nu 3}k_\mu)\Big\}. \tag{2.18}$$

The axial gauge propagator (2.16) can be derived also by imposing the constraint

$$A_3(x, t)=0, \tag{2.19}$$

(see Appendix). In this case, however, $\eta_3(x, t)$ becomes a dependent variable.

In contrast to this or to the case of the covariant gauge (see (2.10)), eq. (2.18) shows that all the components of $\tilde{\eta}_\mu$ are independent of each other. The third component of the potential, A_3 correlates with other components when t is finite.

§ 3. Concluding remark

In stochastic quantization the Green functions of quantum theories are obtained from the correlation functions by taking limit of the fictitious time to intinity. Therefore, at finite t there may exist several types of the formulations and what we have discussed in the previous section and appendix are examples for this fact. This freedom of choice may enable us to expand the territory of quantum theory together with that in stochastic quantization we need neither the Lagrangian nor the Hamiltonian but only the classical equation of motion.

It is inconvenient to apply the procedure discussed here to non-abelian gauge fields for the following reason. If the gauge group is non-abelian, the modified correlation such as (2.18) is not invariant even for t-independent gauge transformation because it is non-local. Therefore the manifestation of the gauge invariance of the theory is lost.

Appendix

Imposing (2.19) on (2.1) we have

$$0=\frac{\partial}{\partial t}\tilde{A}_3(k,\,t)=k_3k_l\tilde{A}_l(k,\,t)+\tilde{\eta}_3(k,\,t)\,,$$

or

$$\tilde{\eta}_3(k,\,t)=-k_3k_l\tilde{A}_l(k,\,t)\,, \tag{A.1}$$

where the Latin index runs from 0 to 2. The above equation implies that the third component of the white noise is not an independent variable. Instead of eqs. (2.1), (2.19) and (A.1) we have equivalently

$$\frac{\partial}{\partial t}\tilde{A}_k(k,\,t)=-(\delta_{kl}k^2-k_kk_l)\tilde{A}_l(k,\,t)+\tilde{\eta}_k(k,\,t)\,, \tag{A.2}$$

with $\tilde{A}_3=0$. This equation is easily solved as

$$\begin{aligned}\tilde{A}_k(k,\,t)=&\int_0^t dt'\left\{\left(\delta_{kl}-\frac{k_kk_l}{k^2}\right)\exp k^2(t'-t)+\frac{k_kk_l}{k^2}\exp k_3^2(t'-t)\right\}\tilde{\eta}_l(k,\,t')\\&+\left\{\left(\delta_{kl}-\frac{k_kk_l}{k^2}\right)\exp(-k^2t)+\frac{k_kk_l}{k^2}\exp(-k_3^2t)\right\}\tilde{A}_l(k,\,0),\end{aligned} \tag{A.3}$$

where $k^2=k_0^2+k_1^2+k_2^2$. Then we have

$$\lim_{t\to\infty} \langle \tilde{A}_k(k, t)\tilde{A}_l(p, t)\rangle = \delta(k+p)\left\{\delta_{kl} + \frac{k_k k_l}{k_3^2}\right\}k^{-2},$$

$$\langle \tilde{A}_3(k, t)\tilde{A}_\mu(p, t)\rangle = 0. \tag{A.4}$$

In the above equation the limit $t\to\infty$ cancels the dependence on the initial values of $\tilde{A}_k$.

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