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Author(s)

Naoyasu Kita, Takeshi Wada

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Sharp asymptotic behavior of solutions to Benjamin-Ono type equations—short range case

Naoyasu Kita[1]

Faculty of Advanced Science and Technology, Kumamoto University
Kumamoto 860-8555, Japan
E-mail: nkita@kumamoto-u.ac.jp

and

Takeshi Wada[2]

Department of Mathematics, Shimane University
Matsue 690-8504, Japan
E-mail: wada@riko.shimane-u.ac.jp

Abstract. This paper studies the large time behavior of a small solution to the generalized Benjamin-Ono equation with a short range nonlinearity, i.e., with the power of nonlinearity greater than 3. In this case, it is well-known that the solution is asymptotically free as $t \to \infty$. We are interested in the asymptotic expansion of the solution, and determine the second asymptotic term. In order to specify the second asymptotic term, we will apply a technique of Fourier series expansion.

1. Introduction

This paper is devoted to the study of the following Benjamin-Ono type equation:

$$\partial_t v + \frac{1}{2}\mathscr{H}\partial_x^2 v + \partial_x f(v) = 0, \tag{1.1}$$

$$v(0,\cdot) = v_0, \tag{1.2}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, the unknown function v is real-valued, $\mathscr{H}$ is the Hilbert transform, and $f(v) = \kappa|v|^{p-1}v$ with $p > 3, \kappa \in \mathbb{R}$. The nonlinearity is a kind of generalization from $f(v) = \kappa v^2$ with which the famous Benjamin-Ono equation is described — it is used to foresee the motion of long internal gravity waves in deep stratified fluids. For detail of physical background of the original Benjamin-Ono equation, refer to [3, 23]. The generalization of nonlinearity as in (1.1) has been conducted, from the mathematical point of view, to see how the dispersive effect caused by $\frac{1}{2}\mathscr{H}\partial_x^2$ controls the nonlinearity, for example, in the problems of local or global well-posedness

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and asymptotic behavior of the solutions. Roughly speaking, if a datum is sufficiently small, the nonlinearity expectedly affects the solution without so serious change and it behaves like a solution to the associated linearized Benjamin-Ono equation. Namely one may expect that $v(t) \sim V(t)\psi$ as $t \to \infty$, where $V(t)\psi$ is a solution to

$$\partial_t w + \frac{1}{2}\mathscr{H}\partial_x^2 w = 0, \tag{1.3}$$

$$w(0,\cdot) = \psi. \tag{1.4}$$

To make this observation reasonable enough, we rewrite (1.1) into

$$\partial_t v + \frac{1}{2}\mathscr{H}\partial_x^2 v + (f'(v)v^{-1}\partial_x v)v = 0, \tag{1.5}$$

where $f'(v) = p\kappa|v|^{p-1}$. Regarding $f'(v)v^{-1}\partial_x v$ in (1.5) as a time-dependent potential term, we must consider the boundedness of

$$\left\| \int_0^t f'(v)v^{-1}\partial_x v(\tau)d\tau \right\|_\infty \tag{1.6}$$

for large $t > 0$, where $\|\cdot\|_\infty$ denotes the usual L^∞-norm. Since the solution $v(t)$ behaves like $V(t)\psi$ a priori and it is well-known that $\|\partial_x^j V(t)\psi\|_\infty = O(t^{-1/2})$ $(j = 0, 1)$ as $t \to \infty$ for some suitable ψ, one may have an estimate like $\|f'(v)v^{-1}\partial_x v\|_\infty = O(t^{-(p-1)/2})$. It suggests that, if $p > 3$, then (1.6) is expectedly bounded for large t. Now, under the constraint of the power of nonlinearity, i.e., $p > 3$, the solution seemingly exhibits the free asymptotic profile :

$$v(t) \sim V(t)\psi \tag{1.7}$$

as $t \to \infty$. In this paper, we are further interested in the asymptotic expansion of $v(t)$. Precisely speaking, when we put $v(t) - V(t)\psi = R(t)$, we want to determine the profile of $R(t)$. Our aim in this paper is to give a rigorous proof to (1.7) and to specify a precise description of $R(t)$. Our goal is the following.

Theorem 1.1. *Let $p > 3$, and let $v_0 \in H^2 \cap H^{1,1}$ be a real-valued function with $\varepsilon_0 \equiv \|v_0\|_{H^2\cap H^{1,1}}$ small enough. Let $v \in C(\mathbb{R}_+; H^2 \cap H^{1,1})$ be the solution to* (1.1)-(1.2). *Then there exists $\psi \in H^2 \cap H^{1,1}$ and $\psi_1 \in L^2$ which satisfy*

$$\|V(-t)v(t) - \psi\|_{H^1\cap H^{0,1}} = O(t^{-(p-3)/2}) \tag{1.8}$$

and

$$\|V(-t)v(t) - \psi - t^{-(p-3)/2}\psi_1\|_2 = o(t^{-(p-3)/2}) \tag{1.9}$$

as $t \to \infty$. Here, $V(t) = \exp(-t\mathscr{H}\partial_x^2/2)$. Furthermore, we have

$$\hat{\psi}_1(\xi) = \frac{ip\kappa\xi}{\pi(p-3)} \int_{-\pi}^{\pi} \left| e^{i\theta}\hat{\psi}(\xi) + e^{-i\theta}\overline{\hat{\psi}(\xi)} \right|^{p-1} \left\{ e^{i\theta}\hat{\psi}(\xi) - e^{-i\theta}\overline{\hat{\psi}(\xi)} \right\} e^{-i\theta}\, d\theta. \tag{1.10}$$

Note that the data-solution map $V(t)$ is unitary. Then, as we see in (1.9), we know that $v(t) - V(t)\psi \sim t^{-(p-3)/2}V(t)\psi_1$ in L^2 as $t \to \infty$. This implies that $t^{-(p-3)/2}V(t)\psi_1$ plays a role of the second asymptotic term.

We do not discuss the local well-posedness issue on (1.1) in this paper, since there have been already several manuscripts on this problem. For example, when $f(v) = \kappa v^2$ (the case of the original Benjamin-Ono equation), Abdelouhab-Bona-Felland-Saut [2] and Iorio [11] proved the local unique existence of a solution in H^s with $s > 3/2$ by applying the energy method. Koch-Tzvetkov [20] solved this problem for $s > 5/4$ by the combination of the energy method and Strichartz type estimate, which was improved thereafter for $s > 9/8$ by Kenig-Koenig [14]. The global existence of a solution was proved by Ponce [24] for $s \geq 3/2$, Tao [25] for $s \geq 1$, Burq-Planchon [4] for $s > 1/4$ and Ionescu-Kenig [10], Molinet-Pilod [21] for $s = 0$, i.e., in L^2.

Benjamin-Ono equation is an integrable model and exhibits solitary wave solutions with polynomial decay at $x = \pm\infty$. Then one may be curious about spatial decay of the solutions. At this research target, the local existence of solutions in weighted Sobolev spaces was proved by Abdelouhab [1], and global existence was proved by Iorio [12]. The order of weight, which indicates the spatial decay of data, is not chosen as much as we like. Fonseca-Ponce [6] proved the global existence of the solution for the data in the weighted Sobolev space $H^{s,r}$ with $r < 5/2$, where s denotes the regularity and r the order of weight. It is remarkable that $r = 5/2$ is critical for the persistence of the solution, i.e., if there exists a solution $v \in C([0,T]; H^{2,2})$ such that $v(t_1), v(t_2) \in H^{5/2,5/2}$ at distinct times $t_1, t_2 \in [0,T]$, then the integral of the initial data must be 0. This means that the strong spatial decay of the solution is not sustainable in general. Also, in the flame of data with the 0-integral, $r = 7/2$ is critical, i.e., if there exists a solution $v \in C([0,T]; H^{3,3})$ and $\int v(t,x)\ dx = 0$ with $v(t_1), v(t_2), v(t_3) \in H^{7/2,7/2}$ at distinct three times $t_1, t_2, t_3 \in [0,T]$, then $v(t,x) \equiv 0$. Fonseca-Linares-Ponce [5] found that the properties of strong spatial decay are required at three distinct times, which can not be reduced.

We next turn our attention to the nonlinearity : $f(v) = \kappa v^p$ for integer $p \geq 3$ (the case of the generalized Benjamin-Ono equation). Kenig-Ponce-Vega [15] proved the local well-posedness in H^s with $s > 1$ if $p = 3$, $s > 5/6$ if $p = 4$ and $s \geq 3/4$ if $p \geq 5$, but the smallness of data was assumed. Their proof was based on the contraction mapping approach with the application of smoothing estimate of Kato type, and the regularity condition on the data arose from the estimate of maximal function, i.e., $\sup_{0\leq t\leq T} |v(t,x)|$. It was improved by Molinet-Ribaud [22]. They proved the local well-posedness of small-data solutions for $s > 1/2$ if $p = 3$, $s > 1/3$ if $p = 4$ and $s > 1/2 - 1/(p-1)$ (the scale-critical regularity) if $p \geq 5$, due to the refined estimate of the maximal function in the inhomogeneous term. Removing the smallness assumption of data had been serious problem. Vento [26] succeeded in this problem, and obtained local well-posedness for arbitrary large initial data with $s > 1/3$ if $p = 4$ and $s \geq 1/2 - 1/(p-1)$ if $p \geq 5$. His idea to overcome the difficulty is to apply a gauge transform, with which a heavy term in the nonlinearity is included into the linear operator. The existence of small-data-global solution is discussed when $p \geq 5$ in [15] and when $p \geq 4$ in [22]. The methods mentioned above are also applicable, under minor modifications, in the case of $f(v) = \kappa|v|^{p-1}v$ with positive real p.

When $f(v) = \kappa v^p$ with $p \geq 5$, the asymptotic profile of the solution is proved in [15], and it is known that there exists some $\psi \in H^1$ such that $\lim_{t\to\infty} \|V(-t)v(t) - \psi\|_{H^1} = 0$. If we are allowed to use the weighted Sobolev space, it is possible to see the asymptotic profile of the solution for smaller, fractional power of nonlinearity. When $f(v) = \kappa|v|^{p-1}v$ with $p > 3$, Hayashi-Naumkin [7] proved that there exist some $\psi \in H^1 \cap H^{0,1}$ such that $\lim_{t\to\infty} \|V(-t)v(t) - \psi\|_{H^1\cap H^{0,1}} = 0$. To obtain this profile, the L^1-L^∞ estimate of $V(t)$ was applied.

In the case of $p \leq 3$, the nonlinearity of (1.1) affects the solution for large time and so it becomes complicated to detect the asymptotic state of $v(t)$. However, when $p = 3$ (we call it the critical power of nonlinearity), Hayashi-Naumkin [8, 7] proved that the solution asymptotically tends to a modified free state, and found that the modification is resulted from the long-range nonlinearity. It is interesting to determine the second asymptotic term in the case of the critical power of nonlinearity, but the estimate turns out to be rather complicated. We will try this problem in another manuscript.

The problem of seeking for the second asymptotic term of solutions has been considered in some other equations. On nonlinear Schrödinger equations (NLS), refer to Kita [16], Kita-Ozawa [17] for short-range nonlinearity and Kita-Wada [18, 19] for long-range nonlinearity. On Hartree type equations (H), refer to Wada [27]. Theorem 1.1 is the statement on the asymptotic expansion of solutions to the generalized Benjamin-Ono equation. Unlike (NLS) and (H), the time-global estimate of $v(t)$ in the weighted Sobolev space is hard to be obtained. One of the reasons is that the free Schrödinger group $U(t) = \exp(it\partial_x^2/2)$ has a nice factorization $U(t) = M(t)D(t)\mathscr{F}M(t)$, but the operator $V(t)$ does not. Here $M(t)$ denotes the multiplication of $\exp(ix^2/2t)$, $D(t)$ the dilation defined by $D(t)\eta(x) = (it)^{-1/2}\eta(x/t)$ and $\mathscr{F}$ the Fourier transform. This is also because the operator $V(t)xV(-t)$ does not work so well in the nonlinearity of (1.1) and the analogy in (NLS) and (H) is not applied so simply. We will reduce (1.1) into a nonlinear Schrödinger equation (2.1) which contains a nonlinearity described as $P\partial_x f(2\,\mathrm{Re}\,Pu)$, where the operator P denotes the projection onto the positive frequency part. Then the nonlinearity is no longer gauge-invariant with respect to u, and so the modified weight operator $J = U(t)xU(-t) = x + it\partial_x$ is not applied so successfully. To avoid this difficulty, we will make use of the auxiliary operator $I = x + 2t\partial_t\partial_x^{-1}$ (a modified scaling generator), following the idea in [8].

We are going to explain the outline of our strategy. Up to the proof of (2.13) in Proposition 2.4, the standard energy estimates will be provided, and the first asymptotic state φ will be obtained. To prove (2.14) in Proposition 2.4, i.e., to specify the second asymptotic state φ_1, we need, after all, handling the quantity :

$$g(\theta,x) = i\kappa p|e^{i\theta}\hat{\varphi}(x) + e^{-i\theta}\overline{\hat{\varphi}(x)}|^{p-1}\{e^{i\theta}\hat{\varphi}(x) - e^{-i\theta}\overline{\hat{\varphi}(x)}\}$$

with $\theta = tx^2/2 - \pi/4$ and positive real p. The idea on estimates is to use the Fourier series expansion of $g(\theta,x)$ with $\{e^{im\theta}\}_{m\in\mathbb{Z}}$. Then, the asymptotic form of $U(-t)P\partial_x f(2\operatorname{Re}Pu)$ integrated with respect to t comes from the term with $m=1$, whereas the other terms rapidly oscillate and turn out to be negligible as $t\to\infty$.

We close this section by introducing several notations used in this paper. The Lebesgue space L^q $(1\le q\le\infty)$ denotes the set of functions with $\|\varphi\|_q<\infty$, where

$$\|\varphi\|_q = \begin{cases} \left(\int_{\mathbb{R}}|\varphi(x)|^q\,dx\right)^{1/q} & (1\le q<\infty),\\ \text{ess. sup}|\varphi(x)| & (q=\infty).\end{cases}$$

The Sobolev space $W^{1,q}$ denotes the set of functions satisfying

$$\|\varphi\|_{W^{1,q}} = \|\varphi\|_q + \|\partial_x\varphi\|_q < \infty.$$

For integers s and α, the weighted Sobolev space $H^{s,\alpha}$ denotes the set of functions satisfying

$$\|\varphi\|_{H^{s,\alpha}} = \sum_{k=0}^{\alpha}\sum_{j=0}^{s}\|x^k\partial_x^j\varphi\|_2 < \infty.$$

When the weight $\alpha=0$, we simply denote $H^{s,0}$ by H^s. The Fourier transform of a function φ is defined by $\mathscr{F}\varphi(\xi) = \hat{\varphi}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi x}\varphi(x)dx$, and the inverse Fourier transform is defined by $\mathscr{F}^{-1}\varphi(x) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{i\xi x}\varphi(\xi)dx$. The commutator of operators A and B is given by $[A,B]=AB-BA$.

2. Proof

Our proof relies on the reduction of (1.1) into nonlinear Schrödinger equation. To this end, let $\lambda(\xi)=1$ for $\xi>0$ and $\lambda(\xi)=0$ for $\xi<0$. We set $P=\mathscr{F}^{-1}\lambda(\xi)\mathscr{F}$ as a projection to the positive frequency part. We consider the equation

$$\partial_t u - \frac{i}{2}\partial_x^2 u + P\partial_x f(2\operatorname{Re}Pu) = 0, \tag{2.1}$$

$$u(0,\cdot) = v_0. \tag{2.2}$$

Since v_0 is real-valued, $2\operatorname{Re}Pv_0 = v_0$. Therefore, if u is a solution to (2.1)-(2.2), then $v=2\operatorname{Re}Pu$ is a solution to (1.1)-(1.2).

We set $L=\partial_t - i\partial_x^2/2$, $I = x+2t\partial_t\partial_x^{-1}$ (a modified scaling generator) and $J = x+it\partial_x$ (a generator of Galilean transform), where $\partial_x^{-1}u = \int_{-\infty}^x u(x')dx'$. We have

$J = U(t)xU(-t)$. Note that fundamental commutator relations and operator identity like

$$[L, I] \equiv LI - IL = 2L\partial_x^{-1}, \quad [L, J] = 0, \quad I - J = 2tL\partial_x^{-1}$$

hold. Since the nonlinearity in (2.1) is not gauge-invariant with respect to u, the operator I is more convenient than J. The third identity in the above three relations will provide the estimates of $\|J\partial_x^j u\|_2$ $(j = 0, 1)$ through those of $\|I\partial_x^j u\|_2$.

Lemma 2.1. *The following two estimates hold.*
(i) *For $\varphi \in L^1$, we have*

$$\|U(t)\varphi\|_\infty + \|PU(t)\varphi\|_\infty \leq C|t|^{-1/2}\|\varphi\|_1. \tag{2.3}$$

(ii) *For $u \in C(\mathbb{R}_+; H^1 \cap H^{0,1})$, we have*

$$\|u(t)\|_\infty + \|Pu(t)\|_\infty \leq C|t|^{-1/2}\|u(t)\|_2^{1/2}\|Ju(t)\|_2^{1/2}.$$

Proof. (i) The dispersive estimate for the Schrödinger group $U(t)$ is well-known, and that for $PU(t)$ was essentially proved in Hayashi-Naumkin [7]. We will prove the latter for the reader's sake. We have $PU(t)\varphi(x) = \int_{-\infty}^{\infty} K(t, x - y)\varphi(y)\, dy$ with

$$K(t, x) = \frac{1}{2\pi}\int_0^\infty e^{ix\xi - it\xi^2/2}\, d\xi = \frac{1}{2\pi\sqrt{t}} e^{ix^2/2t} \int_{-x/\sqrt{t}}^{\infty} e^{-i\eta^2/2}\, d\eta.$$

Since $|K(t, x)| \leq C|t|^{-1/2}$, we obtain (2.3).

(ii) It follows from (2.3) that

$$\begin{aligned}
\|u(t)\|_\infty + \|Pu(t)\|_\infty &\leq C|t|^{-1/2}\|U(-t)u(t)\|_1 \\
&\leq C|t|^{-1/2}\|U(-t)u(t)\|_2^{1/2}\|xU(-t)u(t)\|_2^{1/2} \\
&= C|t|^{-1/2}\|u(t)\|_2^{1/2}\|Ju(t)\|_2^{1/2}.
\end{aligned}$$

□

Applying Lemma 2.1, we will obtain time-global estimates of $u(t)$.

Lemma 2.2. *Let $p > 3$, and let $v_0 \in H^2 \cap H^{1,1}$ be a real-valued function with $\varepsilon_0 \equiv \|v_0\|_{H^2\cap H^{1,1}}$ small enough. Then the Cauchy problem* (2.1)-(2.2) *has a unique solution $u \in C(\mathbb{R}_+; H^2 \cap H^{1,1})$ which satisfies*

$$\|U(-t)u(t)\|_{H^2\cap H^{1,1}} + \|Iu(t)\|_2 + \|I\partial_x u(t)\|_2 \leq C\varepsilon_0, \tag{2.4}$$

$$\langle t\rangle^{1/2}(\|u(t)\|_{W^{1,\infty}} + \|Pu(t)\|_{W^{1,\infty}}) \leq C\varepsilon_0. \tag{2.5}$$

Proof. The unique existence of time-local solutions to (2.1)-(2.2) is shown due to the energy method (refer to [9]). In what follows, we will concentrate ourselves in proving the time-global estimates. Let $u \in C([0,T); H^2 \cap H^{1,1})$ be a solution to (2.1)-(2.2) and set

$$\begin{aligned}
|||u|||_{2,T} &= \sup_{0\le t<T} \{\|u(t)\|_{H^2}^2 + \|Iu(t)\|_2^2 + \|I\partial_x u(t)\|_2^2\}^{1/2}, \\
|||v|||_{\infty,T} &= \sup_{0\le t<T} \langle t\rangle^{1/2} \|v(t)\|_{W^{1,\infty}}.
\end{aligned}$$

We take the H^2-inner-product of the both sides of (2.1) and $u(t)$. Then, the real part gives

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{H^2}^2 &= -\int_{-\infty}^{\infty} \{(1-\partial_x^2) f'(v)\partial_x v\}(1-\partial_x^2)v\,dx \\
&= -\int_{-\infty}^{\infty} \{[(1-\partial_x^2), f'(v(t))]\partial_x v(t)\}(1-\partial_x^2)v(t)\,dx \\
&\quad -\frac{1}{2}\int_{-\infty}^{\infty} f'(v)\partial_x\{(1-\partial_x^2)v\}^2\,dx.
\end{aligned}$$

Using the commutator estimate $\|[(1-\partial_x^2), f'(v)]\partial_x v\|_2 \le C\|v\|_{W^{1,\infty}}^{p-1}\|v\|_{H^2}$ (see Kato-Ponce [13]) and the integration by parts, we obtain the differential inequality

$$\frac{d}{dt}\|u(t)\|_{H^2}^2 \le C\|v(t)\|_{W^{1,\infty}}^{p-1}\|u(t)\|_{H^2}^2. \tag{2.6}$$

Next, multiplying $I\partial_x$ by (2.1) and using the relation $[L, I] = 2L\partial_x^{-1}$ together with $I\partial_x f(v) = f'(v)I\partial_x v$, we have

$$LI\partial_x u = -P\partial_x\{f'(v)I\partial_x v + f(v)\}.$$

Taking the L^2-inner-product of the both sides of this equality and $I\partial_x u$, we obtain from the real part

$$\begin{aligned}
\frac{d}{dt}\|I\partial_x u(t)\|_2^2 &= -\int_{-\infty}^{\infty} \partial_x\{f'(v)I\partial_x v + f(v)\}I\partial_x v\,dx \\
&= -\int_{-\infty}^{\infty} \{[\partial_x, f'(v)]I\partial_x v\}I\partial_x v\,dx - \frac{1}{2}\int_{-\infty}^{\infty} f'(v)\partial_x(I\partial_x v)^2\,dx \\
&\quad - \int_{-\infty}^{\infty} f'(v)\partial_x v I\partial_x v\,dx \\
&\le C\|v\|_{W^{1,\infty}}^{p-1}\{\|u\|_{H^2}^2 + \|I\partial_x u\|_2^2\}.
\end{aligned} \tag{2.7}$$

Here, we have again used the commutator estimate and the integration by parts. Similarly, and actually more easily, we can obtain

$$\frac{d}{dt}\|Iu(t)\|_2^2 \le C\|v\|_{W^{1,\infty}}^{p-1}\{\|u\|_{H^2}^2 + \|Iu\|_2^2 + \|I\partial_x u\|_2^2\}. \tag{2.8}$$

Adding (2.6)-(2.8) and applying Gronwall inequality, we obtain

$$\|u(t)\|_{H^2}^2 + \|Iu(t)\|_2^2 + \|I\partial_x u\|_2^2 \le \varepsilon_0^2 \exp\left\{C\int_0^t \|v(\tau)\|_{W^{1,\infty}}^{p-1} d\tau\right\},$$

which yields

$$|||u|||_{2,T} \le \varepsilon_0 \exp(C_1 |||v|||_{\infty,T}^{p-1}), \tag{2.9}$$

where the constant C_1 is independent of T. On the other hand, it follows from the relation $J - I = 2tL\partial_x^{-1}$ that

$$\|J\partial_x^j u(t)\|_2 \le \|I\partial_x^j u(t)\|_2 + C\langle t\rangle^{-(p-3)/2} |||v|||_{\infty,T}^{p-1} \|u(t)\|_{H^1}$$

for $j = 0, 1$. Hence, it follows from Lemma 2.1 (ii) that

$$\begin{aligned} |||u|||_{\infty,T} + |||v|||_{\infty,T} &\le C \sup_{0\le t<T} \{\|u(t)\|_{H^1} + \|Ju(t)\|_2 + \|J\partial_x u(t)\|\} \\ &\le C_1(1 + |||v|||_{\infty,T}^{p-1}) |||u|||_{2,T}. \end{aligned} \tag{2.10}$$

We put $\delta = \min\{1, (2C_1)^{-1/(p-1)}\}$ and choose ε_0 such that $\varepsilon_0 \le \delta/8C_1$. Let T^* be the supremum of T such that $|||u|||_{2,T} \le 2\varepsilon_0$ and $|||v|||_{\infty,T} \le \delta$ hold. To prove the lemma, it suffices to show that $T^* = \infty$ by contradiction. We assume that $T^* < \infty$. Then, it follows from (2.9) and (2.10) that $|||u|||_{2,T^*} \le \sqrt{e}\,\varepsilon_0$ and $|||v|||_{\infty,T^*} \le 4C_1\varepsilon_0 \le \delta/2$, which contradict the definition of T^*. □

Corollary 2.3. *Let $p > 3$, and let $v_0 \in H^2 \cap H^{1,1}$ be a real-valued function with $\varepsilon_0 \equiv \|v_0\|_{H^2\cap H^{1,1}}$ small enough. Then the Cauchy problem* (1.1)-(1.2) *has a unique solution $v \in C(\mathbb{R}_+; H^2 \cap H^{1,1})$ which satisfies*

$$\|v(t)\|_{H^2} + \|Iv(t)\|_2 + \|I\partial_x v(t)\|_2 \le C\varepsilon_0, \tag{2.11}$$

$$\langle t\rangle^{1/2} \|v(t)\|_{W^{1,\infty}} \le C\varepsilon_0. \tag{2.12}$$

Proof. Let $u(t)$ be the solution to (2.1)-(2.2) obtained in Lemma 2.2, and let $v(t) = 2\,\mathrm{Re}\, Pu(t)$. Then $v(t)$ satisfies (1.1)-(1.2). By definition, $\hat{v}(t,\xi) = \hat{u}(t,\xi)$ for $\xi > 0$, and $\hat{v}(t,\xi) = \overline{\hat{u}(t,-\xi)}$ for $\xi < 0$. Since

$$\hat{u}(t,0) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u(t,x)\,dx = (2\pi)^{-1/2} \int_{-\infty}^{\infty} v_0(x)\,dx$$

is real-valued, we see that $\hat{v}(t,-0) = \hat{v}(t,+0)$, and hence $v \in C(\mathbb{R}_+; H^2 \cap H^{1,1})$. The estimates (2.11) and (2.12) clearly follow from (2.4) and (2.5) respectively. □

Proposition 2.4. *Let $p > 3$, and let $v_0 \in H^2 \cap H^{1,1}$ with $\varepsilon_0 \equiv \|v_0\|_{H^2\cap H^{1,1}}$ small enough. Let $u \in C(\mathbb{R}_+; H^2 \cap H^{1,1})$ be the solution to* (2.1)-(2.2)*. Then there exists $\varphi \in H^2 \cap H^{1,1}$ and $\varphi_1 \in L^2$ which satisfy*

$$\|U(-t)u(t) - \varphi\|_{H^1\cap H^{0,1}} = O(t^{-(p-3)/2}) \tag{2.13}$$

and

$$\|U(-t)u(t) - \varphi - t^{-(p-3)/2}\varphi_1\|_2 = o(t^{-(p-3)/2}) \tag{2.14}$$

as $t \to \infty$.

Proof. Using Corollary 2.3, we have

$$\begin{aligned}\|U(-t')u(t') - U(-t)u(t)\|_{H^1} &\le C\int_t^{t'} \|U(-\tau)P\partial_x f(v(\tau))\|_{H^1}\, d\tau\\ &\le C\int_t^{t'} \|v(\tau)\|_{W^{1,\infty}}^{p-1}\|v(\tau)\|_{H^2}\, d\tau\\ &\le C\int_t^{t'} \tau^{-(p-1)/2}\, d\tau \le Ct^{-(p-3)/2}\end{aligned}$$

for $1 < t < t'$. From this estimate, we see that there exist $\varphi \in H^1$ such that $\operatorname{s-lim}_{t\to\infty} U(-t)u(t) = \varphi$ in H^1. Moreover, from the relations $[L, I]u = 2L\partial_x^{-1}u$ and $[I, P]\partial_x = 0$, we have

$$\begin{aligned}&\|U(-t')Iu(t') - U(-t)Iu(t)\|_2\\ &\quad\le C\int_t^{t'} \|PI\partial_x f(v) + 2Pf(v)\|_2\, d\tau\\ &\quad\le C\int_t^{t'} \|v(\tau)\|_\infty^{p-1}(\|I\partial_x v(\tau)\|_2 + \|v(\tau)\|_2)\, d\tau \le Ct^{-(p-3)/2}.\end{aligned}$$

From this estimate, we see that $U(-t)Iu(t)$ converges in L^2. Since $I - J = 2tL\partial_x^{-1}$, it follows that

$$\|U(-t)Iu(t) - xU(-t)u(t)\|_2 = 2t\|Pf(v)\|_2 \le Ct^{-(p-3)/2}.$$

Hence, $xU(-t)u(t)$ converges to $x\varphi$ in L^2. Thus we have proved (2.13). Furthermore, since $U(-t)u(t)$ is bounded in $H^2 \cap H^{1,1}$ and converges strongly to φ in $H^1 \cap H^{0,1}$, it converges weakly in $H^2 \cap H^{1,1}$. Therefore, we have $\varphi \in H^2 \cap H^{1,1}$.

We put $w(t) = \mathscr{F}U(-t)u(t)$, so that $\|w(t) - \hat{\varphi}\|_{H^1\cap H^{0,1}} = O(t^{-(p-3)/2})$ by (2.13). We can write

$$v(t) = 2\operatorname{Re} U(t)\mathscr{F}^{-1}\lambda w(t), \quad \partial_x v(t) = -2\operatorname{Im} U(t)\mathscr{F}^{-1}x\lambda w(t).$$

Since $U(t) = M(t)D(t)\mathscr{F}M(t)$ where $M(t)$ denotes the multiplication of $\exp(ix^2/2t)$, $D(t)$ the dilation defined by $D(t)\eta(x) = (it)^{-1/2}\eta(x/t)$ and $\mathscr{F}$ the Fourier transform, expected profiles of $v(t)$ and $\partial_x v(t)$ are

$$v_a(t) = 2\operatorname{Re} M(t)D(t)\lambda\hat{\varphi} \quad \text{and} \quad v_{a1}(t) = -2\operatorname{Im} M(t)D(t)x\lambda\hat{\varphi}$$

respectively. Indeed, we can show

$$\|v(t) - v_a(t)\|_2 + \|\partial_x v(t) - v_{a1}(t)\|_2 \to 0$$

as $t \to \infty$, for we have

$$\begin{aligned}\|v(t) - v_a(t)\|_2 &\le 2\|U(t)\mathscr{F}^{-1}\lambda(w(t) - \hat{\varphi})\|_2 + 2\|U(t)(1 - M(-t))P\varphi\|_2\\ &\le 2\|w(t) - \hat{\varphi}\|_2 + 2\|(1 - M(-t))P\varphi\|_2,\end{aligned}$$

and the right-hand side goes to zero by (2.13) and the Lebesgue dominated convergence theorem. Similarly we can show $\|\partial_x v(t) - v_{a1}(t)\|_2 \to 0$. Clearly, we have $\|v_a(t)\|_\infty + \|v_{a1}(t)\|_\infty = O(t^{-1/2})$. Therefore,

$$\begin{aligned}
&\|\partial_x f(v(t)) - f'(v_a(t))v_{a1}(t)\|_2 \\
&\quad \le \|\{f'(v(t)) - f'(v_a(t))\}\partial_x v(t)\|_2 + \|f'(v_a(t))\{\partial_x v(t) - v_{a1}(t)\}\|_2 \\
&\quad \le C(\|v(t)\|_{W^{1,\infty}} + \|v_a(t)\|_\infty)^{p-1}(\|v(t) - v_a(t)\|_2 + \|\partial_x v(t) - v_{a1}(t)\|_2) \\
&\quad = o(t^{-(p-1)/2}).
\end{aligned}$$

From this estimate, we obtain

$$U(-t)u(t) - \varphi = \int_t^\infty PU(-\tau)f'(v_a(\tau))v_{a1}(\tau)\,d\tau + o(t^{-(p-3)/2})$$

in L^2 as $t \to \infty$. We will find the precise behavior of the integral in the right-hand side. Let $D_0(t)g = g(\cdot/t)$, so that $D(t) = e^{-i\pi/4}t^{-1/2}D_0(t)$. We write

$$U(-t)f'(v_a(t))v_{a1}(t) = t^{-p/2}U(-t)D_0(t)x\lambda(x)g(\theta, x),$$

where

$$g(\theta, x) = i\kappa p|e^{i\theta}\hat{\varphi}(x) + e^{-i\theta}\overline{\hat{\varphi}(x)}|^{p-1}\{e^{i\theta}\hat{\varphi}(x) - e^{-i\theta}\overline{\hat{\varphi}(x)}\}$$

with $\theta = tx^2/2 - \pi/4$. Clearly, g is 2π-periodic with respect to θ, and satisfies the estimate

$$\left|\frac{\partial^{j+k}}{\partial\theta^j\partial x^k}g(\theta, x)\right| \le C|\hat{\varphi}(x)|^{p-k}|\partial_x\hat{\varphi}(x)|^k, \quad 0 \le j + k \le 2,\ 0 \le k \le 1. \tag{2.15}$$

We consider the Fourier series expansion of g:

$$g(\theta, x) = \sum_{n=-\infty}^{\infty} a_{2n+1}(x)e^{i(2n+1)\theta}, \quad a_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\tilde{\theta}}g(\tilde{\theta}, x)\,d\tilde{\theta}. \tag{2.16}$$

Here, we have used the property $g(\theta + \pi, x) = -g(\theta, x)$, so that $a_{2n}(x) = 0$. It follows from (2.15) that $\langle n\rangle^{2-k}|\partial_x^k a_n(x)| \le C|\hat{\varphi}(x)|^{p-k}|\partial_x\hat{\varphi}(x)|^k$, and hence

$$\langle n\rangle^{2-k}\|\langle x\rangle\partial_x^k a_n(x)\|_2 \le C\|\varphi\|_{H^{1,1}}^p \tag{2.17}$$

for $0 \le k \le 1$. Taking (2.16) into account, with the notation $\sqrt{m} = i\sqrt{|m|}$ for $m < 0$, we write

$$
\begin{aligned}
&\mathscr{F}U(-t)f'(v_a(t))v_{a1}(t) \\
&\quad = \sum_{n=-\infty}^{\infty} (-i)^n \frac{t^{-(p-1)/2} e^{int\xi^2/(2n+1)}}{\sqrt{2n+1}} D_0(2n+1)U(1/(2n+1)t)\xi\lambda(\xi)a_{2n+1}(\xi) \\
&\quad = t^{-(p-1)/2}\xi\lambda(\xi)a_1(\xi) \\
&\quad + \sum_{n\neq 0} (-i)^n \frac{t^{-(p-1)/2} e^{int\xi^2/(2n+1)}}{\sqrt{2n+1}} D_0(2n+1)\xi\lambda(\xi)a_{2n+1}(\xi) \\
&\quad + \sum_{n=-\infty}^{\infty} (-i)^n \frac{t^{-(p-1)/2} e^{int\xi^2/(2n+1)}}{\sqrt{2n+1}} D_0(2n+1)(U(1/(2n+1)t) - \mathbb{1})\xi\lambda(\xi)a_{2n+1}(\xi) \\
&\quad = \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t).
\end{aligned}
$$

Integrating by parts, for $n \neq 0$ we can show

$$
\begin{aligned}
\left| \xi \int_t^{\infty} \tau^{-(p-1)/2} e^{i\tau n\xi^2/(2n+1)}\, d\tau \right| &\le C \min\{|\xi|^{-1} t^{-(p-1)/2}; |\xi| t^{-(p-3)/2}\} \\
&\le C t^{-(p-2)/2}.
\end{aligned}
\tag{2.18}
$$

Using (2.17) and (2.18) together with the relation $D_0(m)\xi = m^{-1}\xi D_0(m)$, we obtain

$$
\left\| \int_t^{\infty} \mathrm{II}(\tau)\, d\tau \right\|_2 \le C \sum_{n\neq 0} \langle n \rangle^{-3} \|\varphi\|_{H^{1,1}}^p t^{-(p-2)/2} \le C t^{-(p-2)/2}.
$$

On the other hand, it follows from (2.17) that

$$
\begin{aligned}
\|\mathrm{III}(t)\|_2 &\le t^{-(p-1)/2} \sum_{n=-\infty}^{\infty} \|(U(1/(2n+1)t) - \mathbb{1})\xi\lambda(\xi)a_n(\xi)\|_2 \\
&\le C t^{-(p-1)/2} \sum_{n=-\infty}^{\infty} \|((2n+1)t)^{-1/2} \partial_\xi \{\xi\lambda(\xi)a_{2n+1}(\xi)\}\|_2 \\
&\le C t^{-(p-1)/2} \sum_{n=-\infty}^{\infty} \langle n \rangle^{-3/2} t^{-1/2} \le C t^{-p/2},
\end{aligned}
$$

so that $\| \int_t^{\infty} \mathrm{III}(\tau)\, d\tau \|_2 \le C t^{-(p-2)/2}$. Collecting the estimates above, we obtain

$$
\begin{aligned}
U(-t)u(t) - \varphi &= \int_t^{\infty} \tau^{-(p-1)/2} \mathscr{F}^{-1} \xi\lambda(\xi)a_1(\xi)\, d\tau + o(t^{-(p-3)/2}) \\
&= t^{-(p-3)/2}\varphi_1 + o(t^{-(p-3)/2})
\end{aligned}
$$

with

$$\begin{aligned}\hat{\varphi}_1(\xi) &= \frac{2}{p-3}\xi\lambda(\xi)a_1(\xi)\\ &= \frac{p\kappa i\xi\lambda(\xi)}{(p-3)\pi}\int_{-\pi}^{\pi}\left|e^{i\theta}\hat{\varphi}(\xi)+e^{-i\theta}\overline{\hat{\varphi}(\xi)}\right|^{p-1}\left\{e^{i\theta}\hat{\varphi}(\xi)-e^{-i\theta}\overline{\hat{\varphi}(\xi)}\right\}e^{-i\theta}\,d\theta. \qquad (2.19)\end{aligned}$$

Thus we have proved (2.14). □

Proof of Theorem 1.1. Let $u(t)$ be the solution to (2.1)-(2.2), and let φ, φ_1 be as in Proposition 2.4. Then the solution to (1.1)-(1.2) is given by $v(t) = 2\,\mathrm{Re}\,Pu(t)$. We set $\psi = 2\,\mathrm{Re}\,P\varphi$ and $\psi_1 = 2\,\mathrm{Re}\,P\varphi_1$. Note that

$$\hat{\psi}(\xi) = \begin{cases} \hat{\varphi}(\xi) & (\xi > 0)\\ \overline{\hat{\varphi}(-\xi)} & (\xi < 0). \end{cases}$$

Similarly as in the proof of Corollary 2.3, we have $\hat{\psi}(-0) = \hat{\psi}(+0)$, so that $\psi \in H^2 \cap H^{1,1}$. Since $V(-t)v(t) = 2\,\mathrm{Re}\,PU(-t)u(t)$, the asymptotics (1.8) and (1.9) follow from (2.13) and (2.14) respectively. The equality (1.10) follows from (2.19) □

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References

[1] L. Abdelouhab, *Nonlocal dispersive equations in weighted Sobolev spaces*, Differential Integral Equations, **5** (1992), 307–338.

[2] L. Abdelouhab, J. L. Bona, M. Felland, J. C. Saut, *Nonlocal models for nonlinear dispersive waves*, Phys. D, **40** (1989), 360–392.

[3] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid. Mech., **29** (1967), 559–592.

[4] N. Burq, F. Planchon, *On the well-posedness of the Benjamin-Ono equation*, Math. Ann. **340** (2008), 497–542.

[5] G. Fonseca, F. Linares, G. Ponce, *The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II*, J. Funct. Anal. **262** (2012), 2031–2049.

[6] G. Fonseca, G. Ponce, *The IVP for the Benjamin-Ono equation in weighted Sobolev spaces*, J. Funct. Anal. **260** (2011) 436–459.

[7] N. Hayashi, P. I. Naumkin, *Large time asymptotics of solutions to the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc., **351** (1999), 109–130.

[8] N. Hayashi, P. I. Naumkin, *On the reduction of the modified Benjamin-Ono equation to the cubic derivative nonlinear Schrödinger equation*, Discrete Contin. Dyn. Syst., **8** (2002), 237–255.

[9] N. Hayashi, T. Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, Differential Integral Equations, **7** (1994), 453–461.

[10] A. D. Ionescu, C. E. Kenig, *Global well-posedness of the Benjamin-Ono equation on low regularity spaces*, J. Amer. Math. Soc., **20** (2007), 753–798.

[11] R. J. Iorio, *On the Cauchy problem for the Benjamin-Ono equation*, Comm. Partial Differential Equations, **11** (1986), 1031–1081.

[12] R. J. Iorio, *The Benjamin-Ono equations in weighted Sobolev spaces*, J. Math. Anal. Appl. **157** (1991), 577–590.

[13] T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41** (1988), 891–907.

[14] C. E. Kenig, K. D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Lett., **10** (2003), 879–895.

[15] C. E. Kenig, G. Ponce, L. Vega, *On the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc., **342** (1994), 155–172.

[16] N. Kita, *Sharp L^r asymptotics of the small solutions to the nonlinear Schrödinger equations*, Nonlinear Anal., **52** (2003), 1365–1377.

[17] N. Kita, T. Ozawa, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations with repulsive interactions*, Commun. Contemp. Math., **7** (2005), 167–176.

[18] N. Kita, T. Wada, *Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations in one space dimension*, Funkcial. Ekvac., **45** (2002), 53–69.

[19] N. Kita, T. Wada, *Sharp asymptotics of the small solutions to the nonlinear Schrödinger equations of derivative type*, Differential Integral Equations, **15** (2002), 367–384.

[20] H. Koch, N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation on $H^s(\mathbb{R})$*, Int. Math. Res. Not., **26** (2003), 1449–1464.

[21] L. Molinet, D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in L^2 revisited*, Anal. PDE, **5** (2012), 365–395.

[22] L. Molinet, F. Ribaud, *Well-posedness results for the generalized Benjamin-Ono equation with small initial data*, J. Math. Pures Appl., **83** (2004), 277–311.

[23] H. Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan, **39** (1975), 1082–1091.

[24] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Differential Integral Equations, **4** (1991), 527–542.

[25] T. Tao, *Global well-posedness of the Benjamin-Ono equation on H^1*, J. Hyperbolic Differ. Equ., **1** (2004), 27–49.

[26] S. Vento, *Well-posedness of the generalized Benjamin-Ono equations with arbitrary large initial data in the critical space*, Int. Math. Res. Not. IMRN **2010**, no. 2, 297–319.

[27] T. Wada, *Asymptotic expansion of the solution to the nonlinear Schrödinger equation with nonlocal interaction*, J. Funct. Anal., **180** (2001), 11–30.