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LONG RANGE SCATTERING FOR THE MAXWELL–SCHRÖDINGER SYSTEM IN THE LORENZ GAUGE WITHOUT ANY RESTRICTION ON THE SIZE OF DATA

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Abstract. This paper concerns the scattering theory for the Maxwell–Schrödinger (MS) system in the Lorenz gauge, or more precisely, the existence of the modified wave operators for this system in $\mathbb{R}^{3+1}$ space-time. We construct solutions to the MS system which behave as free Maxwell and Schrödinger waves with prescribed asymptotic states when $t \to \infty$, without any restriction on the size thereof. Since this system belongs to the borderline between the short range case and the long range case, we need modification of phase for the Schrödinger function.

1. Introduction

In this paper, we study the scattering theory for the Maxwell–Schrödinger (MS) system under the Lorenz gauge condition in $3+1$ dimensional space-time, and more precisely the existence of modified wave operators for this system. This system describes the interaction between a charged nonrelativistic quantum mechanical particle and the (classical) electromagnetic field generated by the motion of the particle. Generally, considered Maxwell–Schrödinger system is written as follows:

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + A_e u, \\ \Box A + \nabla(\nabla\cdot A + \partial_t A_e) = J \equiv \operatorname{Im} \overline{u}\nabla_A u, \\ \Box A_e - \partial_t(\nabla\cdot A + \partial_t A_e) = J_e \equiv |u|^2, \end{cases} \tag{1.1}$$

where (A, A_e) is an $\mathbb{R}^{3+1}$-valued function defined in space-time $\mathbb{R}^{3+1}, \nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A^2$ are the covariant gradient and covariant Laplacian respectively, and $\Box = \partial_t^2 - \Delta$ is the d'Alembertian. By using the first equation of the system (1.1), we have the current conservation $\nabla\cdot J + \partial_t J_e = 0$, so that we can regard J and J_e as current density and charge

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density, respectively. The system (1.1) is gauge invariant, namely (1.1) is invariant under the transformation

$$(u, A, A_e) \to (u\exp(-i\theta), A - \nabla\theta, A_e + \partial_t\theta),$$

where θ is an arbitrary real-valued function defined in $\mathbb{R}^{3+1}$. Therefore, the system (1.1) is underdetermined as an evolution system, so we should impose an additional equation called a gauge condition. There are two gauge conditions which are commonly used, one is the Coulomb gauge condition $\nabla \cdot A = 0$, and the other one is the Lorenz gauge condition $\nabla \cdot A + \partial_t A_e = 0$. In this paper we will exclusively study the Lorenz gauge case. Then the system (1.1) can be written as

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + A_e u, \\ \Box A = J, \\ \Box A_e = J_e. \end{cases} \tag{1.2}$$

In three-dimensional case, the MS system (1.1) is known to be locally well-posed both in Lorenz gauge and Coulomb gauge in sufficiently regular spaces (Nakamitsu–Tsutsumi [16], Nakamura–Wada [17]). Guo–Nakamitsu–Strauss [14] proved that the MS system has weak global solutions in the energy space. The MS system has been shown to be globally well-posed in a space smaller than the energy space by Nakamura–Wada [18], and in the energy space by Bejenaru–Tataru [1]. On the other hand, in two-dimensional case, Wada [29] extended the Kato-type smoothing estimates for solutions to the MS system in the Lorenz gauge and proved unique solvability in the energy space.

There is a large amount of research concerning the theory of long-range scattering, or more precisely the existence of modified wave operators for nonlinear equations and systems centering on the Schrödinger equation, especially for the nonlinear Schrödinger equation [2, 15, 21], the Hartree equation [2–5, 19, 20], the Klein–Gordon–Schrödinger system [22–24], the Wave–Schrödinger system [6–8, 11, 25] and the MS system [9, 12, 13, 26, 28]. In scattering theory, we aim to know the interaction between more than one particles, waves etc., by comparing scattering states (unperturbed system, without interaction) and interacting states (perturbed system, with nonlinear interaction) as $t \to \pm\infty$. In the case of the linear Schrödinger equation, we need to distinguish the short range case from the long range case. In the short range case, where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation, ordinary wave operators are expected. In the long range case, unlike the short range case, ordinary wave operators are not expected and have to be replaced with modified wave operators, which include suitable phase corrections in their definition. From this point of view, the MS system (1.1) in $\mathbb{R}^{3+1}$ belongs to the borderline long range case, because of the t^{-1} decay in L^∞ norm of solutions of the wave equation. The two dimensional Klein–Gordon–Schrödinger system and the three dimensional Wave–Schrödinger system also belong to the same case. In the case of the MS system, the existence of modified wave operators was first proved by Tsutsumi [28], and under weaker assumption by Shimomura [26], Ginibre–Velo [12]. In these works, the smallness condition for the scattering data was assumed. Later, Ginibre–Velo [9, 13] removed the smallness condition on the scattering data and proved the existence of modified wave operators for large data. They mainly work under the Coulomb gauge (Shimomura [26] also treated the Lorenz gauge case). So, in the present paper, we consider scattering problem in the Lorenz gauge and prove the existence of modified wave operators. Compared with the Coulomb gauge, the Lorenz gauge is more difficult to treat because of the presence of the term $\nabla \cdot A$, so that we need higher order approximation in the construction of asymptotic function for the Schrödinger part.

To state the main theorem in this paper, we introduce the Fourier transform

$$\hat{u}(\xi) = (Fu)(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\xi}\, u(x)\, dx$$

and the free Schrödinger group $U(t) = \exp(it\Delta/2) = F^* \exp(-it|\xi|^2/2)F$. We also need the free propagator for the Maxwell equations. Let $\omega = (-\Delta)^{1/2}$. For a couple $(A_+, \dot{A}_+)$ of $\mathbb{R}^3$-valued functions and a couple $(A_{e+}, \dot{A}_{e+})$ of real-valued functions, we set

$$A_0(t) = (\cos \omega t)A_+ + \omega^{-1}(\sin \omega t)\dot{A}_+, \tag{1.3}$$

$$A_{e0}(t) = (\cos \omega t)A_{e+} + \omega^{-1}(\sin \omega t)\dot{A}_{e+}. \tag{1.4}$$

The main theorem in this paper is the following:

Theorem 1.1. *Let $u_+, A_+, \dot{A}_+, A_{e+}, \dot{A}_{e+}$ be sufficiently smooth functions decaying at infinity. We assume the support condition $\operatorname{supp} \hat{u}_+ \subset \{\xi : ||\xi| - 1| \geq \eta\}$ for some $0 < \eta < 1$, and the compatibility conditions $\nabla \cdot A_+ + \dot{A}_{e+} = \nabla \cdot \dot{A}_+ + \Delta A_{e+} = 0$. We set $\tilde{J}(t,x) = t^{-4}x|\hat{u}_+(x/t)|^2$ and $\tilde{J}_e(t,x) = t^{-3}|\hat{u}_+(x/t)|^2$, and we define*

$$\tilde{A}_1(t) = -\int_t^\infty dt'\ \omega^{-1} \sin(\omega(t-t'))\tilde{J}(t'), \quad \tilde{A}_{e1}(t) = -\int_t^\infty dt'\ \omega^{-1} \sin(\omega(t-t'))\tilde{J}_e(t'),$$

so that $\Box \tilde{A}_1 = \tilde{J}$ with $(\tilde{A}_1, \partial_t \tilde{A}_1) \to 0$ as $t \to \infty$, and $\Box \tilde{A}_{e1} = \tilde{J}_e$ with $(\tilde{A}_{e1}, \partial_t \tilde{A}_{e1}) \to 0$ as $t \to \infty$ respectively. We define $S(t,x) = \int_1^t dt' \{-x \cdot \tilde{A}_1(t', t'x) + \tilde{A}_{e1}(t', t'x)\}$. Then there *exists a solution* (u, A, A_e) *to* (1.2) *such that*

$$\|u(t) - e^{-iS(t,x/t)}U(t)u_+\|_2 + \|A(t) - A_0(t); \dot{H}^1\| + \|A_e(t) - A_{e0}(t); \dot{H}^1\| \to 0 \tag{1.5}$$

as $t \to \infty$.

We will state our result in precise, stronger but more complicated form as Proposition 4.3 in Section 4.

This paper is organized as follows. The construction of the modified wave operator is performed by the use of the transform called pseudo-conformal inversion. In Section 2, we first introduce this transform and replace the problem at $t = \infty$ with the problem at $t = 0$. Next we introduce a parametrization by phase and complex-amplitude for the Schrödinger part. With these procedures, we change the variables from (u, A, A_e) to new variables $(e^{i\phi}w, B, B_e)$, and derive the auxiliary problem for these variables. After summarizing fundamental estimates, we prove the uniqueness of solutions to the auxiliary system. Section 3 is concentrated on the analysis of Cauchy problem at time zero for the auxiliary system. We first assume a desired asymptotic behaviour of dynamical variables (w, s, B, B_e) for the auxiliary system, where $s = \nabla\phi$, and solve the system by contraction mapping principle. After that, we construct such an asymptotic function from the prescribed scattering state u_+. The existence of modified wave operators for the original system is proved in Section 4.

We conclude this introduction by giving some notation which will be used throughout the paper.

Notation. For any $1 \le r \le \infty$, $L^r \equiv L^r(\mathbb{R}^n)$ denotes the Lebesgue space equipped with the norm $\|u\|_r = (\int_{\mathbb{R}^n} |u(x)|^r \, dx)^{1/r}$ for $r < \infty$, $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |u(x)|$. For any nonnegative integer k and for $1 < r < \infty$, $H^k_r \equiv H^k_r(\mathbb{R}^n)$ denotes the Sobolev space:

$$H^k_r = \{u \in \mathscr{S}'(\mathbb{R}^n) : \|u; H^k_r\| = \sum_{|\alpha|\le k} \|\partial^\alpha u\|_r \sim \|\langle\omega\rangle^k u\|_r < \infty\},$$

where $\omega = (-\Delta)^{1/2}$ and $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$. The subscript r in H^k_r will be omitted in the case $r = 2$. We also use the homogeneous Sobolev space $\dot{H}^k_r$ with norm $\|u; \dot{H}^k_r\| = \|\omega^k u\|_r$. In particular it will be understood that $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. In addition we shall use the notation

$$\ddot{H}^k = \dot{H}^1 \cap \dot{H}^k \quad \text{for } \forall\, k \ge 1.$$

For any Banach space $X \subset \mathscr{S}'(\mathbb{R}^n)$, we denote by $\langle x\rangle X$ the space defined by

$$\langle x\rangle X = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle x\rangle^{-1} u \in X\}.$$

For any interval I and for any Banach space X we denote by $\mathscr{C}(I,X)$ (resp. $\mathscr{C}_w(I,X)$) the space of strongly (resp. weakly) continuous functions from I to X. For a given interval I, we denote by (X,f) the set

$$(X,f) = \{u \in \mathscr{C}(I,X) : \|u(t);X\| \le f(t) \quad \forall\, t \in I\}, \tag{1.6}$$

where X is a Banach space and $f \in \mathscr{C}(I,\mathbb{R}^+)$. For real numbers a and b we use the notation $a \vee b = \mathrm{Max}(a,b)$ and $a \wedge b = \mathrm{Min}(a,b)$.

2. Preliminaries

2.1. **Formulation of the problem.** We will first perform a change of variables, which transform the problem at $t=\infty$ into the problem at $t=0$. This transform is called pseudo-conformal inversion, and is well adapted to the study of the asymptotic behaviour in time of solutions to (1.2).

Let $U(t) = \exp\left(i(t/2)\Delta\right)$ be the free Schrödinger group. We use the decomposition

$$U(t) = M(t)D(t)FM(t),$$

where $M(t) = e^{ix^2/2t}$ is the operator of multiplication, F is the Fourier transform and $D(t)$ is the dilation operator

$$(D(t)f)(x) = (it)^{-3/2}(D_0(t)f)(x) = (it)^{-3/2}f(x/t), \tag{2.1}$$

normalized to be unitary in L^2. Taking this factorization into account, we change the variable for the Schrödinger function from u to v, or its parametrization by a complex amplitude w and a phase ϕ, according to

$$u(t) = M(t)D(t)\overline{v}(1/t) = M(t)D(t)\exp(i\phi(1/t))\overline{w}(1/t). \tag{2.2}$$

Correspondingly, we change the variable for the electro-magnetic potentials from A and A_e to B and B_e, according to

$$A(t) = -t^{-1}D_0(t)B(1/t), \quad A_e(t) = -t^{-1}D_0(t)B_e(1/t). \tag{2.3}$$

Substituting (2.2) and (2.3) into the first equation of (1.2) and commuting the Schrödinger operator with MD, we obtain

$$\begin{aligned}&\{(i\partial_t + (1/2)\Delta_A - A_e)u\}(t)\\ &\quad = t^{-2}M(t)D(t)\{\overline{(i\partial_{t'} + (1/2)\Delta_{B(t')} - \check{B}(t') + t'^{-1}B_e(t'))v(t')}\}_{t'=(1/t)}.\end{aligned}$$

Here, for an $\mathbb{R}^3$-valued function f of space-time we define

$$\check{f}(t,x) = t^{-1}x \cdot f(t,x). \tag{2.4}$$

Then the first equation of (1.2) becomes

$$(i\partial_t + (1/2)\Delta_B - \check{B} + t^{-1}B_e)v = 0.$$

Next, we rewrite the Maxwell part. We write the second and the third equations of (1.2) by the associated integral equation, namely

$$A = A_0 + A' \equiv A_0 - \int_t^\infty dt'\ \omega^{-1}\sin(\omega(t-t'))J(t'), \tag{2.5}$$

$$A_e = A_{e0} + A'_e \equiv A_{e0} - \int_t^\infty dt'\ \omega^{-1}\sin(\omega(t-t'))J_e(t'). \tag{2.6}$$

Here, we recall that A_0 and A_{e0} are solutions of the free wave equations given by (1.3) and (1.4) respectively, and $\omega = (-\Delta)^{1/2}$. In order to ensure the condition $\nabla \cdot A + \partial_t A_e = 0$, we assume that

$$\nabla \cdot A_+ + \dot{A}_{e+} = \nabla \cdot \dot{A}_+ + \Delta A_{e+} = 0. \tag{2.7}$$

Furthermore, let $P = \mathbb{1} - \nabla\Delta^{-1}\mathrm{div}$ be the projector on divergence free vector fields. Then, by the current conservation we have

$$(\mathbb{1} - P)J = \nabla\Delta^{-1}\nabla \cdot J = -\nabla\Delta^{-1}\partial_t J_e.$$

Integrating by parts, we can rewrite the equation for A as

$$A = A_0 - \int_t^\infty dt'\ \omega^{-1}\sin(\omega(t-t'))PJ(t') - \int_t^\infty dt'\ \nabla\omega^{-2}\cos(\omega(t-t'))J_e(t'). \tag{2.8}$$

We put

$$M_1 = -x|v|^2, \quad M_2 = \mathrm{Im}\,\overline{v}\nabla_B v = \mathrm{Im}\,\overline{w}\nabla_K w, \quad M_e = |v|^2. \tag{2.9}$$

Then we have

$$\begin{aligned} J(t) &= t^{-3}D_0(t)\{(x|v(t')|^2) - t^{-1}\,\mathrm{Im}\,\overline{v}(t')(\nabla - iB(t'))v(t')\}_{t'=1/t} \\ &= -t^{-3}D_0(t)(M_1 + t^{-1}M_2)(1/t), \end{aligned} \tag{2.10}$$

and

$$J_e(t) = t^{-3}D_0(t)|v(1/t)|^2 = t^{-3}D_0(t)M_e(1/t). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.8) and (2.6) respectively, and letting $t' = t\nu$ we obtain

$$\begin{aligned} A'(t) &= -t^{-1}D_0(t)\int_1^\infty d\nu\ \nu^{-3}\omega^{-1}\sin(\omega(\nu-1))PD_0(\nu)(M_1+(t\nu)^{-1}M_2)(1/t\nu) \\ &\quad - t^{-1}D_0(t)\int_1^\infty d\nu\ \nu^{-3}\nabla\omega^{-2}\cos(\omega(\nu-1))D_0(\nu)M_e(1/t\nu), \\ A_e'(t) &= t^{-1}D_0(t)\int_1^\infty d\nu\ \nu^{-3}\omega^{-1}\sin(\omega(\nu-1))D_0(\nu)M_e(1/t\nu). \end{aligned}$$

From these equalities, we change the variables from (A', A_e') to (B', B_e') by

$$A' = -t^{-1}D_0(t)B'(1/t), \quad A_e' = -t^{-1}D_0(t)B_e'(1/t), \tag{2.12}$$

and similarly from (A_0, A_{e0}) to (B_0, B_{e0}) for the homogeneous part, in accordance with (2.3). Therefore

$$\begin{aligned} &B' = B_1 + B_2, \quad B_1 = F_1(PM_1) + E_1(M_e), \\ &B_2 = tF_2(PM_2), \quad B_e' = B_3 = -F_1(M_e), \end{aligned} \tag{2.13}$$

with M_1, M_2 and M_e defined by (2.9), and with F_j, $j = 1, 2$, and E_1 defined by

$$\begin{aligned} F_j(M) &= \int_1^\infty \frac{d\nu}{\nu^{j+2}}\frac{\sin\omega(\nu-1)}{\omega}D_0(\nu)M(t/\nu), \\ E_1(M) &= \int_1^\infty \frac{d\nu}{\nu^3}\frac{\nabla\cos\omega(\nu-1)}{\omega^2}D_0(\nu)M(t/\nu). \end{aligned} \tag{2.14}$$

Hence the system (1.2) becomes

$$\begin{cases} i\partial_t v = -(1/2)\Delta_B v + (\check{B} - t^{-1}B_e)v, \\ B = B_0 + B_1 + B_2, \\ B_e = B_{e0} + B_3. \end{cases} \tag{2.15}$$

The first equation of (2.15) is also parametrized by (w, ϕ) as (2.2). In terms of (w, ϕ), this equation becomes

$$(i\partial_t + \partial_t\phi + (1/2)\Delta_K - \check{B} + t^{-1}B_e)w = 0, \tag{2.16}$$

where $\Delta_K = \nabla_K^2 = (\nabla - iK)^2$, $K := B + s$ with $s := \nabla\phi$. Since we have only one equation for two functions w and ϕ, we impose an equation for the phase function ϕ, which should be specified later.

In the Maxwell part, $B_1 = B_1(w) = B_1(w,w)$ and $B_3 = B_3(w) = B_3(w,w)$ are explicitly defined quadratic form of w. Here,

$$B_1(w_1,w_2) = -F_1(Px\,\mathrm{Re}\,\overline{w}_1 w_2) + E_1(\mathrm{Re}\,\overline{w}_1 w_2), \quad B_3(w_1,w_2) = -F_1(\mathrm{Re}\,\overline{w}_1 w_2). \tag{2.17}$$

On the other hand, B_2 is determined by

$$B_2 = \mathscr{B}_2(w,w,s+B) \quad \text{with} \quad \mathscr{B}_2(w_1,w_2,K) \equiv tF_2(P\,\mathrm{Im}\,\overline{w}_1 \nabla_K w_2). \tag{2.18}$$

Here, the right-hand side contains B_2 through B, so that (2.18) is an equation for B_2, and that we should regard B_2 as a dynamical variable.

There is a large amount of freedom in the choice of ϕ, and we choose ϕ so as to get rid of the long range terms in (2.16) coming from the interaction. All the terms coming from the covariant Laplacian are expected (and will turn out) to be short range, if we assume the support condition for $w_+ = \overline{Fu_+}$. The contribution of B_2 to $\check{B}$ is also short range because of the factor t in (2.18). The terms $\check{B}_1$ in $\check{B}$ and $t^{-1}B_3$ are also of long range. Let $\chi \in \mathscr{C}^\infty(\mathbb{R}^3,\mathbb{R})$, $0 \le \chi \le 1$, $\chi(\xi) = 1$ for $|\xi| \le 1$, $\chi(\xi) = 0$ for $|\xi| \ge 2$, and let $0 < \beta < 1$. We define

$$\chi^L \equiv F^* \chi(\cdot t^\beta) F, \quad \chi^S = \mathbb{1} - \chi^L.$$

Then we can split $\check{B}_1$ and $t^{-1}B_3$ into short range and long range parts as

$$\begin{cases} \check{B}_1 = \check{B}_1^S + \check{B}_1^L, \ \check{B}_1^S = \chi^S \check{B}_1, \ \check{B}_1^L = \chi^L \check{B}_1, \\ B_3 = B_3^S + B_3^L, \ B_3^S = \chi^S B_3, \ B_3^L = \chi^L B_3. \end{cases} \tag{2.19}$$

$\check{B}_1^L$ and $t^{-1}B_3^L$ are of long range, but smoother than $\check{B}_1$ and $t^{-1}B_3$ themselves. Corresponding to the fact that $\check{B}_0 - t^{-1}B_{e0}$ and $\check{B}_2$ are regarded as short range, we denote

$$B_m^S = \check{B}^S - t^{-1} B_e^S, \tag{2.20}$$

where $\check{B}^S := \check{B}_0 + \check{B}_1^S + \check{B}_2, B_e^S := B_{e0} + B_3^S$. Note that $\check{B}_0$ and $t^{-1}B_{e0}$ may not be short range separately. As the equation for ϕ, we can impose

$$\partial_t \phi = \check{B}_1^L - t^{-1} B_3^L. \tag{2.21}$$

Thus, we obtain a closed system of equations for (w,s,B_2), namely

$$\begin{cases} i\partial_t w = Hw \equiv -(1/2)\Delta_K w + B_m^S w, \\ \partial_t s = \nabla \check{B}_1^L - t^{-1} \nabla B_3^L, \\ B_2 = \mathscr{B}_2(w,w,K) \end{cases} \tag{2.22}$$

with $\check{B}_1^L$, B_3^L and $\mathscr{B}_2$ defined by (2.19) and (2.18). Here, in the first and the third equations of (2.22), the phase ϕ appears only through its gradient $s = \nabla\phi$, so that we have replaced the equation (2.21) for ϕ by that for s, by taking the gradient of the both-sides of (2.21).

The problem of constructing the wave operators is transformed into the problem of solving (2.22) with suitable asymptotic forms of (w, s, B_1, B_2, B_3) as $t \to 0$. We will solve this transformed problem in Section 3.

In order to solve the problem, apart from the system (2.22), it is useful to consider also a partly linearized system for (w, B_2), namely

$$\begin{cases} i\partial_t w' = Hw', \\ B_2' = \mathscr{B}_2(w, w, K) \end{cases} \tag{2.23}$$

for new variables (w', B_2'), where K still corresponds to (w, B_2). We do not need to introduce a new variable s', since s is explicitly determined by the second equation of (2.22) as an explicit function of w.

2.2. **Estimates for the Maxwell part.** We summarize properties of the homogeneous term (A_0, A_{e0}) of the Maxwell part (A, A_e), defined by (1.3) and (1.4). From the Lorenz gauge condition $\nabla \cdot A_0 + \partial_t A_{e0} = 0$, we easily see that $x \cdot A_0 - tA_{e0}$ satisfies the homogeneous wave equation as well as A_0 and A_{e0}, that is,

$$\Box(x \cdot A_0 - tA_{e0}) = 0$$

with initial data $(x \cdot A_0 - tA_{e0})|_{t=0} = x \cdot A_+$ and $\partial_t(x \cdot A_0 - tA_{e0})|_{t=0} = x \cdot \dot{A}_+ - A_{e+}$. We shall need the dilation operator

$$Q = t\partial_t + x \cdot \nabla + \mathbb{1}. \tag{2.24}$$

Since we have the relation $Q\partial_t^j\nabla^k = \partial_t^j\nabla^k(Q - j - k)$, QA_0 and QA_{e0} also satisfy the homogeneous wave equations, so that QA_0 and QA_{e0} can be written as

$$(QA_0)(t) = (\cos\omega t)(1 + x \cdot \nabla)A_+ + \omega^{-1}(\sin\omega t)(2 + x \cdot \nabla)\dot{A}_+, \tag{2.25}$$

$$(QA_{e0})(t) = (\cos\omega t)(1 + x \cdot \nabla)A_{e+} + \omega^{-1}(\sin\omega t)(2 + x \cdot \nabla)\dot{A}_{e+}. \tag{2.26}$$

If we change variables from A_0 and A_{e0} to B_0 and B_{e0} according to (2.3), we obtain

$$(x \cdot A_0 - tA_{e0})(t) = -t^{-1}D_0(t)(\check{B}_0 - t^{-1}B_{e0})(1/t), \tag{2.27}$$

and

$$\nabla^k Q^j A_0(t) = (-1)^{j+1} t^{-1-k} D_0(t)(\nabla^k (t\partial_t)^j B_0)(1/t), \tag{2.28}$$

$$\nabla^k Q^j A_{e0}(t) = (-1)^{j+1} t^{-1-k} D_0(t)(\nabla^k (t\partial_t)^j B_{e0})(1/t). \tag{2.29}$$

We introduce some decay estimates of A_0 and A_{e0}. For that purpose, we need assumptions for initial data. We shall say that a pair of functions $(\mathscr{A}, \dot{\mathscr{A}})$ defined on $\mathbb{R}^3$ satisfies the condition (D) if

$$\mathscr{A} \in L^2, \quad \nabla^2 \mathscr{A} \in L^1, \quad \omega^{-1}\dot{\mathscr{A}} \in L^2, \quad \nabla \dot{\mathscr{A}} \in L^1.$$

Moreover, we shall say that $(\mathscr{A}, \dot{\mathscr{A}})$ satisfies the condition $(\mathrm{D})_{jk}$ if $(x\cdot\nabla)^{j'}\nabla^k(\mathscr{A}, \dot{\mathscr{A}})$ satisfies (D) for $0 \le j' \le j$.

Lemma 2.1. *Let j, k be nonnegative integers. Let (A_0, A_{e0}) be a pair of solutions of the homogeneous wave equations defined by* (1.3) *and* (1.4) *satisfying the compatibility conditions* (2.7)*. Let $(A_+, \dot{A}_+)$, $(A_{e+}, \dot{A}_{e+})$, and $(x\cdot A_+, x\cdot \dot{A}_+ - A_{e+})$ satisfy the condition $(\mathrm{D})_{jk}$. Then (A_0, A_{e0}) satisfies the following estimates:*

$$\|(Q^{j'}\nabla^k A_0)(t)\|_r \vee \|(Q^{j'}\nabla^k A_{e0})(t)\|_r \vee \|(Q^{j'}\nabla^k (x\cdot A_0 - tA_{e0}))(t)\|_r \le b_0 t^{-1+2/r} \tag{2.30}$$

for $0 \le j' \le j$, for $2 \le r \le \infty$ and for all $t > 0$.

Let B_0, B_{e0} and $\check{B}_0$ be defined by (2.3) *and* (2.4)*. Then B_0, B_{e0} and $\check{B}_0$ satisfy the following estimates:*

$$\|\partial_t^j \nabla^k B_0(t)\|_r \vee \|\partial_t^j \nabla^k B_{e0}(t)\|_r \vee \|\partial_t^j \nabla^k (\check{B}_0 - t^{-1}B_{e0})(t)\|_r \le b_0 t^{-j-k+1/r} \tag{2.31}$$

for $2 \le r \le \infty$ and for all $t > 0$.

Proof. For the proof of (2.30), see [27]. If $j > 0$, we also use the relation $Q\partial_t^j\nabla^k = \partial_t^j\nabla^k(Q - j - k)$. The estimate (2.31) follows from (2.27), (2.28), (2.29) and (2.30). □

We next give some estimates for various components of B_1 and B_3 expressed by (2.19). It follows immediately therefrom that

$$\|\omega^m \check{B}_1^S\|_2 \le t^{\beta(p-m)}\|\omega^p \check{B}_1^S\|_2 \le t^{\beta(p-m)}\|\omega^p \check{B}_1\|_2, \tag{2.32}$$

$$\|\omega^m (t^{-1}B_3^S)\|_2 \le t^{\beta(p-m)-1}\|\omega^p B_3^S\|_2 \le t^{\beta(p-m)-1}\|\omega^p B_3\|_2 \tag{2.33}$$

for $m \le p$, and similarly

$$\|\omega^m \check{B}_1^L\|_2 \le (2t^{-\beta})^{(m-p)}\|\omega^p \check{B}_1^L\|_2 \le (2t^{-\beta})^{(m-p)}\|\omega^p \check{B}_1\|_2, \tag{2.34}$$

$$\|\omega^m (t^{-1}B_3^L)\|_2 \le (2t^{-\beta})^{(m-p)}\|\omega^p (t^{-1}B_3^L)\|_2 \le (2t^{-\beta})^{(m-p)}\|\omega^p (t^{-1}B_3)\|_2 \tag{2.35}$$

for $m \geq p$.

We now estimate $F_j(M)$ and $E_j(M)$ defined by (2.14). From (2.14) it follows that

$$\omega F_j(M) = F_{j+1}(\omega M), \quad \omega E_j(M) = E_{j+1}(\omega M), \tag{2.36}$$

$$\partial_t F_j(M) = F_{j+1}(\partial_t M), \quad \partial_t E_j(M) = E_{j+1}(\partial_t M), \tag{2.37}$$

and from the identity $[x, f(\omega)] = f'(\omega)\omega^{-1}\nabla$ together with $\nabla P = 0$, it follows that

$$\begin{aligned} x \cdot F_j(PM) &= F_{j-1}(x \cdot PM), \\ x \cdot E_j(M) &= E_{j-1}(x \cdot M) - \omega^{-2}\nabla \cdot E_j(M) + F_{j-1}(M) - F_j(M). \end{aligned} \tag{2.38}$$

In order to estimate F_j and E_j, we define

$$I_j(f)(t) = \int_1^\infty d\nu\ \nu^{-j-3/2} f(t/\nu) \tag{2.39}$$

for any $j \in \mathbb{R}$ and for any nonnegative function f in $\mathbb{R}^+$. Note that $I_j(f)$ is decreasing for j and increasing for f, namely, if $j \geq k$ and $f(t) \leq g(t)$ a.e. in I, then $I_j(f) \leq I_k(g)$. Let $\alpha \in \mathbb{R}$ with $\alpha + j + 1/2 \geq 0$. If $\overline{f}(t) = t^{-\alpha} f(t)$ satisfies $\int_0^t dt'\, t'^{-1}\overline{f}(t') \leq c\overline{f}(t)$, then we have $I_j(f)(t) \leq cf(t)$. Indeed, by the change of the variable, we have

$$I_j(f)(t) = t^{-j-1/2} \int_0^t dt'\ t'^{j+\alpha-1/2}\overline{f}(t') \leq t^\alpha \int_0^t dt'\, t'^{-1}\overline{f}(t') \leq cf(t).$$

Lemma 2.2. *For any $m, j \in \mathbb{R}$ the following estimates hold:*

(1) About $\|\omega^m F_j(M)\|_2$ and $\|\omega^m E_j(M)\|_2$, we obtain

$$\begin{aligned} \|\omega^m F_j(M)\|_2 &\leq cI_{j+m-2}(\|\omega^{m-1}M\|_2 \wedge \|\omega^m M\|_2), \\ \|\omega^m E_j(M)\|_2 &\leq cI_{j+m-2}(\|\omega^{m-1}M\|_2). \end{aligned} \tag{2.40}$$

(2) About $\|\omega^m x \cdot F_j(PM)\|_2$ and $\|\omega^m x \cdot E_j(M)\|_2$, we obtain

$$\begin{aligned} \|\omega^m x \cdot F_j(PM)\|_2 &\leq cI_{j+m-3}(\|\langle x\rangle \omega^{m-1}M\|_2), \\ \|\omega^m x \cdot E_j(M)\|_2 &\leq c(I_{j+m-3}(\|\langle x\rangle \omega^{m-1}M\|_2) + I_{j+m-3}(\|\omega^{m-2}M\|_2)). \end{aligned} \tag{2.41}$$

(3) For any r, r_1 with $2 \leq r \leq 4$ and $3/r_1 = 2 + 1/r$, we obtain

$$\|F_j(M)\|_r \vee \|E_j(M)\|_r \leq c \int_1^\infty d\nu\ (\nu - 1)^{-1+2/r} \nu^{-j+1/r} \|M(t/\nu)\|_{r_1}. \tag{2.42}$$

Proof. The proof of the estimates for $F_j(M)$, see Lemma 3.6 in [13]. We can prove the estimates for $E_j(M)$ in the same way, taking account of the estimate $|\cos|\xi|(\nu - 1)| \leq 1$ and the monotonicity of $I_j(f)$. □

Hereafter, in all of the estimates in this paper, we denote by C a positive constant whose specific value is not required but depends on the asymptotic functions (w_a, K_a) through the available norms. Absolute constants will be in general omitted, except in special arguments where they are explicitly needed, in which case they are denoted by c.

2.3. **Uniqueness of Solutions for the Auxiliary System.** We shall derive a uniqueness result for the solutions of (2.22) under suitable assumptions on their behaviour at time zero. We begin with some estimates of the difference of two solutions of the system (2.23). For two functions or operators $f_i, i = 1, 2$, we define $f_\pm = (1/2)(f_1 \pm f_2)$, so that $f_1 = f_+ + f_-, f_2 = f_+ - f_-$ and $(fg)_\pm = f_+ g_\pm + f_- g_\mp$. Let $(w'_i, B'_{2i}), i = 1, 2$, be a pair of solutions of the linearized system (2.23) associated with a pair $(w_i, s_i, B_{2i}), i = 1, 2$. Then, taking the difference for the equations for (w'_i, B'_{2i}), we see that (w'_-, B'_{2-}) satisfies the equations

$$\begin{cases} i\partial_t w'_- = H_+ w'_- + H_- w'_+, \\ B'_{2-} = 2\mathscr{B}_2(w_+, w_-, K_+) - tF_2(PK_-(|w_+|^2 + |w_-|^2)), \end{cases} \tag{2.43}$$

where

$$H_+ = -(1/2)\Delta_{K_+} + (1/2)K_-^2 + B^S_{m+}, \tag{2.44}$$

$$H_- = iK_- \cdot \nabla_{K_+} + (i/2)(\nabla \cdot K_-) + B^S_{m-}. \tag{2.45}$$

By definition, B_{1-} and B_{3-} satisfy

$$B_{1-} = 2B_1(w_+, w_-), \quad \check{B}^{S/L}_{1-} = 2\check{B}^{S/L}_1(w_+, w_-), \tag{2.46}$$

$$B_{3-} = 2B_3(w_+, w_-), \quad B^{S/L}_{3-} = 2B^{S/L}_3(w_+, w_-). \tag{2.47}$$

If $s_i, i = 1, 2$, satisfy the second equation of (2.22), then

$$\partial_t s_- = \nabla \check{B}^L_{1-} - t^{-1}\nabla B^L_{3-}. \tag{2.48}$$

By the following lemma, we can estimate the difference of two solutions of the linearized system (2.23). The estimates of w'_- are stated in differential form for brevity, but should be understood in integral form, in the same way as the conservation laws of Proposition 4.1 in [13].

Lemma 2.3. *Let $0 < \beta < 1$. Let $I = (0, \tau]$ with $0 < \tau \le 1$. Let $h_1 \in \mathscr{C}(I, \mathbb{R}^+)$ satisfy*

$$\int_0^\tau dt\ t^{-3/2} h_1(t) < \infty. \tag{2.49}$$

Let $w_i, i = 1, 2$ satisfy $w_i \in L^\infty(I, H^3)$, $xw_i \in L^\infty(I, H^2)$ and

$$\|\langle x\rangle w_-(t)\|_2 \le Ch_1(t) \tag{2.50}$$

for all $t \in I$.

(1) *Let $B_1(w_i), B_3(w_i)$, $i = 1, 2$, be defined by (2.17). Then $B_1(w_i), B_3(w_i) \in (\mathscr{C} \cap L^\infty)(I, \ddot{H}^4)$, $t\nabla \check{B}_1(w_i) \in (\mathscr{C} \cap L^\infty)(I, \ddot{H}^2)$ and B_{1-}, B_{3-} satisfy the estimates*

$$\|\nabla B_{1-}\|_2 \le CI_0(\|\langle x\rangle w_-\|_2), \tag{2.51}$$

$$\|\nabla\nabla \cdot B_{1-}\|_2 \le CI_1(\|\nabla w_-\|_2) + CI_1(\|w_-\|_2), \tag{2.52}$$

$$\|\nabla B_{3-}\|_2 \le CI_0(\|w_-\|_2), \tag{2.53}$$

$$\|\nabla \check{B}_{1-}\|_2 \le Ct^{-1} I_{-1}(\|\langle x\rangle w_-\|_2). \tag{2.54}$$

(2) *Let s_i satisfy the second equation of (2.22) with $w = w_i, i = 1, 2$, with $s_i(t_0) \in \ddot{H}^2$ for some $t_0 \in I$. Then $s_i \in \mathscr{C}(I, \ddot{H}^2), s_- \in \mathscr{C}(I, H^2)$ and s_- satisfies the estimates*

$$\|\nabla^k \partial_t s_-\|_2 \le Ct^{-1-k\beta} I_{-1}(\|\langle x\rangle w_-\|_2) \tag{2.55}$$

for $k = 0, 1, 2$. Furthermore, $s_-(t)$ has an L^2 limit as $t \to 0$.

(3) *Let B_0 and B_{e0} satisfy (2.31) for $0 \le j, k, l + k \le 1$ and $r = \infty$. Let in addition $t\partial_t w_i \in L^\infty(I, H^1), i = 1, 2$. Let B_{2i} satisfy $B_{2i} \in L^\infty(I, \ddot{H}^2), t\partial_t B_{2i} \in L^\infty(I, \dot{H}^1), t\nabla \check{B}_{2i} \in L^\infty(I, \dot{H}^1), i = 1, 2$. Let (w'_i, B'_{2i}), $i = 1, 2$ be solutions of the linearized system (2.23) satisfying the same conditions as (w_i, B_{2i}), $i = 1, 2$. Then the following estimates hold:*

$$\begin{aligned} |\partial_t \|w'_-\|_2| \le\ & C(\|\nabla \cdot s_-\|_2 + (1 - \ln t)(\|s_-\|_3 + \|\nabla B_-\|_2) \\ & + t^\beta(\|\nabla \check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2) + \|\nabla \check{B}_{2-}\|_2), \end{aligned} \tag{2.56}$$

$$\begin{aligned} |\partial_t \|xw'_-\|_2| \le\ & \|\nabla_{K_+} w'_-\|_2 + C(\|\nabla \cdot s_-\|_2 + (1 - \ln t)(\|s_-\|_3 + \|\nabla B_-\|_2) \\ & + t^\beta(\|\nabla \check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2) + \|\nabla \check{B}_{2-}\|_2), \end{aligned} \tag{2.57}$$

$$\begin{aligned} |\partial_t \|\nabla_{K_+} w'_-\|_2| \le\ & C(t^{-1}(\|w'_-\|_2 + \|w'_-\|_3) + t^{-1}\|s_-\|_2 + (1 - \ln t)\|\nabla s_-\|_2 \\ & + \|\nabla\nabla \cdot B_-\|_2 + \|\nabla\nabla \cdot s_-\|_2 + t^{-1}\|\nabla B_-\|_2 \\ & + \|\nabla \check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2 + (1 - \ln t)\|\nabla \check{B}_{2-}\|_2), \end{aligned} \tag{2.58}$$

$$\|\nabla B'_{2-}\|_2 \le CtI_1((1 - \ln t)\|w_-\|_2 + \|s_-\|_2 + \|\nabla B_-\|_2), \tag{2.59}$$

$$\|\nabla \check{B}'_{2-}\|_2 \le CI_0((1 - \ln t)\|\langle x\rangle w_-\|_2 + \|s_-\|_2 + \|\nabla B_-\|_2). \tag{2.60}$$

Proof. **Part (1).** From (2.13), (2.14), (2.39) and (2.40), and from Lemma 3.2 in [13], we obtain the estimates

$$\|\omega^{m+1}B_1(w_i)\|_2 \le I_m(\|\omega^m x|w_i|^2\|_2) + I_m(\|\omega^m|w_i|^2\|_2) \le C, \tag{2.61}$$

$$\|\omega^{m+1}B_3(w_i)\|_2 \le I_m(\|\omega^m|w_i|^2\|_2) \le C \tag{2.62}$$

for $0 \le m \le 3$. Similarly from (2.41), we obtain

$$\begin{aligned}\|\omega^{m+1}\check{B}_1(w_i)\|_2 &\le t^{-1}(I_{m-1}(\|\langle x\rangle\omega^m x|w_i|^2\|_2) + I_{m-1}(\|\langle x\rangle\omega^m|w_i|^2\|_2) \\ &\quad + I_{m-1}(\|\omega^{m-1}|w_i|^2\|_2)) \\ &\le Ct^{-1} \end{aligned}\tag{2.63}$$

for $1 \le m \le 2$. These estimates show that $B_1(w_i), B_3(w_i)$ and $\nabla\check{B}_1(w_i)$ belong to the class stated in part (1) of the lemma. The estimates (2.51)–(2.54) for the differences follow immediately from (2.40), (2.41), (2.46) and (2.47). From (2.49), (2.50) and the change of variable, it follows that $I_{-1}(\|\langle x\rangle w_-\|_2) \le Ct^{1/2}$, and that the right-hand side of (2.54) is finite.

We note that we can similarly obtain the estimates

$$\|\omega^{m+1}\partial_t B_1(w_i)\|_2 \le I_{m+1}(\|\omega^m x\partial_t|w_i|^2\|_2) + I_{m+1}(\|\omega^m\partial_t|w_i|^2\|_2) \le Ct^{-1}, \tag{2.64}$$

$$\|\omega^{m+1}\partial_t B_3(w_i)\|_2 \le I_{m+1}(\|\omega^m\partial_t|w_i|^2\|_2) \le Ct^{-1} \tag{2.65}$$

for $0 \le m \le 1$, to be used in the proof of part (3).

Part (2). From (2.62) and (2.63), we obtain

$$\|\omega^{m+1}\partial_t s_i\|_2 = \|\omega^{m+1}(\nabla\check{B}_1^L(w_i) - t^{-1}\nabla B_3^L(w_i))\|_2 \le Ct^{-1} \tag{2.66}$$

for $0 \le m \le 1$. Integrating (2.66) over time, we obtain

$$\|\omega^{m+1}s_i\|_2 \le C(1 - \ln t) \tag{2.67}$$

for $0 \le m \le 1$ and for all $t \in I$. These estimates show the property $s_i \in \mathscr{C}(I, \ddot{H}^2)$. The estimate (2.55) follows from (2.34), (2.35) and (2.54). Since $\|\partial_t s_-(t)\|_2 \le Ct^{-1/2}$, the limit $s_-(0) = \lim_{t\to 0} s_-(t)$ exists in L^2.

Part (3). We note that from (2.61)–(2.67) we obtain

$$\begin{aligned}&\|B_{1+}\|_\infty \vee \|\nabla B_{1+}\|_\infty \vee t\|\nabla\check{B}_{1+}\|_\infty \vee t\|\partial_t B_{1+}\|_\infty \\ &\quad \vee \|B_{3+}\|_\infty \vee \|\nabla B_{3+}\|_\infty \vee t\|\partial_t B_{3+}\|_\infty \le C, \end{aligned}\tag{2.68}$$

$$t\|\partial_t s_+\|_\infty \le C, \quad \|s_+\|_\infty \vee \|\nabla s_+\|_6 \le C(1 - \ln t). \tag{2.69}$$

We first estimate $\|w'_-\|_2$. From (2.43), by using (2.32) and (2.33), we obtain

$$\begin{aligned}
|\partial_t\|w'_-\|_2| \le \|H_- w'_+\|_2 &= \| -i(s_- + B_-)\cdot\nabla_{K_+}w'_+ + (i/2)(\nabla\cdot(s_- + B_-))w'_+ \\
&\quad + (\check{B}^S_- - t^{-1}B^S_{e-})w'_+\|_2 \\
&\le \|s_-\|_3\|\nabla_{K_+}w'_+\|_6 + \|B_-\|_6\|\nabla_{K_+}w'_+\|_3 \\
&\quad + (\|\nabla\cdot s_-\|_2 + \|\nabla\cdot B_-\|_2)\|w'_+\|_\infty \\
&\quad + t^\beta(\|\nabla\check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2)\|w'_+\|_\infty + \|\check{B}_{2-}\|_6\|w'_+\|_3,
\end{aligned}$$

which implies (2.56) from the assumptions on w_i, B_0, B_{e0}, B_{2i} and the estimate (2.68).

We next estimate $\|xw'_-\|_2$. From (2.32), (2.33), (2.43) and the commutation relation $[x, H_+] = \nabla_{K_+}$, we similarly obtain

$$\begin{aligned}
|\partial_t\|xw'_-\|_2| &\le \|\nabla_{K_+}w'_-\|_2 + \|xH_-w'_+\|_2 \\
&\le \|\nabla_{K_+}w'_-\|_2 + \|s_-\|_3\|x\nabla_{K_+}w'_+\|_6 + \|B_-\|_6\|x\nabla_{K_+}w'_+\|_3 \\
&\quad + (\|\nabla\cdot s_-\|_2 + \|\nabla\cdot B_-\|_2)\|xw'_+\|_\infty \\
&\quad + t^\beta(\|\nabla\check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2)\|xw'_+\|_\infty + \|\check{B}_{2-}\|_6\|xw'_+\|_3,
\end{aligned}$$

which implies (2.57).

We next estimate $\|\nabla_{K_+}w'_-\|_2$. We take the covariant gradient of (2.43) to obtain

$$\begin{aligned}
i\partial_t\nabla_{K_+}w'_- &= -(1/2)\nabla_{K_+}\Delta_{K_+}w'_- + ((1/2)K_-^2 + B^S_{m+})\nabla_{K_+}w'_- \\
&\quad + (\partial_t K_+ + \nabla B^S_{m+} + K_-\nabla K_-)w'_- \\
&\quad + iK_-\cdot\nabla^2_{K_+}w'_+ + i(\nabla K_-)\cdot\nabla_{K_+}w'_+ + (i/2)(\nabla\cdot K_-)\nabla_{K_+}w'_+ \\
&\quad + (i/2)(\nabla\nabla\cdot K_-)w'_+ + (\nabla B^S_{m-})w'_+ + B^S_{m-}\nabla_{K_+}w'_+,
\end{aligned}$$

so that we have

$$\begin{aligned}
|\partial_t\|\nabla_{K_+}w'_-\|_2| &\le \|(\partial_t K_+ + \nabla B^S_{m+})w'_-\|_2 + \|K_-\cdot\nabla^2_{K_+}w'_+\|_2 \\
&\quad + \|\nabla K_-\|_2(\|\nabla_{K_+}w'_+\|_\infty + \|K_-w'_-\|_\infty) \\
&\quad + \|\nabla\nabla\cdot K_-\|_2\|w'_+\|_\infty + \|\nabla\check{B}_{2-}\|_2(\|\nabla_{K_+}w'_+\|_3 + \|w'_+\|_\infty) \\
&\quad + (\|\nabla\check{B}_{1-}\|_2 + t^{-1}\|\nabla B_{3-}\|_2)(t^\beta\|\nabla_{K_+}w'_+\|_\infty + \|w'_+\|_\infty). \qquad (2.70)
\end{aligned}$$

We next estimate the first two terms in the right-hand side of (2.70).

$$\begin{aligned}\|(\partial_t K_+ + \nabla B^S_{m+})w'_-\|_2 &\le \|\partial_t(s_+ + B_0 + B_{1+}) + \nabla(\check{B}_0 + \check{B}_{1+} - t^{-1}B_{e0} - t^{-1}B_{3+})\|_\infty \|w'_-\|_2 \\ &\quad + \|\partial_t B_{2+} + \nabla \check{B}_{2+}\|_6 \|w'_-\|_3 \\ &\le Ct^{-1}(\|w'_-\|_2 + \|w'_-\|_3), \end{aligned} \tag{2.71}$$

$$\begin{aligned}\|K_-\nabla^2_{K_+} w'_+\|_2 &\le \|s_-\|_3(\|\nabla^2 w'_+\|_6 + \|\nabla(s_+ + B_{2+})\|_6\|w'_+\|_\infty) \\ &\quad + \|s_-\|_2(\|K_+\|_\infty\|\nabla w'_+\|_\infty + (\|\nabla(B_0 + B_{1+})\|_\infty + \|K_+\|^2_\infty)\|w'_+\|_\infty) \\ &\quad + \|B_-\|_6(\|\nabla^2 w'_+\|_3 + \|K_+\|_\infty\|\nabla w'_+\|_3 + \|\nabla(B_0 + B_{1+})\|_\infty\|w'_+\|_3 \\ &\quad + \|\nabla(s_+ + B_{2+})\|_6\|w'_+\|_6 + \|K_+\|^2_\infty\|w'_+\|_3) \\ &\le C((1 - \ln t)\|s_-\|_3 + t^{-1}(\|s_-\|_2 + \|\nabla B_-\|_2)). \end{aligned} \tag{2.72}$$

We substitute (2.71) and (2.72) into (2.70), and estimate the other terms similarly. Then we obtain (2.58).

We finally estimate B'_{2-}. From (2.40), (2.41) and (2.43), we obtain

$$\begin{aligned}\|\nabla B'_{2-}\|_2 &\le tI_1(\|w_-\|_2\|\nabla_{K_+}w_+\|_\infty + \|s_-\|_2\|w_+\|^2_\infty + \|B_-\|_6\|w_+\|^2_6), \\ \|\nabla \check{B}'_{2-}\|_2 &\le I_0(\|\langle x\rangle w_-\|_2\|\nabla_{K_+}w_+\|_\infty + \|s_-\|_2\|w_+\|_\infty\|\langle x\rangle w_+\|_\infty \\ &\quad + \|B_-\|_6\|\langle x\rangle w_+\|_6\|w_+\|_6),\end{aligned}$$

which implies (2.59) and (2.60). □

We apply Lemma 2.3 to obtain a uniqueness result for the nonlinear system (2.22) with initial condition at time zero.

Proposition 2.4. *Let $0 < \beta < 1$. Let $I = (0, \tau]$ with $0 < \tau \le 1$. Let $h_1 \in \mathscr{C}(I, \mathbb{R}^+)$ be such that $\overline{h}_1(t) = (t^{-2\beta} \vee t^{-1/2})h_1(t)$ be non decreasing and satisfy*

$$\int_0^t dt'\ t'^{-1}\overline{h}_1(t') \le c\overline{h}_1(t) \tag{2.73}$$

for some $c > 0$ and for all $t \in I$. Let B_0 and B_{e0} satisfy (2.31) for $r = \infty$ and $0 \le j, k, j+k \le 1$. Let $(w_i, s_i, B_{2i}), i = 1, 2,$ be two solutions of the system (2.22) such that

$$\begin{cases} w_i \in L^\infty(I, H^3), \quad xw_i \in L^\infty(I, H^2), \quad t\partial_t w_i \in L^\infty(I, H^1), \\ B_{2i} \in L^\infty(I, \ddot{H}^2), \quad t\partial_t B_{2i} \in L^\infty(I, \dot{H}^1), \quad t\nabla\check{B}_{2i} \in L^\infty(I, \dot{H}^1). \end{cases}$$

Assume in addition that $s_-(0) = 0$ *and that*

$$\|\langle x\rangle w_-(t)\|_2 \le Ch_1(t) \tag{2.74}$$

for all $t \in I$*. Then* $(w_1, s_1, B_{21}) = (w_2, s_2, B_{22})$*.*

Proof. Since (2.73) implies (2.49), we can apply Lemma 2.3. Especially, we see that $s_-(t)$ is convergent in L^2 as $t \to 0$, and more strongly, the limit exists in H^2. Indeed, from (2.55), (2.73) and (2.74), we obtain $\|\nabla^2 \partial_t s_-\|_2 \le Ct^{-1-2\beta}h_1(t)$, which ensures the existence of H^2-limit.

We first prove the proposition for τ small enough. Once we have proved the uniqueness for small τ, we can prove the uniqueness for general τ by similar but more standard arguments. We set

$$v_0 = \|\langle x\rangle w_-\|_2, \quad v_1 = \|\nabla_{K_+} w_-\|_2, \quad V_0 = \sup_{t\in I} h_1(t)^{-1} v_0(t).$$

From Lemma 2.3, especially (2.51)–(2.55) and from (2.73) with $(w_i', B_{2i}') = (w_i, B_{2i})$, we obtain

$$\|\nabla B_{1-}\|_2 \vee \|\nabla B_{3-}\|_2 \le CV_0 h_1, \tag{2.75}$$

$$\|\nabla\nabla\cdot B_{1-}\|_2 \le C\int_0^t dt'\ t'^{-1} v_1(t') + CV_0 h_1, \tag{2.76}$$

$$\|\nabla \check{B}_{1-}\|_2 \le CV_0 t^{-1} h_1, \tag{2.77}$$

$$\|\nabla^k s_-\|_2 \le CV_0 t^{-k\beta} h_1 \tag{2.78}$$

for all $t \in I$ and for $k = 0, 1, 2$. In the proof of (2.76), we have used the estimate

$$I_1(\|\nabla w_-\|_2) = \int_0^t dt'\ t'^{-1}(t'/t)^{3/2}\|\nabla w_-(t')\|_2 \le \int_0^t dt'\ t'^{-1}\|\nabla w_-(t')\|_2.$$

The time integral in the right-hand side converges because of the estimate

$$\|\nabla w_-\|_2 \le (\|w_-\|_2\|\Delta w_-\|_2)^{1/2} \le C(V_0 h_1)^{1/2}.$$

We obtain (2.76) if we replace the ordinary derivative by the covariant derivative in that integral, by adding a harmless term $V_0 h_1$. On the other hand, from (2.59), (2.75) and (2.78), we obtain

$$\|\nabla B_{2-}\|_2 \le CV_0 t(1 - \ln t)h_1 + CtI_1(\|\nabla B_{2-}\|_2).$$

From the assumptions on B_{2i}, it follows that $B_{2-} \in L^\infty(I, \dot{H}^1)$. Therefore, if τ is sufficiently small, the last term in the right-hand side of can be absorbed in the left-hand side, so that we can derive

$$\|\nabla B_{2-}\|_2 \le CV_0 t(1 - \ln t)h_1. \tag{2.79}$$

Substituting (2.75) and (2.79) into (2.60), we obtain

$$\|\nabla \check{B}_{2-}\|_2 \le CV_0(1 - \ln t)h_1. \tag{2.80}$$

Substituting (2.75)–(2.80) into (2.58), we obtain

$$|\partial_t v_1| \le C\left(V_0(t^{-1} + t^{-2\beta})h_1 + \int_0^t dt'\ t'^{-1} v_1(t') + t^{-1}(V_0 h_1 v_1)^{1/2}\right).$$

Setting $v_1 = V_0 v$, $g = C(t^{-1} + t^{-2\beta})h_1$ and $l = Ct^{-1}h_1^{1/2}$, we find that this inequality becomes

$$|\partial_t v| \le g + \int_0^t dt'\ t'^{-1} v(t') + l v^{1/2}. \tag{2.81}$$

Integrating (2.81) in t and applying the Schwarz inequality for the last term, we obtain

$$\begin{aligned} v(t) &\le G(t) + C\int_0^t dt'\ t'^{-1}(t - t')v(t') + \int_0^t dt'\ (lv^{1/2})(t') \\ &\le G(t) + Ctz(t) + z(t)^{1/2}\left(\int_0^t dt'\ t' l^2(t')\right)^{1/2} \\ &\le G(t) + Ctz(t) + Cz(t)^{1/2} h_1^{1/2}(t), \end{aligned}$$

where $G(t) := \int_0^t dt'\ g(t')$, $z(t) := \int_0^t dt'\ t'^{-1} v(t')$, so that $v = t\partial_t z$. Therefore we have

$$\partial_t z \le t^{-1} G + Cz + l z^{1/2}. \tag{2.82}$$

Applying Lemma 2.3 in [10] to (2.82), we obtain

$$\begin{aligned} z &\le e^{Ct}\left\{\int_0^t dt'\ l(t') + \left(\int_0^t dt'\ t'^{-1} G(t')\right)^{1/2}\right\}^2 \\ &\le C(1 + t^{1-2\beta})h_1(t). \end{aligned}$$

Substituting this result into (2.82), we obtain

$$v_1 \le CV_0(1 + t^{1-2\beta})h_1(t). \tag{2.83}$$

Substituting (2.75), (2.77)–(2.80) and (2.83) into (2.56) and (2.57), we obtain

$$\partial_t v_0 \le CV_0(t^{-\beta} + t^{-1+\beta})h_1(t).$$

Integrating this inequality in time, we obtain

$$V_0 \leq CV_0(\tau^{1-\beta} + \tau^{\beta}).$$

From this inequality, we see that $V_0 = 0$ for τ small enough. By definition of V_0 together with (2.79), we can conclude $\|\langle x\rangle w_-(t)\|_2 = \|\nabla B_{2-}(t)\|_2 = 0$ for $0 < t \leq \tau$. □

3. Cauchy problem at Time Zero for the Auxiliary system

In this section, we aim to solve the Cauchy problem at time zero for auxiliary system (2.22). We choose a set of asymptotic functions $\boldsymbol{w}_a = (w_a, s_a, B_{1a}, B_{2a}, B_{3a})$ which are expected to be suitable asymptotic forms of (w, s, B_1, B_2, B_3) at $t = 0$, and we try to construct solutions of (2.22) that are asymptotic to $\boldsymbol{w}_a$ in a suitable sense at $t = 0$. Note that although B_1 and B_3 are explicit functions of w, we do not assume that $B_{1a} = B_1(w_a)$, $B_{3a} = B_3(w_a)$ and $s_a = s(w_a)$; we only assume that the difference of both sides decay sufficiently fast (see (3.14) and the assumption (A3), especially (3.31), (3.32) and (3.34)). We also define

$$B_a = B_0 + B_{1a} + B_{2a}, \quad B_{ea} = B_{e0} + B_{3a}, \quad K_a = s_a + B_a, \quad B_{ma} = \check{B}_a - t^{-1}B_{ea}. \tag{3.1}$$

In order to solve the auxiliary system (2.22) with the previous asymptotic behaviour at $t = 0$, we define the difference variables

$$(q, \sigma, G_1, G_2, G_3) \equiv (w - w_a, s - s_a, B_1 - B_{1a}, B_2 - B_{2a}, B_3 - B_{3a}). \tag{3.2}$$

We also define

$$G = G_1 + G_2, \quad L = \sigma + G, \tag{3.3}$$

so that

$$B = B_a + G, \quad B_e = B_{ea} + G_3, \quad K = K_a + L. \tag{3.4}$$

We define in addition

$$Q_{K_1}(K_2, \cdot) = K_2 \cdot \nabla_{K_1} + (1/2)(\nabla \cdot K_2), \tag{3.5}$$

so that

$$\Delta_{K_1+K_2} = \Delta_{K_1} - 2iQ_{K_1}(K_2, \cdot) - K_2^2. \tag{3.6}$$

The separation of B_a, B_{ea} and of G, G_3 into short range and long range parts follows the same pattern as that of B and B_e, namely

$$\begin{cases} \check{B}^S_{1a} = \chi^S \check{B}_{1a},\ \check{B}^L_{1a} = \chi^L \check{B}_{1a}, \\ \check{B}^S_a = \check{B}_0 + \check{B}^S_{1a} + \check{B}_{2a}, \\ B^S_{3a} = \chi^S B_{3a},\ B^L_{3a} = \chi^L B_{3a}, \end{cases} \tag{3.7}$$

$$\begin{cases} \check{G}^S_1 = \chi^S \check{G}_1, \qquad \check{G}^L_1 = \chi^L \check{G}_1, \\ G^S_3 = \chi^S G_3, \qquad G^L_3 = \chi^L G_3, \\ \check{G}^S = \check{G}^S_1 + \check{G}_2 \end{cases} \tag{3.8}$$

with χ^S and χ^L defined by (2.19). By the relation $[\chi(t^\beta\omega), x] = -t^\beta \chi'(t^\beta\omega)\omega^{-1}\nabla$, we have

$$\check{G}^S_1 = t^{-1+\beta}\chi'(t^\beta\omega)\omega^{-1}\nabla \cdot G_1 + t^{-1}x \cdot (\chi^S G_1), \tag{3.9}$$

$$\check{G}^L_1 = -t^{-1+\beta}\chi'(t^\beta\omega)\omega^{-1}\nabla \cdot G_1 + t^{-1}x \cdot (\chi^L G_1). \tag{3.10}$$

Using the definitions (3.2)–(3.6), we rewrite the auxiliary system (2.22) in terms of the difference variables. We take $(q, \sigma, G_1, G_2, G_3)$ as independent dynamical variables and we consider G_1, G_3 and σ with initial condition $\sigma(0) = 0$ as functions of q. The auxiliary system for $(q, \sigma, G_1, G_2, G_3)$ then becomes

$$\begin{cases} i\partial_t q = Hq - \widetilde{R}_1, \\ \partial_t \sigma = \nabla \check{G}^L_1 - t^{-1}\nabla G^L_3 - R_2, \\ G_1 = B_1(q, 2w_a + q) - R_3, \\ G_2 = \mathscr{B}_2(q, 2w_a + q, K) - tF_2(PL|w_a|^2) - R_4, \\ G_3 = B_3(q, 2w_a + q) - R_5, \end{cases} \tag{3.11}$$

where

$$\widetilde{R}_1 = R_1 - H_1 w_a, \tag{3.12}$$

$$H = (-1/2)\Delta_K + B^S_m,$$

$$H_1 = iQ_{K_a}(L, \cdot) + (1/2)L^2 + \check{G}^S - t^{-1}G^S_3, \tag{3.13}$$

and the remainders R_j, $1 \le j \le 5$ are defined by

$$\begin{cases} R_1 = i\partial_t w_a + (1/2)\Delta_{K_a} w_a - B^S_{ma} w_a, \\ R_2 = \partial_t s_a - \nabla \check{B}^L_{1a} + t^{-1}\nabla B^L_{3a}, \\ R_3 = B_{1a} - B_1(w_a), \\ R_4 = B_{2a} - \mathscr{B}_2(w_a, K_a), \\ R_5 = B_{3a} - B_3(w_a), \end{cases} \tag{3.14}$$

where $B^S_{ma} = \check{B}^S_a - t^{-1}B^S_{ea}$. In the equation for G_2 in (3.11), we have used the identity

$$\mathscr{B}_2(w, K) = \mathscr{B}_2(w_a, K_a) + \mathscr{B}_2(q, 2w_a + q, K) - tF_2(PL|w_a|^2). \tag{3.15}$$

The remainders R_j, $1 \le j \le 5$, express the accuracy of the set of asymptotic solutions $(w_a, s_a, B_{1a}, B_{2a}, B_{3a})$ to the original system (2.22).

The resolution of the new auxiliary system (3.11) proceeds in two steps.

Firstly, we solve the system (3.11) for $(q, \sigma, G_1, G_2, G_3)$ tending to zero as $t \to 0$ with general boundedness properties of $w_a, s_a, B_{1a}, B_{2a}, B_{3a}$ and general decay assumptions on the remainders $R_j, 1 \le j \le 5$, as $t \to 0$. For that purpose, we shall need a partly linearized version of the system (3.11) for the independent dynamical variables (q, G_2). With

$$w' = w_a + q', \quad B'_2 = B_{2a} + G'_2, \tag{3.16}$$

the linearized version of (3.11) corresponding to (2.23) becomes

$$\begin{cases} i\partial_t q' = Hq' - \widetilde{R}_1, \\ G'_2 = \mathscr{B}_2(q, 2w_a + q, K) - tF_2(PL|w_a|^2) - R_4. \end{cases} \tag{3.17}$$

Again we do not need new variables G'_1, G'_3 and σ, since G_1, G_3 and σ (with the initial condition $\sigma(0) = 0$) are explicit functions of q. We solve the linearized system (3.17) for (q', G'_2) with given (q, G_2), thereby defining a mapping $\Gamma : (q, G_2) \mapsto (q', G'_2)$. We then prove that the mapping Γ is a contraction in a suitable function space $X(I)$ for $I = (0, \tau]$ and τ sufficiently small.

Secondly, we construct asymptotic functions $(w_a, s_a, B_{1a}, B_{2a}, B_{3a})$ satisfying the assumption needed for the previous step. For that purpose, one solves the auxiliary system (3.11) approximately by an iteration procedure. Unlike the Coulomb gauge case, in the Lorenz gauge, the term $\nabla \cdot B_0$ appears in H. This term loses one power of t for both one time or space derivatives (see Lemma 2.1), which makes our analysis more difficult. To overcome this difficulty, we need higher order approximation for w. It turns out that the third approximation for w and the second approximation for (s, B_1, B_2, B_3) are sufficient. (In the Coulomb gauge, the second approximation for w is sufficient.)

We now define the spaces where to look for solutions of the auxiliary system. For any interval $I \subset (0, 1]$, we denote by $X_0(I)$ the Banach space

$$\begin{aligned} X_0(I) = \{(w, B_2) : w &\in \mathscr{C}(I, H^3) \cap \mathscr{C}^1(I, H^1), \\ xw \in \mathscr{C}(I, H^2) &\cap \mathscr{C}^1(I, L^2), B_2, \check{B}_2 \in \mathscr{C}(I, \dot{H}^1 \cap \dot{H}^2) \cap \mathscr{C}^1(I, \dot{H}^1)\}, \end{aligned} \tag{3.18}$$

where $\check{B}_2$ is defined by (2.4). In order to take into account the time decay of the norms of the variables q and G_2 (see (3.2)) as t tends to zero, we introduce a function $h \in \mathscr{C}(I, \mathbb{R}^+)$ where $I = (0, \tau]$ with $0 < \tau \leq 1$, such that the function $\overline{h}(t) = t^{-3/2}h(t)$ be non decreasing in I and satisfy

$$\int_0^t dt'\ t'^{-1}\overline{h}(t') \leq c\ \overline{h}(t) \tag{3.19}$$

for some $c > 0$ and for all $t \in I$. A typical example of such an h is $h(t) = t^{3/2+\lambda}$, with $\lambda > 0$, which satisfies (3.19) with $c = \lambda^{-1}$. The function $h(t)$ will characterize the time decay of $\|q(t)\|_2$ as $t \to 0$. We then define the Banach space

$$X(I) = \{(q, G_2) \in X_0(I) : \|(q, G_2); X(I)\| < \infty\} \tag{3.20}$$

with

$$\begin{aligned} \|(q, G_2); X(I)\| = \sup_{t \in I} h(t)^{-1}\{&t^{-1/2}\|q(t)\|_2 \vee \|\langle x\rangle q(t)\|_2 \vee t(\|\langle x\rangle \partial_t q(t)\|_2 \vee \|\langle x\rangle \Delta q(t)\|_2) \\ &\vee t^{3/2}(\|\nabla\partial_t q(t)\|_2 \vee \|\nabla\Delta q(t)\|_2) \vee t^{-1/2}\|\nabla G_2(t)\|_2 \\ &\vee t^{1/2}(\|\nabla^2 G_2(t)\|_2 \vee \|\nabla\partial_t G_2(t)\|_2 \vee \|\nabla\check{G}_2(t)\|_2) \\ &\vee t^{3/2}(\|\nabla^2\check{G}_2(t)\|_2 \vee \|\nabla\partial_t\check{G}_2(t)\|_2)\}. \end{aligned}$$

In Subsection 3.2, it turns out that the choice $h(t) = t^2(1 - \ln t)^6$ will suffice. This type of space has previously been used in the Coulomb gauge case in [13], but in the Lorenz gauge case, we need more subtle argument, so that we characterize that $\|q(t)\|_2$ decays faster than $\|\langle x\rangle q(t)\|_2$ as $t \to 0$.

In some stages of the first step, we need smallness conditions on τ which depend on (w_a, s_a, B_a, B_{ea}). Such conditions are called asymptotic region conditions. They are essentially imposed in order to prove Γ to be a contraction mapping, but they are also used to eliminate higher order terms with respect to the dynamical variables and make the estimates simpler.

3.1. **Existence of Solutions of the auxiliary system.** We now turn to the first step. We need the general assumptions made on (w_a, s_a, B_a, B_{ea}), listed as (A1), (A2) and (A3) below. The final result will need all the assumptions (A1), (A2) and (A3), but some intermediate ones will need only part of them. In these assumptions, let $I_0 = (0, \tau_0]$ with $0 < \tau_0 \leq 1$.

(A1) (boundedness properties of w_a)

w_a satisfies the following properties:

$$w_a \in (\mathscr{C} \cap L^\infty)(I_0, H^3), \qquad xw_a \in (\mathscr{C} \cap L^\infty)(I_0, H^2), \tag{3.21}$$

$$t^{1/2}\partial_t w_a \in (\mathscr{C} \cap L^\infty)(I_0, H^2), \qquad t^{1/2}x\partial_t w_a \in (\mathscr{C} \cap L^\infty)(I_0, H^1). \tag{3.22}$$

In order to state (A2), we recall that $B_a = B_0 + B_{1a} + B_{2a}$, $B_{ea} = B_{e0} + B_{3a}$, $K_a = s_a + B_a$, and that $\check{B}_a^S = \check{B}_0 + \check{B}_{1a}^S + \check{B}_{2a}$, $B_{ea}^S = B_{e0} + B_{3a}^S$ (see (3.7)).

(A2) (boundedness properties of (s_a, B_a, B_{ea}))

$s_a, B_a, B_{ea} \in \mathscr{C}(I_0, H_\infty^1)$ with sufficient additional regularity, and the following estimates hold for all $t \in I_0$:

$$\|K_a\|_\infty \leq C(1 - \ln t), \tag{3.23}$$

$$\begin{aligned} &\|\partial_t K_a\|_\infty \vee \|\nabla K_a\|_\infty \vee t\|\nabla\nabla K_a\|_\infty \\ &\qquad \vee t\|\nabla\partial_t K_a\|_\infty \vee t^2\|\nabla\partial_t\nabla K_a\|_\infty \leq Ct^{-1}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\|\nabla s_a\|_\infty \vee \|\nabla\nabla\cdot s_a\|_3 \vee \|\nabla(B_{1a} + B_{2a})\|_\infty \\ &\vee \|\nabla B_{3a}\|_\infty \vee t(\|\nabla\partial_t s_a\|_\infty \vee \|\nabla\partial_t\nabla\cdot s_a\|_3 \\ &\qquad \vee \|\nabla\partial_t(B_{1a} + B_{2a})\|_\infty \vee \|\nabla\partial_t B_{3a}\|_\infty) \leq Ct^{-1/2}, \end{aligned} \tag{3.25}$$

$$\|\nabla B_{ma}\|_\infty \vee t\|\nabla\partial_t B_{ma}\|_\infty \leq Ct^{-1}, \tag{3.26}$$

$$\|B_{ma}^S\|_\infty \vee t^{1/2}\|\nabla B_{ma}^S\|_\infty \vee t\|\partial_t B_{ma}^S\|_\infty \leq Ct^{-1/2}. \tag{3.27}$$

Note that by Lemma 2.1, B_0 and B_{e0} satisfy the assumptions made on B_a and B_{ea} under suitable assumptions on $(A_+, \dot{A}_+)$ and $(A_{e+}, \dot{A}_{e+})$.

(A3) (decay properties of the remainders R_j)

The remainders R_j, $1 \le j \le 5$, satisfy the following estimates for all $t \in I_0$:

$$\|R_1\|_2 \le r_1 t^{-1/2} h(t), \tag{3.28}$$

$$(\|\langle x\rangle R_1\|_2 \le)\|\langle x\rangle \partial_t R_1; L^1((0,t], L^2)\| \le r_1 t^{-1} h(t), \tag{3.29}$$

$$(\|\nabla R_1\|_2 \le)\|\nabla \partial_t R_1; L^1((0,t], L^2)\| \le r_1 t^{-3/2} h(t), \tag{3.30}$$

$$\|\nabla^k R_2\|_2 \le r_2 t^{-1-k\beta} h(t) \text{ for } k = 0,1,2, \tag{3.31}$$

$$\begin{aligned}&\|\nabla R_3\|_2 \vee t^{1/2}\|\nabla^2 R_3\|_2 \vee t(\|\nabla \partial_t R_3\|_2 \vee \|\nabla \check{R}_3\|_2)\\ &\quad \vee t^{3/2}\|\nabla^2 \partial_t R_3\|_2 \vee t^{3/2}\|\nabla^2 \check{R}_3\|_2 \vee t^2\|\nabla \partial_t \check{R}_3\|_2 \le r_3 h(t),\end{aligned} \tag{3.32}$$

$$\begin{aligned}&\|\nabla R_4\|_2 \vee t(\|\nabla^2 R_4\|_2 \vee \|\nabla \partial_t R_4\|_2 \vee \|\nabla \check{R}_4\|_2)\\ &\qquad \vee t^2(\|\nabla^2 \check{R}_4\|_2 \vee \|\nabla \partial_t \check{R}_4\|_2) \le r_4 t^{1/2} h(t),\end{aligned} \tag{3.33}$$

$$\|\nabla R_5\|_2 \vee t^{1/2}\|\nabla^2 R_5\|_2 \vee t\|\nabla \partial_t R_5\|_2 \le r_5 h(t) \tag{3.34}$$

for some positive constants $r_j, 1 \le j \le 5$, where $h \in \mathscr{C}(I_0, \mathbb{R}^+)$ satisfies the condition introduced above.

Now we derive preliminary estimates of $G_1, G_3, \sigma, G_2', H_1$ and q', which are functions defined by (q, G_2).

Lemma 3.1. *Let $0 < \beta < 1$ and $I = (0, \tau]$ with $0 < \tau \le \tau_0$. Let w_a satisfy (A1) and let $(q, 0) \in X_0(I)$ with*

$$\|q; L^\infty(I, H^3)\| \le \|w_a; L^\infty(I, H^3)\|, \quad \|xq; L^\infty(I, H^2)\| \le \|xw_a; L^\infty(I, H^2)\|. \tag{3.35}$$

Then the following estimates hold for all $t \in I$:

$$\|\nabla G_1\|_2 \le C I_0(\|q\|_2) + \|\nabla R_3\|_2, \tag{3.36}$$

$$\|\nabla \check{G}_1\|_2 \le C t^{-1}(I_{-1}(\|\langle x\rangle q\|_2) + I_0(\|q\|_2)) + \|\nabla \check{R}_3\|_2, \tag{3.37}$$

$$\|\nabla^2 G_1\|_2 \le C I_1(\|\nabla q\|_2) + \|\nabla^2 R_3\|_2, \tag{3.38}$$

$$\|\nabla^2 \check{G}_1\|_2 \le C t^{-1}(I_0(\|\langle x\rangle \nabla q\|_2 + \|q\|_2) + I_1(\|\nabla q\|_2)) + \|\nabla^2 \check{R}_3\|_2, \tag{3.39}$$

$$\|\nabla \partial_t G_1\|_2 \le C I_1(\|\partial_t q\|_2 + t^{-1/2}\|q\|_3) + \|\nabla \partial_t R_3\|_2, \tag{3.40}$$

$$\|\nabla^2 \partial_t G_1\|_2 \le C I_2(\|\nabla \partial_t q\|_2 + t^{-1/2}\|\nabla q\|_3) + \|\nabla^2 \partial_t R_3\|_2, \tag{3.41}$$

$$\begin{aligned}\|\nabla \partial_t \check{G}_1\|_2 \le{}& C t^{-2}(I_{-1}(\|\langle x\rangle q\|_2) + I_0(\|q\|_2)) + C t^{-1}(I_0(\|\langle x\rangle \partial_t q\|_2\\ &+ t^{-1/2}\|\langle x\rangle q\|_3) + I_1(\|\partial_t q\|_2 + t^{-1/2}\|q\|_3)) + \|\nabla \partial_t \check{R}_3\|_2,\end{aligned} \tag{3.42}$$

$$\|\nabla G_3\|_2 \le C I_0(\|q\|_2) + \|\nabla R_5\|_2, \tag{3.43}$$

$$\|\nabla^2 G_3\|_2 \le C I_1(\|\nabla q\|_2) + \|\nabla^2 R_5\|_2, \tag{3.44}$$

$$\|\nabla \partial_t G_3\|_2 \le C I_1(\|\partial_t q\|_2 + t^{-1/2}\|q\|_3) + \|\nabla \partial_t R_5\|_2, \tag{3.45}$$

$$\|\nabla^k \partial_t \sigma\|_2 \le C t^{-k\beta}(\|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2) + \|\nabla^k R_2\|_2 \quad \text{for } k = 0,1,2, \tag{3.46}$$

$$\|\langle x\rangle^{-1}\nabla^k \partial_t \sigma\|_2 \le C t^{-1-k\beta}(\|\nabla G_1\|_2 + \|\nabla G_3\|_2) + \|\nabla^k R_2\|_2 \quad \text{for } k = 0,1,2. \tag{3.47}$$

Proof. From the definitions of B_1 and B_3, we obtain

$$G_1 = F_1(PN_1) + E_1(N_2) - R_3, \quad G_3 = -F_1(N_2) - R_5,$$

where $N_1 = -x \operatorname{Re} \overline{q}(2w_a + q)$, $N_2 = \operatorname{Re} \overline{q}(2w_a + q)$, so that

$$\begin{aligned}
\partial_t G_1 &= F_2(P\partial_t N_1) + E_2(\partial_t N_2) - \partial_t R_3,\\
\check{G}_1 &= t^{-1}(F_0(x\cdot PN_1) + E_0(x\cdot N_2) + \frac{\nabla}{\omega^2}E_1(N_2) + F_0(N_2) - F_1(N_2)) - \check{R}_3,\\
\partial_t \check{G}_1 &= -t^{-2}(F_0(x\cdot PN_1) + E_0(x\cdot N_2) + \frac{\nabla}{\omega^2}E_1(N_2) + F_0(N_2) - F_1(N_2))\\
&\quad + t^{-1}(F_1(x\cdot P\partial_t N_1) + E_1(x\cdot \partial_t N_2) + \frac{\nabla}{\omega^2}E_2(\partial_t N_2) + F_1(\partial_t N_2) - F_2(\partial_t N_2)) - \partial_t \check{R}_3.
\end{aligned}$$

By Lemma 2.2, especially by the estimates (2.40) and (2.41), we obtain the following:

$$\begin{aligned}
\|\nabla G_1\|_2 &\le I_0(\|N_1\|_2) + I_0(\|N_2\|_2) + \|\nabla R_3\|_2\\
&\le I_0(\|q\|_2\|x(2w_a+q)\|_\infty) + I_0(\|q\|_2\|2w_a+q\|_\infty) + \|\nabla R_3\|_2,\\
\|\nabla \check{G}_1\|_2 &\le t^{-1}\{I_{-1}(\|\langle x\rangle N_1\|_2) + I_{-1}(\|\langle x\rangle N_2\|_2) + I_{-1}(\|\omega^{-1}N_2\|_2)\\
&\quad + I_{-1}(\|N_2\|_2) + I_0(\|N_2\|_2)\} + \|\nabla \check{R}_3\|_2\\
&\le t^{-1}\{I_{-1}(\|xq\|_2\|\langle x\rangle(2w_a+q)\|_\infty) + I_{-1}(\|q\|_2\|\langle x\rangle(2w_a+q)\|_\infty)\\
&\quad + I_{-1}(\|q\|_2\|2w_a+q\|_3) + I_{-1}(\|q\|_2\|2w_a+q\|_\infty)\\
&\quad + I_0(\|q\|_2\|2w_a+q\|_\infty)\} + \|\nabla \check{R}_3\|_2,\\
\|\nabla^2 G_1\|_2 &\le I_1(\|\nabla N_1\|_2) + I_1(\|\nabla N_2\|_2) + \|\nabla^2 R_3\|_2,\\
\|\nabla N_1\|_2 &\le 2\|\nabla q\|_2\|x(w_a+q)\|_\infty + \|q\|_6(\|2w_a+q\|_3 + 2\|x\nabla w_a\|_3),\\
\|\nabla N_2\|_2 &\le 2(\|\nabla q\|_2\|w_a+q\|_\infty + \|q\|_6\|\nabla w_a\|_3),\\
\|\nabla^2 \check{G}_1\|_2 &\le t^{-1}\{I_0(\|\langle x\rangle\nabla N_1\|_2) + I_0(\|\langle x\rangle\nabla N_2\|_2) + I_0(\|N_2\|_2)\\
&\quad + I_0(\|\nabla N_2\|_2) + I_1(\|\nabla N_2\|_2)\} + \|\nabla^2 \check{R}_3\|_2,\\
\|\langle x\rangle\nabla N_1\|_2 &\le 2(\|x\nabla q\|_2\|\langle x\rangle(w_a+q)\|_\infty + \|xq\|_6\|\langle x\rangle\nabla w_a\|_3) + \|q\|_2\|\langle x\rangle(2w_a+q)\|_\infty,
\end{aligned}$$

$$
\begin{aligned}
\|\langle x\rangle \nabla N_2\|_2 &\le 2(\|\nabla q\|_2 \|\langle x\rangle (2w_a+q)\|_\infty + \|q\|_6 \|\langle x\rangle \nabla w_a\|_3),\\
\|\nabla \partial_t G_1\|_2 &\le I_1(\|\partial_t N_1\|_2) + I_1(\|\partial_t N_2\|_2) + \|\nabla \partial_t R_3\|_2\\
&\le 2I_1(\|\partial_t q\|_2 \|x(w_a+q)\|_\infty + \|q\|_3 \|x\partial_t w_a\|_6)\\
&\quad + 2I_1(\|\partial_t q\|_2 \|w_a+q\|_\infty + \|q\|_3 \|\partial_t w_a\|_6) + \|\nabla \partial_t R_3\|_2,\\
\|\nabla^2 \partial_t G_1\|_2 &\le I_2(\|\nabla \partial_t N_1\|_2) + I_2(\|\nabla \partial_t N_2\|_2) + \|\nabla^2 \partial_t R_3\|_2\\
&\le 2I_2(\|\nabla \partial_t q\|_2 \|x(w_a+q)\|_\infty + \|\partial_t q\|_6 \|w_a+q\|_3 + \|\partial_t q\|_6 \|x\nabla w_a\|_3\\
&\quad + \|\nabla q\|_3 \|x\partial_t w_a\|_6 + \|q\|_3 \|\partial_t w_a\|_6 + \|q\|_3 \|\nabla \partial_t w_a\|_6)\\
&\quad + 2I_2(\|\nabla \partial_t q\|_2 \|w_a+q\|_\infty + \|\partial_t q\|_6 \|\nabla w_a\|_3\\
&\quad + \|\nabla q\|_3 \|\partial_t w_a\|_6 + \|q\|_3 \|\nabla \partial_t w_a\|_6) + \|\nabla^2 \partial_t R_3\|_2,\\
\|\nabla \partial_t \check{G}_1\|_2 &\le t^{-2}(I_{-1}(\|\langle x\rangle N_1\|_2) + I_{-1}(\|\langle x\rangle N_2\|_2) + I_{-1}(\|\omega^{-1} N_2\|_2)\\
&\quad + I_{-1}(\|N_2\|_2) + I_0(\|N_2\|_2)) + t^{-1}(I_0(\|\langle x\rangle \partial_t N_1\|_2) + I_0(\|\langle x\rangle \partial_t N_2\|_2)\\
&\quad + I_0(\|\partial_t \omega^{-1} N_2\|_2) + I_0(\|\partial_t N_2\|_2) + I_1(\|\partial_t N_2\|_2)) + \|\nabla \partial_t \check{R}_3\|_2,\\
\|\langle x\rangle \partial_t N_1\|_2 &\le 2(\|x\partial_t q\|_2 \|\langle x\rangle (w_a+q)\|_\infty + \|xq\|_3 \|\langle x\rangle \partial_t w_a\|_6),\\
\|\langle x\rangle \partial_t N_2\|_2 &\le 2(\|\partial_t q\|_2 \|\langle x\rangle (w_a+q)\|_\infty + \|q\|_3 \|\langle x\rangle \partial_t w_a\|_6),\\
\|\partial_t \omega^{-1} N_2\|_2 &\le \|\partial_t N_2\|_{6/5} \le 2(\|\partial_t q\|_2 \|w_a+q\|_3 + \|q\|_3 \|\partial_t w_a\|_2),\\
\|\partial_t N_2\|_2 &\le 2(\|\partial_t q\|_2 \|w_a+q\|_\infty + \|q\|_3 \|\partial_t w_a\|_6).
\end{aligned}
$$

From these estimates together with (A1) and (3.35), we obtain (3.36)–(3.42). In the same way we can obtain (3.43)–(3.45).

Finally, we obtain (3.46) and (3.47) from (2.34), (2.35), (3.11), (A1) and (3.35). In the proof of (3.47), we also use (3.10) and Hardy's inequality. □

Lemma 3.2. *Let* $0 < \beta < 1$ *and* $I = (0, \tau]$ *with* $0 < \tau \le \tau_0$. *Let* w_a *satisfy (A1). Let* K_a *satisfy* (3.23) *and*

$$\|\nabla B_a\|_\infty \vee \|\nabla B_{ea}\|_\infty \vee \|\partial_t K_a\|_\infty \le Ct^{-1}. \tag{3.48}$$

Let $(q, G_2) \in X_0(I)$ *satisfy* (3.35) *and*

$$\|L\|_\infty \le C(1 - \ln t). \tag{3.49}$$

Then the following estimates hold for all $t \in I$:

$$\|\nabla G_2'\|_2 \le CtI_1(\|q\|_2(1-\ln t) + \|\sigma\|_2 + \|\nabla G\|_2) + \|\nabla R_4\|_2, \tag{3.50}$$

$$\|\nabla \check{G}_2'\|_2 \le CI_0(\|\langle x\rangle q\|_2(1-\ln t) + \|\sigma\|_2 + \|\nabla G\|_2) + \|\nabla \check{R}_4\|_2, \tag{3.51}$$

$$\|\nabla^2 G_2'\|_2 \le CtI_2(\|\nabla_K q\|_2(1-\ln t) + t^{-1}\|q\|_2 + \|\nabla L\|_2) + \|\nabla^2 R_4\|_2, \tag{3.52}$$

$$\|\nabla^2 \check{G}_2'\|_2 \le CI_1(\|\langle x\rangle \nabla_K q\|_2(1-\ln t) + t^{-1}\|q\|_2 + \|\nabla L\|_2) + \|\nabla^2 \check{R}_4\|_2, \tag{3.53}$$

$$\begin{aligned}\|\nabla\partial_t G_2'\|_2 \le\, & CI_1(\|q\|_2(1-\ln t) + \|\sigma\|_2 + \|\nabla G\|_2)\\ &+ CtI_2(\|\partial_t q\|_2(1-\ln t) + \|\nabla_K q\|_2 t^{-1/2} + t^{-1}\|q\|_2\\ &+ \|\partial_t\sigma\|_2 + \|\nabla\partial_t G\|_2 + (\|\sigma\|_2 + \|\nabla G\|_2)t^{-1/2}) + \|\nabla\partial_t R_4\|_2,\end{aligned} \tag{3.54}$$

$$\begin{aligned}\|\nabla\partial_t \check{G}_2'\|_2 \le\, & CI_1(\|\langle x\rangle\partial_t q\|_2(1-\ln t) + \|\langle x\rangle\nabla_K q\|_2 t^{-1/2} + t^{-1}\|q\|_2\\ &+ \|\partial_t\sigma\|_2 + \|\nabla\partial_t G\|_2 + (\|\sigma\|_2 + \|\nabla G\|_2)t^{-1/2}) + \|\nabla\partial_t \check{R}_4\|_2.\end{aligned} \tag{3.55}$$

Proof. The proof is the same as that of Lemma 5.2 in [13]. □

Lemma 3.3. *Let $0 < \beta < 1$ and $I = (0, \tau]$ with $0 < \tau \le \tau_0$. Let w_a satisfy (A1). Let K_a satisfy* (3.23) *and* (3.24)*. Let $(q, G_2) \in X_0(I)$ satisfy* (3.35) *and*

$$\|L\|_\infty \vee \|\nabla L\|_3 \le C(1-\ln t) \tag{3.56}$$

for all $t \in I$. Then the following estimates hold for all $t \in I$:

$$\begin{aligned}\|H_1 w_a\|_2 \le\, & C((\|\langle x\rangle^{-1}\sigma\|_3 + \|\nabla G\|_2)(1-\ln t) + \|\langle x\rangle^{-1}\nabla\cdot\sigma\|_2\\ &+ t^{-1+\beta}(\|\nabla G_1\|_2 + \|\nabla G_3\|_2) + \|\nabla \check{G}_2\|_2),\end{aligned} \tag{3.57}$$

$$\begin{aligned}\|\langle x\rangle H_1 w_a\|_2 \le\, & C((\|\sigma\|_3 + \|\nabla G\|_2)(1-\ln t) + \|\nabla\cdot\sigma\|_2\\ &+ t^{\beta}(\|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2) + \|\nabla \check{G}_2\|_2),\end{aligned} \tag{3.58}$$

$$\begin{aligned}\|\langle x\rangle H_1\partial_t w_a\|_2 \le\, & C(\|L\|_\infty + \|L\|_6(1-\ln t) + \|\nabla\cdot\sigma\|_3 + \|\nabla\cdot G\|_3\\ &+ t^{\beta/2}(\|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2) + \|\nabla \check{G}_2\|_2)t^{-1/2},\end{aligned} \tag{3.59}$$

$$\begin{aligned}\|\langle x\rangle(\partial_t H_1)w_a\|_2 \le\, & C((\|\partial_t\sigma\|_3 + \|\nabla\partial_t G\|_2)(1-\ln t) + \|\partial_t\nabla\cdot\sigma\|_2\\ &+ t^{\beta}(\|\nabla\partial_t \check{G}_1\|_2 + \|\nabla\partial_t(t^{-1}G_3)\|_2) + \|\nabla\partial_t \check{G}_2\|_2\\ &+ t^{-1}(\|\sigma\|_2 + \|\nabla G\|_2)),\end{aligned} \tag{3.60}$$

$$\begin{aligned}\|\nabla_K H_1 w_a\|_2 \le\, & C(\|\nabla L\|_2(1-\ln t)^2 + \|\nabla\cdot\sigma\|_3(1-\ln t) + \|\nabla\nabla\cdot L\|_2\\ &+ t^{-1}(\|\sigma\|_2 + \|\nabla G\|_2) + \|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2\end{aligned}$$

$$+ \|\nabla \check{G}_2\|_2(1 - \ln t)), \tag{3.61}$$

$$\|\nabla_K H_1 \partial_t w_a\|_2 \le C((\|L\|_\infty + \|\nabla L\|_3)(1 - \ln t) + \|\nabla L\|_2(1 - \ln t)^2$$
$$+ t^{-1}(\|\sigma\|_2 + \|\nabla G\|_2) + \|\nabla \cdot \sigma\|_3(1 - \ln t) + \|\nabla\nabla \cdot L\|_2$$
$$+ \|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2 + \|\nabla \check{G}_2\|_2(1 - \ln t))t^{-1/2}, \tag{3.62}$$

$$\|\nabla_K(\partial_t H_1)w_a\|_2 \le C(\|\nabla \partial_t L\|_2(1 - \ln t)^2 + \|\nabla \partial_t \nabla \cdot L\|_2 + t^{-1}(\|\partial_t \sigma\|_2$$
$$+ \|\nabla \partial_t G\|_2 + \|\nabla L\|_2(1 - \ln t)) + t^{-2}(\|\sigma\|_2 + \|\nabla G\|_2)$$
$$+ \|\nabla \partial_t \check{G}_1\|_2 + \|\nabla \partial_t(t^{-1}G_3)\|_2 + \|\nabla \partial_t \check{G}_2\|_2(1 - \ln t)). \tag{3.63}$$

Proof. We rewrite H_1 defined by (3.13) as

$$H_1 = iQ_{K_a}(L, \cdot) + (1/2)L^2 + \check{G}^S - t^{-1}G_3^S. \tag{3.64}$$

Using (2.32) and (2.33) together with (3.9), we obtain

$$\|H_1 w_a\|_2 \le \|\langle x\rangle^{-1}\sigma\|_3\|\langle x\rangle \nabla_{K_a} w_a\|_6 + \|G\|_6\|\nabla_{K_a} w_a\|_3$$
$$+ \|\langle x\rangle^{-1}\nabla \cdot \sigma\|_2\|\langle x\rangle w_a\|_\infty + \|\nabla \cdot G\|_2\|w_a\|_\infty$$
$$+ t^{-1+\beta}\|\nabla G_1\|_2(\|w_a\|_3 + \|\langle x\rangle w_a\|_\infty) + t^{-1+\beta}\|\nabla G_3\|_2\|w_a\|_\infty$$
$$+ \|\check{G}_2\|_6\|w_a\|_3 + \|L\|_\infty(\|\langle x\rangle^{-1}\sigma\|_3\|\langle x\rangle w_a\|_6 + \|G\|_6\|w_a\|_3),$$

$$\|\langle x\rangle H_1 w_a\|_2 \le \|\sigma\|_3\|\langle x\rangle \nabla_{K_a} w_a\|_6 + \|G\|_6\|\langle x\rangle \nabla_{K_a} w_a\|_3$$
$$+ (\|\nabla \cdot \sigma\|_2 + \|\nabla \cdot G\|_2 + t^\beta(\|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2))\|\langle x\rangle w_a\|_\infty$$
$$+ \|\check{G}_2\|_6\|\langle x\rangle w_a\|_3 + \|L\|_\infty(\|\sigma\|_3\|\langle x\rangle w_a\|_6 + \|G\|_6\|\langle x\rangle w_a\|_3),$$

$$\|\langle x\rangle H_1 \partial_t w_a\|_2 \le \|L\|_\infty\|\langle x\rangle \nabla \partial_t w_a\|_2 + \|L\|_6\|K_a\|_\infty\|\langle x\rangle \partial_t w_a\|_3$$
$$+ (\|\nabla \cdot \sigma\|_3 + \|\nabla \cdot G\|_3 + t^{\beta/2}(\|\nabla \check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2))\|\langle x\rangle \partial_t w_a\|_6$$
$$+ \|\check{G}_2\|_6\|\langle x\rangle \partial_t w_a\|_3 + \|L\|_\infty\|L\|_6\|\langle x\rangle \partial_t w_a\|_3.$$

Hence we obtain (3.57), (3.58) and (3.59) by (A1), (3.23) and (3.56).

Taking the time derivative of (3.64), we obtain

$$\partial_t H_1 = i(\partial_t L) \cdot \nabla_{K_a} + (i/2)(\partial_t \nabla \cdot L) + L \cdot (\partial_t K) + (\partial_t \check{G}^S) - \partial_t(t^{-1}G_3^S). \tag{3.65}$$

Using (2.32) and (2.33), we obtain

$$\begin{aligned}\|\langle x\rangle(\partial_t H_1)w_a\|_2 &\le \|\partial_t\sigma\|_3\|\langle x\rangle\nabla_{K_a}w_a\|_6 + \|\partial_t G\|_6\|\langle x\rangle\nabla_{K_a}w_a\|_3\\ &\quad + (\|\partial_t\nabla\cdot\sigma\|_2 + \|\partial_t\nabla\cdot G\|_2 + t^\beta\|\nabla\partial_t\check{G}_1\|_2 + t^\beta\|\nabla\partial_t(t^{-1}G_3)\|_2)\|\langle x\rangle w_a\|_\infty\\ &\quad + \|\partial_t\check{G}_2\|_6\|\langle x\rangle w_a\|_3 + \|\partial_t K_a\|_\infty(\|\sigma\|_2\|\langle x\rangle w_a\|_\infty + \|G\|_6\|\langle x\rangle w_a\|_3)\\ &\quad + \|L\|_\infty(\|\partial_t\sigma\|_3\|\langle x\rangle w_a\|_6 + \|\partial_t G\|_6\|\langle x\rangle w_a\|_3).\end{aligned}$$

Hence we obtain (3.60) by (A1), (3.23), (3.24) and (3.56).

Taking the covariant gradient of H_1v, using (3.64) and the identity

$$\nabla_K\nabla_{K_a} = \nabla_{K_a}\nabla - iK\nabla_{K_a} - i(\nabla K_a), \tag{3.66}$$

we obtain

$$\begin{aligned}\nabla_K H_1 v &= iL\cdot\nabla_{K_a}\nabla v + (i(\nabla L) + KL)\cdot\nabla_{K_a}v\\ &\quad + ((i/2)(\nabla\cdot L) + (1/2)L^2 + (\check{G}^S - t^{-1}G_3^S))\nabla_K v\\ &\quad + ((i/2)(\nabla\nabla\cdot L) + L\cdot(\nabla K) + \nabla(\check{G}^S - t^{-1}G_3^S))v.\end{aligned} \tag{3.67}$$

Using (2.32) and (2.33), we obtain

$$\begin{aligned}\|\nabla_K H_1 v\|_2 &\le \|\nabla L\|_2(\|\nabla_{K_a}\nabla v\|_3 + \|\nabla_{K_a}v\|_\infty + \|K\|_\infty\|\nabla_{K_a}v\|_3)\\ &\quad + \|\nabla\cdot L\|_3\|\nabla_K v\|_6 + \|\nabla\nabla\cdot L\|_2\|v\|_\infty\\ &\quad + \|\nabla L\|_2\|L\|_\infty(\|\nabla_K v\|_3 + \|v\|_\infty)\\ &\quad + \|\nabla K_a\|_\infty(\|\sigma\|_2\|v\|_\infty + \|G\|_6\|v\|_3)\\ &\quad + (\|\nabla\check{G}_1\|_2 + t^{-1}\|\nabla G_3\|_2)(t^{\beta/2}\|\nabla_K v\|_6 + \|v\|_\infty)\\ &\quad + \|\nabla\check{G}_2\|_2(\|\nabla_K v\|_3 + \|v\|_\infty).\end{aligned} \tag{3.68}$$

Applying (3.68) with $v = w_a$ and using (A1), (3.23), (3.24) and (3.56), we obtain (3.61). On the other hand, (3.62) follows from (3.67) with $v = \partial_t w_a$, except for a slightly different estimate of the contribution of the first two terms in the right-hand side of (3.67), namely

$$\|L\cdot\nabla_{K_a}\nabla\partial_t w_a\|_2 + \|(\nabla L)\nabla_{K_a}\partial_t w_a\|_2 \le \|L\|_\infty\|\nabla_{K_a}\nabla\partial_t w_a\|_2 + \|\nabla L\|_3\|\nabla_{K_a}\partial_t w_a\|_6.$$

Taking the covariant gradient of $(\partial_t H_1)v$, using (3.65) and (3.66), we obtain

$$\begin{aligned}\nabla_K(\partial_t H_1)v &= i(\partial_t L)\cdot\nabla_{K_a}\nabla v + (i(\nabla\partial_t L) + K(\partial_t L))\cdot\nabla_{K_a} v\\ &+ ((i/2)(\partial_t\nabla\cdot L) + L\cdot(\partial_t K) + \partial_t(\check{G}^S - t^{-1}G_3^S))\nabla_K v\\ &+ ((i/2)(\nabla\partial_t\nabla\cdot L) + (\partial_t L)\cdot(\nabla K_a) + (\nabla L)(\partial_t K)\\ &+ L(\nabla\partial_t K) + (\nabla\partial_t(\check{G}^S - t^{-1}G_3^S)))v.\end{aligned}$$

Using (2.32) and (2.33), we obtain

$$\begin{aligned}\|\nabla_K(\partial_t H_1)v\|_2 \le\ & \|\nabla\partial_t L\|_2(\|\nabla_{K_a}\nabla v\|_3 + \|\nabla_{K_a}v\|_\infty + \|K\|_\infty\|\nabla_{K_a}v\|_3)\\ &+ \|\partial_t\nabla\cdot L\|_2\|\nabla_K v\|_\infty + \|\nabla\partial_t\nabla\cdot L\|_2\|v\|_\infty\\ &+ \|\nabla K_a\|_\infty(\|\partial_t\sigma\|_2\|v\|_\infty + \|\partial_t G\|_6\|v\|_3)\\ &+ \|\partial_t K_a\|_\infty\|\nabla L\|_2(\|\nabla_K v\|_3 + \|v\|_\infty)\\ &+ \|\nabla\partial_t K_a\|_\infty(\|\sigma\|_2\|v\|_\infty + \|G\|_6\|v\|_3)\\ &+ \|\nabla\partial_t L\|_2(\|L\|_\infty\|\nabla_K v\|_3 + (\|\nabla L\|_3 + \|L\|_\infty)\|v\|_\infty)\\ &+ (\|\nabla\partial_t\check{G}_1\|_2 + \|\nabla\partial_t(t^{-1}G_3)\|_2)(t^\beta\|\nabla_K v\|_\infty + \|v\|_\infty)\\ &+ \|\nabla\partial_t\check{G}_2\|_2(\|\nabla_K v\|_3 + \|v\|_\infty).\end{aligned}$$

Substituting w_a for v and using (A1), (3.23), (3.24) and (3.56), we obtain (3.63). □

We next estimate the solutions q' of the Schrödinger equation in (3.17). The estimates are given in differential form, but should be understood in integral form. The derivation in the proof is formal, but the estimates in integral form are justified by Proposition 4.1 in [13].

Lemma 3.4. *Let* $0 < \beta < 1$ *and* $I = (0, \tau]$ *with* $0 < \tau \le \tau_0$. *Let* (w_a, K_a) *satisfy (A1), (A2) and let* B_0, B_{e0} *satisfy* (2.31) *for* $0 \le j \le 1, 0 \le k \le 2$ *and* $r = \infty$. *Let* $(q, G_2) \in X_0(I)$ *satisfy* (3.35), (3.56) *and in addition*

$$\|\nabla\check{G}^S\|_2 \vee t^{-1}\|\nabla G_3^S\|_2 \le Ct^{-1/4}, \tag{3.69}$$

$$\|\nabla\partial_t L\|_2 \vee \|\nabla\check{G}^S\|_6 \vee t^{-1}\|\nabla G_3^S\|_6 \le Ct^{-3/4}, \tag{3.70}$$

$$\|\nabla\partial_t\nabla\cdot L\|_2 \vee \|\nabla\partial_t\check{G}^S\|_2 \vee \|\nabla\partial_t(t^{-1}G_3^S)\|_2 \le Ct^{-5/4} \tag{3.71}$$

for all $t \in I$. Let q' with $(q', 0) \in X_0(I)$ be a solution of the Schrödinger equation in (3.17)*. Then the following estimates hold for all $t \in I$:*

$$|\partial_t \|q'\|_2| \le \|\tilde{R}_1\|_2, \tag{3.72}$$

$$|\partial_t \|xq'\|_2| \le \|\nabla_K q'\|_2 + \|x\tilde{R}_1\|_2, \tag{3.73}$$

$$\begin{aligned} |\partial_t \|\partial_t q'\|_2| \le\, & C(t^{-1}\|\nabla_K q'\|_2 + t^{-3/4}(\|\nabla_K q'\|_3 + \|q'\|_\infty) + t^{-3/2}\|q'\|_2) \\ & + b_0 t^{-2}\|q'\|_2 + \|\partial_t \tilde{R}_1\|_2, \end{aligned} \tag{3.74}$$

$$\begin{aligned} |\partial_t \|x\partial_t q'\|_2| \le\, & \|\nabla_K \partial_t q'\|_2 + C(t^{-1}\|x\nabla_K q'\|_2 + t^{-3/4}(\|x\nabla_K q'\|_3 \\ & + \|xq'\|_\infty) + t^{-3/2}\|xq'\|_2) + b_0 t^{-2}\|xq'\|_2 + \|x\partial_t \tilde{R}_1\|_2, \end{aligned} \tag{3.75}$$

$$\begin{aligned} |\partial_t \|\nabla_K \partial_t q'\|_2| \le\, & C(t^{-1}(\|\nabla_K^2 q'\|_2 + \|\partial_t q'\|_2) \\ & + t^{-3/4}(\|\nabla_K q'\|_\infty + \|\partial_t q'\|_3 + \|\nabla_K^2 q'\|_3) \\ & + t^{-3/2}\|\nabla_K q'\|_2 + t^{-5/4}\|q'\|_\infty + t^{-5/2}\|q'\|_2) \\ & + b_0 t^{-2}\|\nabla_K q'\|_2 + b_0 t^{-3}\|q'\|_2 + \|\nabla_K \partial_t \tilde{R}_1\|_2, \end{aligned} \tag{3.76}$$

$$\|\langle x\rangle \Delta_K q'\|_2 \le \|\langle x\rangle \partial_t q'\|_2 + Ct^{-1/2}\|\langle x\rangle q'\|_2 + \|\langle x\rangle \tilde{R}_1\|_2, \tag{3.77}$$

$$\|\nabla_K \Delta_K q'\|_2 \le \|\nabla_K \partial_t q'\|_2 + C(t^{-1/2}\|\nabla_K q'\|_2 + t^{-1}\|q'\|_2) + \|\nabla_K \tilde{R}_1\|_2. \tag{3.78}$$

Proof. Applying standard L^2 estimate for the first equation of (3.17), we immediately obtain (3.72).

We next estimate xq'. From the commutation relation $[x, H] = \nabla_K$, we obtain

$$i\partial_t xq' = \nabla_K q' + Hxq' - x\tilde{R}_1, \tag{3.79}$$

which implies (3.73).

We next estimate $\partial_t q'$. We take the time derivative of the first equation of (3.17) to obtain

$$i\partial_t \partial_t q' = H\partial_t q' + (\partial_t H)q' - \partial_t \tilde{R}_1 \tag{3.80}$$

with

$$\partial_t H = i(\partial_t K) \cdot \nabla_K + (i/2)(\partial_t \nabla \cdot K) + \partial_t B_m^S. \tag{3.81}$$

Hence we obtain

$$\begin{aligned}
|\partial_t \|\partial_t q'\|_2| &\le \|(\partial_t H)q'\|_2 + \|\partial_t \tilde{R}_1\|_2,\\
\|(\partial_t H)q'\|_2 &\le \|\partial_t K_a\|_\infty \|\nabla_K q'\|_2 + \|\partial_t L\|_6 \|\nabla_K q'\|_3 + \|\partial_t \nabla\cdot L\|_2 \|q'\|_\infty\\
&\quad + (\|\partial_t \nabla\cdot K_a\|_\infty + \|\partial_t B^S_{ma}\|_\infty)\|q'\|_2\\
&\quad + (\|\partial_t \check{G}^S\|_6 + \|\partial_t (t^{-1}G^S_3)\|_6)\|q'\|_3.
\end{aligned}$$

Using (A2), (3.70), (3.71) and a covariant Sobolev inequality, we obtain (3.74).

We next estimate $x\partial_t q'$. Multiplying x by (3.80) and using the commutation relation $[x,H] = \nabla_K$, we obtain

$$i\partial_t x\partial_t q' = \nabla_K \partial_t q' + Hx\partial_t q' + x(\partial_t H)q' - x\partial_t \tilde{R}_1, \tag{3.82}$$

and hence we obtain

$$\partial_t \|x\partial_t q'\|_2 \le \|\nabla_K \partial_t q'\|_2 + \|x(\partial_t H)q'\|_2 + \|x\partial_t \tilde{R}_1\|_2.$$

Estimating $\|x(\partial_t H)q'\|_2$ as before with $\nabla_K q'$ and q' replaced by $x\nabla_K q'$ and xq' respectively, we obtain (3.75).

We next estimate $\nabla_K \partial_t q'$. We take the covariant gradient of (3.80) to obtain

$$\begin{aligned}
i\partial_t \nabla_K \partial_t q' &= (\partial_t K + \nabla B^S_m)\partial_t q' - (1/2)\nabla_K \Delta_K \partial_t q' + B^S_m \nabla_K \partial_t q'\\
&\quad + \nabla_K (\partial_t H)q' - \nabla_K \partial_t \tilde{R}_1.
\end{aligned} \tag{3.83}$$

Hence we have

$$\partial_t \|\nabla_K \partial_t q'\|_2 \le \|(\partial_t K + \nabla B^S_m)\partial_t q'\|_2 + \|\nabla_K(\partial_t H)q'\|_2 + \|\nabla_K \partial_t \tilde{R}_1\|_2 \tag{3.84}$$

with

$$\begin{aligned}
\nabla_K(\partial_t H)q' &= i(\partial_t K)\cdot \nabla_K^2 q' + (i(\nabla \partial_t K) + (i/2)(\partial_t \nabla\cdot K) + (\partial_t B^S_m))\cdot \nabla_K q'\\
&\quad + ((i/2)(\nabla \partial_t \nabla \cdot K) + (\nabla \partial_t B^S_m))q'.
\end{aligned}$$

We estimate each term of (3.84) as follows:

$$\begin{aligned}
\|(\partial_t K + \nabla B^S_m)\partial_t q'\|_2 &\le \|\partial_t K_a + \nabla B^S_{ma}\|_\infty \|\partial_t q'\|_2\\
&\quad + \|\partial_t L + \nabla \check{G}^S - t^{-1}\nabla G^S_3\|_6 \|\partial_t q'\|_3\\
&\le C(t^{-1}\|\partial_t q'\|_2 + t^{-3/4}\|\partial_t q'\|_3),\\
\|(\partial_t K)\nabla_K^2 q'\|_2 &\le C(t^{-1}\|\nabla_K^2 q'\|_2 + t^{-3/4}\|\nabla_K^2 q'\|_3),
\end{aligned}$$

$$\begin{aligned}
&\|(\nabla\partial_t K)\nabla_K q'\|_2 + \|(\partial_t \nabla\cdot K)\nabla_K q'\|_2 + \|(\partial_t B^S_m)\nabla_K q'\|_2 \\
&\qquad \le (\|\nabla\partial_t K_a\|_\infty + \|\partial_t\nabla\cdot K_a\|_\infty + \|\partial_t B^S_{ma}\|_\infty)\|\nabla_K q'\|_2 \\
&\qquad\quad + (\|\nabla\partial_t L\|_2 + \|\partial_t\nabla\cdot L\|_2)\|\nabla_K q'\|_\infty \\
&\qquad\quad + (\|\partial_t \check{G}^S\|_6 + \|\partial_t(t^{-1}G^S_3)\|_6)\|\nabla_K q'\|_3 \\
&\qquad \le (b_0 t^{-2} + Ct^{-3/2})\|\nabla_K q'\|_2 + Ct^{-3/4}\|\nabla_K q'\|_\infty, \\
&\|(\nabla\partial_t\nabla\cdot K)q'\|_2 + \|(\nabla\partial_t B^S_m)q'\|_2 \\
&\qquad \le \|\nabla\partial_t\nabla\cdot K_a\|_\infty\|q'\|_2 + \|\nabla\partial_t\nabla\cdot L\|_2\|q'\|_\infty \\
&\qquad\quad + \|\nabla\partial_t B^S_{ma}\|_\infty\|q'\|_2 + \|\nabla\partial_t\check{G}^S\|_2\|q'\|_\infty \\
&\qquad\quad + \|\nabla\partial_t(t^{-1}G^S_3)\|_2\|q'\|_\infty \\
&\qquad \le (b_0 t^{-3} + Ct^{-2})\|q'\|_2 + Ct^{-5/4}\|q'\|_\infty.
\end{aligned}$$

Here, we have used (A2), (3.41), (3.70) and (3.71). The constant b_0 is the one which appears in Lemma 2.1. Substituting these estimates into (3.81), we obtain (3.76).

We next estimate $\langle x\rangle\Delta_K q'$ and $\nabla_K\Delta_K q'$. Using the first equation of (3.17), we have

$$\begin{aligned}
\|\langle x\rangle\Delta_K q'\|_2 &\le \|\langle x\rangle\partial_t q'\|_2 + \|B^S_m\|_\infty\|\langle x\rangle q'\|_2 + \|\langle x\rangle\tilde{R}_1\|_2, \\
\|\nabla_K\Delta_K q'\|_2 &\le \|\nabla_K\partial_t q'\|_2 + \|B^S_m\|_\infty\|\nabla_K q'\|_2 + \|(\nabla B^S_m)q'\|_2 + \|\nabla_K\tilde{R}_1\|_2.
\end{aligned}$$

On the other hand, from (A2), (3.69) and (3.70), we have

$$\begin{aligned}
\|(\nabla B^S_m)q'\|_2 &\le \|\nabla B^S_{ma}\|_\infty\|q'\|_2 + (\|\nabla\check{G}^S\|_3 + t^{-1}\|\nabla G^S_3\|_3)\|q'\|_6 \\
&\le C(t^{-1}\|q'\|_2 + t^{-1/2}\|\nabla_K q'\|_2).
\end{aligned}$$

From these inequalities, we obtain (3.77) and (3.78). □

Lemma 3.5. *Let $2 \le r \le 3$. We define*

$$\begin{aligned}
n &= \|K\|_\infty^2 + \|\nabla\cdot L\|_3^2 + \|\nabla\cdot K_a\|_\infty + \|K\|_\infty + \|\nabla\cdot L\|_3, \\
\bar{n} &= \|K\|_\infty^2 + \|\nabla L\|_3^2 + \|\nabla K_a\|_\infty, \\
\tilde{n} &= \|K\|_\infty^3 + \|\nabla L\|_3^3 + \|K\|_\infty\|\nabla K_a\|_\infty + \|\nabla\nabla\cdot K_a\|_\infty + \|\nabla\nabla\cdot L\|_2^2.
\end{aligned}$$

Then the following estimates hold:

$$\|\langle x\rangle \nabla_K v\|_3 \le \|\langle x\rangle \nabla v\|_2^{1/2}(\|\langle x\rangle \Delta v\|_2 + \|\nabla v\|_2)^{1/2} + \|K\|_\infty \|\langle x\rangle v\|_2^{1/2}(\|\langle x\rangle \nabla v\|_2 + \|v\|_2)^{1/2}, \tag{3.85}$$

$$\|\langle x\rangle \Delta_K v\|_r \le \|\langle x\rangle \Delta v\|_r + n\|\langle x\rangle v\|_r, \tag{3.86}$$

$$\|\nabla_K^2 v\|_r \le \|\Delta v\|_r + \bar{n}\|v\|_r, \tag{3.87}$$

$$\|\nabla \Delta v\|_2 \le \|\nabla_K \Delta_K v\|_2 + \|\nabla K_a\|_\infty \|\nabla v\|_2 + \tilde{n}\|v\|_2. \tag{3.88}$$

Proof. The proof is the same as that of Lemma 5.5 in [13]. We remark that in our case the term $(\nabla\cdot K)v$ appears in $\Delta_K v$, which disappears in the Coulomb gauge case. □

Now we will show that Γ maps a bounded set of $X(I)$ into itself, where $I = (0,\tau]$ with τ sufficiently small. In what follows we assume that $(q, G_2) \in X(I)$ satisfies

$$\|q(t)\|_2 \le V_{00} t^{1/2} h(t), \quad \|\langle x\rangle q(t)\|_2 \le V_{01} h(t), \tag{3.89}$$

$$\|\langle x\rangle \partial_t q(t)\|_2 \vee \|\langle x\rangle \Delta q(t)\|_2 \le V_2 t^{-1} h(t), \tag{3.90}$$

$$\|\nabla \partial_t q(t)\|_2 \vee \|\nabla \Delta q(t)\|_2 \le V_3 t^{-3/2} h(t), \tag{3.91}$$

$$\|\nabla G_2(t)\|_2 \vee t(\|\nabla^2 G_2(t)\|_2 \vee \|\nabla \partial_t G_2(t)\|_2 \vee \|\nabla \check{G}_2(t)\|_2) \vee t^2(\|\nabla^2 \check{G}_2(t)\|_2 \vee \|\nabla \partial_t \check{G}_2(t)\|_2) \le Z t^{1/2} h(t) \tag{3.92}$$

for some constants $V_{00}, V_{01}, V_2, V_3, Z$ and for all $t \in I$, with h introduced in the definition of $X(I)$. Note that from the definition of V_2, we have

$$\|\langle x\rangle q(t)\|_2 \le \int_0^t dt' \|\langle x\rangle \partial_t q(t')\|_2 \le V_2 h(t),$$

so that we may assume $V_{01} \le V_2$. From (3.89) and (3.90) together with the Sobolev inequality, we have

$$\|\langle x\rangle \nabla q(t)\|_2 \le (\|\langle x\rangle \Delta q(t)\|_2 \|\langle x\rangle q(t)\|_2)^{1/2} \le V_1 t^{-1/2} h(t) \tag{3.93}$$

for some constant V_1. We may assume that $V_{01} \le V_1 \le V_2$.

Lemma 3.6. *Let* $1/4 \le \beta < 3/4$ *and* $I = (0,\tau]$ *with* $0 < \tau \le \tau_0$. *Let* w_a, K_a *and the remainders* R_j, $1 \le j \le 5$, *satisfy the assumptions (A1), (A2) and (A3). Let* $(q, G_2) \in X(I)$ *satisfy the conditions* (3.89)–(3.93), *and let* τ *be sufficiently small so that* (3.35) *holds and that*

$$(V_2 + r_6) h \le t^{3/4} \wedge t^{-1/4+2\beta}, \quad Z h \le t^{3/4} \tag{3.94}$$

for all $t \in I$*, where* $r_6 = r_2 + r_3 + r_5$*. Then the estimates* (3.56)*,* (3.69)*–*(3.71)*, and the following estimates hold for all* $t \in I$*:*

$$\|\nabla \check{G}_1\|_2 \le C(V_{01} + r_3)t^{-1}h, \tag{3.95}$$

$$\|\nabla G_1\|_2 \vee \|\nabla G_3\|_2 \le C((V_{00}t^{1/2}) \wedge V_{01} + r_6)h, \tag{3.96}$$

$$\|\nabla^2 G_1\|_2 \vee t\|\nabla^2 \check{G}_1\|_2 \le C(V_1 + r_3)t^{-1/2}h, \tag{3.97}$$

$$\|\nabla^2 G_3\|_2 \le C(V_1 + r_5)t^{-1/2}h, \tag{3.98}$$

$$\|\nabla \partial_t G_1\|_2 \vee t\|\nabla \partial_t \check{G}_1\|_2 \le C(V_2 + r_3)t^{-1}h, \tag{3.99}$$

$$\|\nabla \partial_t G_3\|_2 \vee t\|\nabla \partial_t (t^{-1}G_3)\|_2 \le C(V_2 + r_5)t^{-1}h, \tag{3.100}$$

$$\|\nabla^2 \partial_t G_1\|_2 \le C(V_2 + V_3 + r_3)t^{-3/2}h, \tag{3.101}$$

$$\|\nabla^k \sigma\|_2 \vee t\|\nabla^k \partial_t \sigma\|_2 \le C(V_{01} + r_6)t^{-k\beta}h \quad \text{for } k = 0, 1, 2, \tag{3.102}$$

$$\|\langle x\rangle^{-1}\nabla^k \sigma\|_2 \le C(V_{00}t^{1/2} + r_6)t^{-k\beta}h \quad \text{for } k = 0, 1, 2, \tag{3.103}$$

$$\|\nabla G_2'\|_2 \vee t\|\nabla \check{G}_2'\|_2 \le \{C((V_1 + r_6)t^{1/2}(1 - \ln t) + Zt) + r_4\}t^{1/2}h, \tag{3.104}$$

$$\|\nabla^2 G_2'\|_2 \vee t\|\nabla^2 \check{G}_2'\|_2 \le \{C((V_1 + r_6)t^{1/2} + Zt^2) + r_4\}t^{-1/2}h, \tag{3.105}$$

$$\|\nabla \partial_t G_2'\|_2 \vee t\|\nabla \partial_t \check{G}_2'\|_2 \le \{C((V_2 + r_6)t^{1/2}(1 - \ln t) + Zt) + r_4\}t^{-1/2}h, \tag{3.106}$$

$$\|H_1 w_a\|_2 \le C\{(V_{00}t^{1/2} + r_6)(t^{-\beta} + t^{-1+\beta}) + Zt^{-1/2}\}h, \tag{3.107}$$

$$\|\langle x\rangle H_1 w_a\|_2 \le C\{(V_{01} + r_6)(t^{-\beta} + t^{-1+\beta}) + Zt^{-1/2}\}h, \tag{3.108}$$

$$\|\langle x\rangle H_1 \partial_t w_a\|_2 \le C\{(V_{01} + r_6)(t^{-3\beta/2} + t^{-1+\beta/2}) + Zt^{-1/2}\}t^{-1/2}h, \tag{3.109}$$

$$\|\langle x\rangle (\partial_t H_1) w_a\|_2 \le C\{(V_2 + r_6)(t^{-\beta} + t^{-1+\beta}) + Zt^{-1/2}\}t^{-1}h, \tag{3.110}$$

$$\|\nabla_K H_1 w_a\|_2 \le C\{(V_1 + r_6)(t^{-2\beta} + t^{-1}) + Zt^{-1/2}(1 - \ln t)\}h, \tag{3.111}$$

$$\|\nabla_K H_1 \partial_t w_a\|_2 \le C\{(V_1 + r_6)(t^{-2\beta} + t^{-1}) + Zt^{-1/2}(1 - \ln t)\}t^{-1/2}h, \tag{3.112}$$

$$\begin{aligned}\|\nabla_K (\partial_t H_1) w_a\|_2 \le{} & C\{(V_2 + r_6)(t^{-2\beta} + t^{-1}) \\ & + V_3 t^{-1/2} + Zt^{-1/2}(1 - \ln t)\}t^{-1}h. \end{aligned}\tag{3.113}$$

Proof. The estimates (3.95)–(3.103) are obtained by substituting the bounds (3.89)–(3.92) on (q, G_2) into the estimates of Lemma 3.1. Then, these estimates together with (3.92) and (3.94) yield the estimates (3.56), (3.69)–(3.71). Once we obtain (3.56), we can use Lemmas 3.2 and 3.3 to prove the estimates (3.104)–(3.113). □

Lemma 3.7. *Let the assumptions of Lemma 3.6 be satisfied and $\tilde{R}_1 = R_1 - H_1 w_a$. Then the following estimates hold:*

$$\begin{aligned}
N_{00} &= \sup_{t\in(0,\tau]} t^{-1/2}h(t)^{-1}\int_0^t dt'\|\tilde{R}_1(t')\|_2 \le r_1 + C\{V_{00}(t^{1/2}+r_6)(\tau^{1/2-\beta}+\tau^{-1/2+\beta}) + Z\},\\
N_{01} &= \sup_{t\in(0,\tau]} h(t)^{-1}\int_0^t dt'\|\langle x\rangle\tilde{R}_1(t')\|_2 \le r_1 + C\{(V_{01}+r_6)(\tau^{1-\beta}+\tau^{\beta}) + Z\tau^{1/2}\},\\
N_2 &= \sup_{t\in(0,\tau]} th(t)^{-1}\int_0^t dt'\|\langle x\rangle\partial_t\tilde{R}_1(t')\|_2 \le r_1 + C\{(V_2+r_6)(\tau^{1-\beta}+\tau^{\beta}) + Z\tau^{1/2}\},\\
N_3 &= \sup_{t\in(0,\tau]} t^{3/2}h(t)^{-1}\{\|\nabla_K\tilde{R}_1(t')\|_2 \vee \int_0^t dt'\|\nabla_K\partial_t\tilde{R}_1(t')\|_2\}\\
&\le r_1(1+C\tau^{1/2}(1-\ln\tau)) + C\{(V_2+r_6)(\tau^{3/2-2\beta}+\tau^{1/2}) + V_3\tau + Z\tau(1-\ln\tau)\}.
\end{aligned}$$

Proof. The estimates follow from (A3) and (3.107)–(3.113). In the proof of the last inequality, we also need the estimate $\|K\partial_t\tilde{R}_1; L^1((0,t], L^2)\| \le Cr_1(1-\ln t)t^{-1}h(t)$. For the proof of this estimate, see that of Lemma 5.7 in [13]. □

We now turn to the construction of solutions (q', G_2') of the linearized system (3.17). We consider (q, G_2) belonging to a bounded set of $X((0,\tau])$, defined by (3.89)–(3.92) for some $\tau, 0 < \tau \le \tau_0$. We shall deal with solutions (q', G_2') of the system (3.17) defined in an interval $I = [t_0,\tau]\cap(0,\tau]$ for some t_0 with $0 \le t_0 < \tau$. We shall need to estimate (q', G_2') in $X(I)$ and for that purpose we define the relevant norms

$$V_{00}' = \sup_{t\in I} t^{-1/2}h(t)^{-1}\|q'(t)\|_2, \quad V_{01}' = \sup_{t\in I} h(t)^{-1}\|\langle x\rangle q'(t)\|_2, \tag{3.114}$$

$$V_2' = \sup_{t\in I} th(t)^{-1}(\|\langle x\rangle\partial_t q'(t)\|_2 \vee \|\langle x\rangle\Delta q'(t)\|_2), \tag{3.115}$$

$$V_3' = \sup_{t\in I} t^{3/2}h(t)^{-1}(\|\nabla\partial_t q'(t)\|_2 \vee \|\nabla\Delta q'(t)\|_2), \tag{3.116}$$

$$\begin{aligned}
Z' = \sup_{t\in I} t^{-1/2}h(t)^{-1}\{&\|\nabla G_2'(t)\|_2 \vee t(\|\nabla^2 G_2'(t)\|_2 \vee \|\nabla\partial_t G_2'(t)\|_2\\
&\vee \|\nabla\check{G}_2'(t)\|_2) \vee t^2(\|\nabla^2\check{G}_2'(t)\|_2 \vee \|\nabla\partial_t\check{G}_2'(t)\|_2)\}.
\end{aligned} \tag{3.117}$$

For technical reasons, we shall also need the following auxiliary norms:

$$V_1' = \sup_{t\in I} t^{1/2}h(t)^{-1}\|\langle x\rangle\nabla q'(t)\|_2, \tag{3.118}$$

$$\tilde{V}_1' = \sup_{t\in I} t^{1/2}h(t)^{-1}\|\langle x\rangle\nabla_K q'(t)\|_2, \tag{3.119}$$

$$\tilde{V}_{3/2}' = \sup_{t\in I} t^{3/4}h(t)^{-1}\|\langle x\rangle\nabla_K q'(t)\|_3, \tag{3.120}$$

$$V'_{3/2} = \sup_{t\in I} t^{3/4}h(t)^{-1}\|\langle x\rangle q'(t)\|_\infty, \tag{3.121}$$

$$V'_{2,t} = \sup_{t\in I} th(t)^{-1}\|\langle x\rangle \partial_t q'(t)\|_2, \tag{3.122}$$

$$\tilde{V}'_{2,x} = \sup_{t\in I} th(t)^{-1}\|\langle x\rangle \Delta_K q'(t)\|_2, \tag{3.123}$$

$$\tilde{V}'_2 = V'_{2,t} \vee \tilde{V}'_{2,x}, \tag{3.124}$$

$$\tilde{V}''_2 = V'_{2,t} \vee \sup_{t\in I} th(t)^{-1}\|\nabla_K^2 q'(t)\|_2, \tag{3.125}$$

$$\tilde{V}'_{5/2} = \sup_{t\in I} t^{5/4}h(t)^{-1}(\|\partial_t q'(t)\|_3 \vee \|\nabla_K q'(t)\|_\infty \vee \|\nabla_K^2 q'(t)\|_3), \tag{3.126}$$

$$\tilde{V}'_{3,t} = \sup_{t\in I} t^{3/2}h(t)^{-1}\|\nabla_K \partial_t q'(t)\|_2, \tag{3.127}$$

$$\tilde{V}'_{3,x} = \sup_{t\in I} t^{3/2}h(t)^{-1}\|\nabla_K \Delta_K q'(t)\|_2, \tag{3.128}$$

$$\tilde{V}'_3 = \tilde{V}'_{3,t} \vee \tilde{V}'_{3,x}. \tag{3.129}$$

We can now state the existence result of solutions of the linearized system (3.17).

Proposition 3.8. *Let* $1/4 \le \beta < 3/4$ *and* $I = (0, \tau]$ *with* $0 < \tau \le \tau_0$. *Let* w_a, K_a *and the remainders* R_j, $1 \le j \le 5$, *satisfy the assumptions (A1), (A2), (A3) and let* B_0, B_{e0} *satisfy* (2.31) *for* $0 \le j \le 1, 0 \le k \le 2$ *and* $r = \infty$. *Let* $(q, G_2) \in X(I)$, *satisfying the bounds* (3.89)–(3.92). *Then, for* τ *sufficiently small, there exists a unique solution* (q', G'_2) *of the system* (3.17) *in* $X(I)$, *and that solution is estimated in the norms* $V'_{00}, V'_{01}, V'_2, V'_3, Z'$ *defined by* (3.114)–(3.117) *by*

$$V'_{00} \le N_{00}, \tag{3.130}$$

$$V'_{01} \le b_0 N_{00}\tau^{1/2} + N_{01} + N_2\tau + N_3\tau^{3/2}, \tag{3.131}$$

$$V'_2 \le b_0 N_{00}\tau^{1/2} + b_0 N_{01} + N_2 + N_3\tau^{1/2}, \tag{3.132}$$

$$V'_3 \le b_0(N_{00} + b_0 N_{01} + N_2) + N_3, \tag{3.133}$$

$$Z' \le C((V_2 + r_6)\tau^{1/2}(1 - \ln\tau) + Z\tau) + r_4. \tag{3.134}$$

Proof. We first choose τ sufficiently small to satisfy the asymptotic region conditions (3.35) of Lemma 3.1 and (3.94) of Lemma 3.6, so that we have (3.56), (3.69)–(3.71) and all the estimates in Lemma 3.6. Furthermore, from (3.94), (3.97) and (3.102), we obtain $\|\nabla\nabla\cdot L\|_2 \le Ct^{-1/4}$. This estimate, together with (A2) and (3.56), we obtain

$$n, \bar{n} \le b_0 t^{-1} + C(1 - \ln t)^2, \quad \tilde{n} \le b_0 t^{-2} + Ct^{-1}(1 - \ln t), \tag{3.135}$$

where $n, \bar{n}$ and $\tilde{n}$ are defined in Lemma 3.5.

Since G_2' is explicitly defined by the expression (3.17) and satisfies the estimates (3.104)–(3.106), it is defined in the same interval as (q, G_2) and satisfies the estimate (3.134).

We proceed to the construction and estimates of q'. Let $0 < t_0 < \tau$ and let q'_{t_0} be the solution of the Schrödinger equation in (3.17) with initial condition $q'_{t_0}(t_0) = 0$. We shall derive estimates of $(q'_{t_0}, 0)$ in $X([t_0, \tau])$ that are uniform in t_0 by the use of Lemmas 3.4 and 3.5. Once we obtain such estimates, we can construct q' as the limit of q'_{t_0} as $t_0 \to 0$. Indeed, let $0 < t_0 < t_1 \le \tau$. Since $q'_- \equiv q'_{t_1} - q'_{t_0}$ satisfies the equation $i\partial_t q'_- = Hq'_-$, we have the conservation of the L^2-norm of q'_-, that is,

$$\|q'_-(t)\|_2 = \|q'_-(t_1)\|_2 = \|q'_{t_0}(t_1)\|_2 \le V'_{00} t_1^{1/2} h(t_1),$$

where V'_{00} is defined by (3.114). Since the right-hand side goes to zero as $0 < t_0 < t_1 \to 0$, we see that $q' = \lim_{t_0 \to 0} q'_{t_0}$ exists in $L^\infty((0,\tau], L^2)$. Furthermore, since $(q'_{t_0}, 0)$ is uniformly bounded in $X([t_0, \tau])$ with respect to t_0, we can show that q'_{t_0} converges to q', strongly in $\mathscr{C}((0,\tau], H^s)$ for $0 \le s < 3$, star-weakly in $\mathscr{C}_w((0,\tau], H^3)$ and pointwise in H^3. We can also show that q' satisfies the Schrödinger equation in (3.17), and that $(q', 0) \in X((0,\tau])$ with the same estimates in $(0,\tau]$ as $(q'_{t_0}, 0)$ in $[t_0, \tau]$, uniformly with respect to t_0. For the detail, see the proof of Proposition 5.1 in [13].

We shall now derive estimates for q'_{t_0}. In the computation below, we omit the subscript t_0 for brevity, and we use the definitions (3.114)–(3.129) with $I = [t_0, \tau]$.

Integrating (3.72) and (3.73) in $[t_0, t]$ with $q(t_0) = 0$, we obtain

$$\|q'(t)\|_2 \le \int_{t_0}^t dt' \|\langle x \rangle \tilde{R}_1(t')\|_2,$$

$$\|\langle x \rangle q'(t)\|_2 \le \tilde{V}_1' \int_{t_0}^t dt'\ t'^{-1/2} h(t') + \int_{t_0}^t dt' \|\langle x \rangle \tilde{R}_1(t')\|_2,$$

so that

$$V'_{00} \le N_{00}, \quad V'_{01} \le \tilde{V}_1' \tau^{1/2} + N_{01}. \tag{3.136}$$

Integrating (3.74) and (3.75) in $[t_0, t]$ with $\partial_t q'(t_0) = i\tilde{R}_1(t_0)$, we obtain

$$\|\langle x \rangle \partial_t q'(t)\|_2 \le (\tilde{V}'_{3,t} + C(\tilde{V}_1' + \tilde{V}'_{3/2} + V'_{01})) \int_{t_0}^t dt'\ t'^{-3/2} h(t')$$
$$+ b_0 V'_{01} \int_{t_0}^t dt'\ t'^{-2} h(t') + \|\langle x \rangle \tilde{R}_1(t_0)\|_2 + \int_{t_0}^t dt' \|\langle x \rangle \partial_t \tilde{R}_1(t')\|_2,$$

so that

$$V'_{2,t} \le b_0 V'_{01} + \tilde{V}'_{3,t}\tau^{1/2} + C(\tilde{V}'_1 + \tilde{V}'_{3/2} + V'_{3/2} + V'_{01})\tau^{1/2} + N_2.$$

Integrating (3.76) in $[t_0, t]$ with $\nabla_K \partial_t q'(t_0) = i\nabla_K \tilde{R}_1(t_0)$, we obtain

$$\begin{aligned}\|\nabla_K \partial_t q'(t)\|_2 \le{}& b_0 \tilde{V}'_1 \int_{t_0}^t dt'\ t'^{-5/2} h(t') \\ &+ C(\tilde{V}''_2 + \tilde{V}'_{5/2} + \tilde{V}'_1 + V'_{3/2} + V'_{00} t^{1/2}) \int_{t_0}^t dt'\ t'^{-2} h(t') \\ &+ b_0 V'_{00} \int_{t_0}^t dt'\ t'^{-5/2} h(t') + \|\nabla_K \tilde{R}_1(t_0)\|_2 + \int_{t_0}^t dt' \|\nabla_K \partial_t \tilde{R}_1(t')\|_2,\end{aligned}$$

so that

$$\tilde{V}'_{3,t} \le b_0(V'_{00} + \tilde{V}'_1) + C(\tilde{V}''_2 + \tilde{V}'_{5/2} + \tilde{V}'_1 + V'_{3/2} + V'_{00}\tau^{1/2})\tau^{1/2} + N_3.$$

From (3.77), (3.78), we then obtain

$$\begin{aligned}\tilde{V}'_{2,x} &\le V'_{2,t} + CV'_{01}\tau^{1/2} + N_2, \\ \tilde{V}'_{3,x} &\le V'_{3,t} + C(\tilde{V}'_1\tau^{1/2} + V'_{00}\tau) + N_3,\end{aligned}$$

so that $\tilde{V}'_2$ and $\tilde{V}'_3$ satisfy the same estimates as $V'_{2,t}, \tilde{V}'_{3,t}$, namely

$$\tilde{V}'_2 \le b_0 V'_{01} + \tilde{V}'_{3,t}\tau^{1/2} + C(\tilde{V}'_1 + \tilde{V}'_{3/2} + V'_{3/2} + V'_{01})\tau^{1/2} + N_2, \tag{3.137}$$

$$\tilde{V}'_3 \le b_0(V'_{00} + \tilde{V}'_1) + C(\tilde{V}''_2 + \tilde{V}'_{5/2} + \tilde{V}'_1 + V'_{3/2} + V'_{00})\tau^{1/2} + N_3. \tag{3.138}$$

We will next replace the covariant derivatives in (3.136), (3.137), (3.138) with usual ones, by the use of assumption (A2) and Lemma 3.5 together with the Sobolev inequality. From (A2), we obtain

$$\tilde{V}'_1 \le V'_1 + CV'_{01}\tau^{1/2}(1 - \ln\tau), \tag{3.139}$$

$$\tilde{V}'_{3,t} \le V'_3 + CV'_2\tau^{1/2}(1 - \ln\tau). \tag{3.140}$$

From Lemma 3.5, especially (3.85), and (3.86), (3.87) with $r = 2$, we have

$$\begin{aligned}\tilde{V}'_{3/2} &\le (V'_1(V'_2 + V'_1\tau^{1/2}))^{1/2} + C(V'_{01}(V'_1 + V'_{01}\tau^{1/2}))^{1/2}\tau^{1/2}(1 - \ln\tau) \\ &\le (V'_1 V'_2)^{1/2} + C(V'_1 + V'_{01}\tau^{1/2})\tau^{1/4}(1 - \ln\tau),\end{aligned} \tag{3.141}$$

$$V'_2 \le \tilde{V}'_2 + CV'_{01}, \tag{3.142}$$

$$\tilde{V}''_2 \le V'_2 + CV'_{01}. \tag{3.143}$$

Here, we have used (3.135). From the Sobolev inequality, the estimate (3.87) with $r = 3$, and (3.135), we obtain

$$\begin{aligned}
\|\partial_t q'\|_3 &\leq (V_2'V_3')^{1/2} t^{-5/4} h(t), \\
\|\nabla_K q'\|_\infty &\leq ((V_2'V_3')^{1/2} + C(V_1'V_2')^{1/2} t^{1/2}(1-\ln t)) t^{-5/4} h(t), \\
\|\nabla_K^2 q'\|_3 &\leq ((V_2'V_3')^{1/2} + C(V_{01}'V_1')^{1/2}) t^{-5/4} h(t).
\end{aligned}$$

Combining these estimates, we obtain

$$\tilde{V}_{5/2}' \leq (V_2'V_3')^{1/2} + C(V_2'\tau(1-\ln\tau)^2 + V_1' + V_{01}'). \tag{3.144}$$

Using (3.88), (3.135) and (A2), and the Sobolev inequality, we estimate

$$\begin{aligned}
\sup_{t\in I} t^{3/2} h(t)^{-1} \|\nabla\Delta q'(t)\|_2 &\leq \tilde{V}_{3,x}' + b_0 \tau^{1/4} (V_{00}'V_2')^{1/2} + (b_0 + C\tau(1-\ln\tau))V_{00}' \\
&\leq \tilde{V}_{3,x}' + b_0(V_2'\tau^{1/2} + V_{00}') + CV_{00}'\tau(1-\ln\tau),
\end{aligned}$$

which together with (3.140) implies

$$V_3' \leq \tilde{V}_3' + b_0 V_{00}' + C(V_2' + V_{00}'\tau^{1/2})\tau^{1/2}(1-\ln\tau). \tag{3.145}$$

Now, we replace the covariant derivatives in (3.136), (3.137) and (3.138) with usual ones by the use of (3.139)–(3.145). First, substituting (3.139) into the second inequality of (3.136), we obtain

$$V_{01}' \leq V_1'\tau^{1/2} + CV_{01}'\tau(1-\ln\tau) + N_{01}. \tag{3.146}$$

Next, substituting (3.139), (3.140), (3.141) into (3.137) and substituting the result into (3.142), we obtain

$$V_2' \leq b_0 V_{01}' + V_3'\tau^{1/2} + C(V_2'\tau(1-\ln\tau) + ((V_1'V_2')^{1/2} + V_1' + V_{01}')\tau^{1/2}) + N_2. \tag{3.147}$$

Substituting (3.139), (3.143), (3.144) into (3.138) and using $V_{3/2}' \leq (V_1'V_2')^{1/2}$, we obtain

$$\tilde{V}_3' \leq b_0(V_{00} + V_1') + C((V_2'V_3')^{1/2} + V_2' + V_1' + V_{01}'(1-\ln\tau) + V_{00}')\tau^{1/2} + N_3. \tag{3.148}$$

Substituting (3.148) into (3.145), we obtain

$$V_3' \leq b_0(V_{00}' + V_1') + C((V_2'V_3')^{1/2} + (V_2' + V_{01}')(1-\ln\tau) + V_1' + V_{00}')\tau^{1/2} + N_3. \tag{3.149}$$

We next simplify the resulting inequalities (3.146), (3.147), (3.149) by using the inequality $V_1' \leq (V_{01}'V_2')^{1/2}$ to eliminate V_1', the obvious inequality $V_{01}' \leq V_2'$ at some harmless places, smallness conditions of the type $C\tau(1-\ln\tau) \le 1$ to eliminate the diagonal terms in V'_{01} and V'_2 in (3.146), (3.147) and some elementary algebraic manipulations. We obtain

$$V'_{00} \le N_{00}, \quad V'_{01} \le V'_2\tau^{1/2} + N_{01}, \tag{3.150}$$

$$V'_2 \le b_0 V'_{01} + V'_3\tau^{1/2} + N_2, \tag{3.151}$$

$$V'_3 \le b_0 V'_{00} + b_0(V'_{01}V'_2)^{1/2} + CV'_2\tau^{1/2}(1-\ln\tau) + N_3. \tag{3.152}$$

Substituting (3.150), (3.152) into (3.151), we obtain

$$V'_2 \le b_0(N_{00}\tau^{1/2} + N_{01}) + CV'_2\tau^{1/2}(1-\ln\tau) + N_3\tau^{1/2} + N_2,$$

which yields (3.132) under an additional smallness condition on τ. Substituting (3.132) into (3.150), (3.152) yields (3.131), (3.133). □

We can now derive the main result of this section, namely the existence of solutions of the nonlinear system (3.11).

Proposition 3.9. *Let $\beta = 1/2$. Let w_a, K_a and the remainders R_j, $1 \le j \le 5$, satisfy the assumptions (A1), (A2), (A3) and let B_0, B_{e0} satisfy* (2.31) *for $0 \le j \le 1, 0 \le k \le 2$ and $r = \infty$. Then there exists τ, $0 < \tau \le \tau_0$ and there exists a unique solution $(q, G_2) \in X(I)$ of the system* (3.11)*, where $I = (0, \tau]$. In particular (q, G_2) satisfies the estimates* (3.89)–(3.92) *for some constants V_{00}, V_{01}, V_2, V_3 and Z depending on w_a, K_a and on the remainders through the norms occurring in the assumptions (A1), (A2) and (A3). The solution (q, G_2) is unique under the assumption that $(q, G_2) \in X_0(I)$, and that (q, G_2) satisfies the following conditions:*

$$q \in L^\infty(I, H^3), \quad xq \in L^\infty(I, H^2), \quad t\partial_t q \in L^\infty(I, H^1),$$

$$\nabla G_2 \in L^\infty(I, H^1), \quad t\nabla\partial_t G_2 \in L^\infty(I, L^2), \quad t\nabla^2\check{G}_2 \in L^\infty(I, L^2),$$

$$\|\langle x\rangle q(t)\|_2 \le Ch_1(t)$$

for all $t \in I$, for some h_1 satisfying the conditions of Proposition 2.4.

Proof. Proposition 3.8 defines a mapping $\Gamma : (q, G_2) \to (q', G'_2)$ from $X(I)$ into itself. For given $\boldsymbol{V} = (V_{00}, V_{01}, V_2, V_3)$ and Z, we define a subset $\mathscr{R}$ of $X(I)$ by (3.89)–(3.92). We show that for sufficiently small τ and for a suitable choice of $(\boldsymbol{V}, Z)$, the mapping Γ is a contraction on $\mathscr{R}$ with respect to the norm

$$\sup_{t\in I} h(t)^{-1}\|\langle x\rangle q(t)\|_2 + \sup_{t\in I} t^{1/2}h(t)^{-1}\|\nabla q(t)\|_2 + \sup_{t\in I} t^{-1/2}h(t)^{-1}(\|\nabla G_2(t)\|_2 \vee t\|\nabla\check{G}_2(t)\|_2),$$

which we have used in Lemma 2.3.

We first show that Γ maps $\mathscr{R}$ into itself. If $\beta = 1/2$, then it follows from Lemma 3.7 that

$$\begin{aligned} N_{00} &\le Cr + C(V_{00}\tau^{1/2} + Z), \\ N_{01} &\le Cr + C(V_{01} + Z)\tau^{1/2}, \\ N_2 &\le Cr + C(V_2 + Z)\tau^{1/2}, \\ N_3 &\le Cr + C(V_2 + V_3 + Z)\tau^{1/2}, \end{aligned}$$

where $r = \max_{1\le i\le 5} r_i$. Therefore, from Proposition 3.8, it follows that $\boldsymbol{V}' = (V'_{00}, V'_{01}, V'_2, V'_3)$ and Z' defined by (3.114)–(3.117) satisfy

$$\begin{aligned} |\boldsymbol{V}'| &\le Cr + C|\boldsymbol{V}|\tau^{1/2} + CZ, \\ Z' &\le c\,r + C((|\boldsymbol{V}|(1 - \ln\tau) + r) + Z)\tau^{1/2} \end{aligned}$$

as long as the conclusion of Proposition 3.8 is satisfied. Here, $|\boldsymbol{V}| = \max\{V_{00}, V_{01}, V_2, V_3\}$, c is an absolute constant, and the various constants C may depend on the asymptotic functions (w_a, s_a, B_a, B_{ea}) but are independent of $(\boldsymbol{V}, Z)$. We shall choose $(\boldsymbol{V}, Z)$ such that

$$Z = 2cr, \quad V_{00} = V_{01} = V_2 = V_3 = 2C(r + Z).$$

We shall take τ sufficiently small, so that the assumption of Proposition 3.8 is satisfied, and that

$$2C|\boldsymbol{V}|\tau^{1/2} \le |\boldsymbol{V}|, \quad 2C((|\boldsymbol{V}|(1 - \ln\tau) + r) + Z)\tau^{1/2} \le Z.$$

Then, we find that $\Gamma(\mathscr{R}) \subset \mathscr{R}$.

We next show that Γ is a contraction mapping on $\mathscr{R}$. We note that $\mathscr{R}$ is closed with respect to this metric. For $i = 1, 2$, let $(q_i, G_{2i}) \in \mathscr{R}$ and $(q'_i, G'_{2i}) = \Gamma(q_i, G_{2i})$. We define $(q_\pm, G_{2\pm})$ and $(q'_\pm, G'_{2\pm})$ by $f_\pm = (1/2)(f_1 \pm f_2)$, so that in particular all those quantities belong to $\mathscr{R}$. We set

$$\begin{aligned} V_{0-} &= \sup_{t\in I} h(t)^{-1}\|\langle x\rangle q_-(t)\|_2, \\ V_{1-} &= \sup_{t\in I} t^{1/2}h(t)^{-1}\|\nabla q_-(t)\|_2, \\ Z_- &= \sup_{t\in I} t^{-1/2}h(t)^{-1}(\|\nabla G_{2-}(t)\|_2 \vee t\|\nabla \check{G}_{2-}(t)\|_2), \end{aligned}$$

and similarly for the primed quantities, and we estimate V'_{0-}, V'_{1-}, Z'_- in terms of V_{0-}, V_{1-}, Z_-. From Lemma 2.3, we obtain

$$\|G_{1-}\|_2 \vee \|G_{3-}\|_2 \le CV_{0-}h, \tag{3.153}$$

$$\|\check{G}_{1-}\|_2 \vee t^{-1}\|G_{3-}\|_2 \le Ct^{-1}V_{0-}h, \tag{3.154}$$

$$\|\nabla^k \sigma_-\|_2 \le CV_{0-}t^{-k/2}h \quad \text{for } k = 0, 1, 2, \tag{3.155}$$

$$|\partial_t \|q'_-\|_2| \le C(V_{0-} + Z_-)t^{-1/2}h, \tag{3.156}$$

$$|\partial_t \|xq'_-\|_2| \le (V'_{1-} + CV'_{0-}t^{1/2}(1 - \ln t) + CV_{0-} + CZ_-)t^{-1/2}h, \tag{3.157}$$

$$\begin{aligned} |\partial_t \|\nabla_{K_+} q'_-\|_2| \le C(&V'_{0-}t^{-1} + (V'_{0-}V'_{1-})^{1/2}t^{-5/4} + V_{0-}t^{-1} \\ &+ V_{1-}t^{-1/2} + Z_-t^{-1/2}(1 - \ln t))h. \end{aligned} \tag{3.158}$$

Integrating (3.156)–(3.158) over time and estimating G'_{2-} by Lemma 2.3, especially (2.59), (2.60), we obtain

$$\begin{aligned} V'_{0-} &\le V'_{1-}\tau^{1/2} + C(V'_{0-}\tau(1 - \ln \tau) + V_{0-}\tau^{1/2} + Z_-\tau^{1/2}), \\ V'_{1-} &\le C(V'_{0-}\tau^{1/2}(1 - \ln \tau) + (V'_{0-}V'_{1-})^{1/2}\tau^{1/4} + V_{0-}\tau^{1/2} + V_{1-}\tau + Z_-\tau(1 - \ln \tau)), \\ Z'_- &\le C(V_{0-}\tau^{1/2}(1 - \ln \tau) + Z_-\tau). \end{aligned}$$

From these estimates, we obtain

$$V'_{0-} + V'_{1-} + Z'_- \le C((V'_{0-} + V'_{1-} + V_{0-})\tau^{1/2}(1 - \ln \tau) + V_{1-}\tau + Z_-\tau^{1/2}),$$

which implies

$$V'_{0-} + V'_{1-} + Z'_- \le (1/2)(V_{0-} + V_{1-} + Z_-)$$

for τ sufficiently small. This proves that Γ is a contraction mapping, and hence Γ has a fixed point by the contraction mapping principle. □

3.2. **Analysis of the Asymptotic Functions.** We now turn to the second step, namely the construction of (w_a, s_a, B_a, B_{ea}), which is an asymptotic solution of the auxiliary system (2.22), satisfying the assumptions made in the previous step and in particular the remainder estimates needed for the Cauchy problem at $t = 0$ for that system. We note that as regards B_a and B_{ea}, we separately construct asymptotic forms B_{1a}, B_{2a} and B_{3a}, namely

$$B'_a = B_{1a} + B_{2a}, \quad B'_{ea} = B_{3a}$$

in accordance with (2.13). We construct (w_a, s_a, B_a, B_{ea}) by solving the system (2.22) by iteration in the form $w_a = \sum_j w_{aj}$ etc., with $w_{aj} = O(t^j)$ modulo logarithms. Then, we first define the approximate solution of order 0 by

$$\begin{cases} i\partial_t w_{a0} + (1/2)\Delta w_{a0} = 0, \quad w_{a0}(0) = w_+, \\ B_{1a0} = B_1(w_{a0}), \quad B_{2a0} = 0, \quad B_{3a0} = B_3(w_{a0}), \\ \partial_t s_{a0} = \nabla \check{B}_1^L(w_{a0}) - t^{-1}\nabla B_3^L(w_{a0}), \quad s_{a0}(1) = 0. \end{cases} \tag{3.159}$$

Here, the choice $B_{2a0} = 0$ reflects the fact that B_2 itself is of order t. We next define the terms of order t by

$$\begin{cases} B_{2a1} = \mathscr{B}_2(w_{a0}, w_{a0}, s_{a0} + B_{1a0}), \\ i\partial_t w_{a1} = i(s_{a0} + B_{1a0}) \cdot \nabla w_{a0} + (i/2)(\nabla \cdot (s_{a0} + B_{1a0}))w_{a0} \\ \qquad + (1/2)(s_{a0} + B_{1a0})^2 w_{a0} + (\check{B}_{1a0}^S + \check{B}_{2a1} - t^{-1}B_{3a0}^S)w_{a0}, \quad w_{a1}(0) = 0, \\ B_{1a1} = 2B_1(w_{a0}, w_{a1}), \quad B_{3a1} = 2B_3(w_{a0}, w_{a1}), \\ \partial_t s_{a1} = 2\nabla \check{B}_1^L(w_{a0}, w_{a1}) - 2t^{-1}\nabla B_3^L(w_{a0}, w_{a1}), \quad s_{a1}(0) = 0. \end{cases} \tag{3.160}$$

For w_a and B_{2a}, we need the approximate solution of order 2 defined by

$$\begin{cases} B_{2a2} = 2\mathscr{B}_2(w_{a0}, w_{a1}, s_{a0} + B_{1a0}) - tF_2(P(s_{a1} + B_{1a1} + B_{2a1})|w_{a0}|^2), \\ i\partial_t w_{a2} = -(1/2)\Delta w_{a1} + i\sum_{j+k=1}\{(s_{aj} + B_{1aj} + B_{2aj}) \cdot \nabla w_{ak} \\ \qquad + (1/2)(\nabla \cdot (s_{aj} + B_{1aj} + B_{2aj}))w_{ak}\} \\ \qquad + (1/2)\sum_{j+k+l=1}(s_{aj} + B_{1aj} + B_{2aj})(s_{ak} + B_{1ak} + B_{2ak})w_{al} \\ \qquad + \sum_{j+k=1}(\check{B}_{1aj}^S + \check{B}_{2a(j+1)} - t^{-1}B_{3aj}^S)w_{ak}, \quad w_{a1}(0) = 0. \end{cases} \tag{3.161}$$

We define

$$\begin{cases} w_a = w_{a0} + w_{a1} + w_{a2}, \quad s_a = s_{a0} + s_{a1}, \\ B_{1a} = B_{1a0} + B_{1a1}, \quad B_{2a} = B_{2a1} + B_{2a2}, \quad B_{3a} = B_{3a0} + B_{3a1}. \end{cases} \tag{3.162}$$

Then (w_a, s_a, B_a, B_{ea}) defined above turns out to be an adequate approximate solution. In the construction of w_a, we omit the contribution of B_0 and B_{e0}, so that they appear only in R_1 and R_4. For any polynomial function $f(w_a, s_a, B_{1a}, B_{2a}, B_{3a})$ and any nonnegative integer p, we define

$$f(w_a, s_a, B_{1a}, B_{2a}, B_{3a})_{\geq p} = \sum_{j+k+l+m+n \geq p} f(w_{aj}, s_{ak}, B_{1al}, B_{2am}, B_{3an}).$$

The remainders $R_j, 1 \leq j \leq 5$, defined by (3.14) then become

$$R_1 = R_{10} + R_{11}, \tag{3.163}$$

where

$$\begin{aligned} R_{10} &= -iB_0 \cdot \nabla w_a - (i/2)(\nabla \cdot B_0) w_a \\ &\quad - (B_0(s_a + B_{1a} + B_{2a}) + (1/2)B_0^2 + \check{B}_0 - t^{-1}B_{e0}) w_a, \qquad (3.164) \\ R_{11} &= (1/2)\Delta w_{a2} - \{ i(s_a + B_{1a} + B_{2a}) \cdot \nabla w_a + (i/2)(\nabla \cdot (s_a + B_{1a} + B_{2a})) w_a \\ &\quad + (1/2)(s_a + B_{1a} + B_{2a})^2 w_a + \check{B}_{1a}^S w_a - t^{-1} B_{3a}^S w_a \}_{\geq 2} - (\check{B}_{2a} w_a)_{\geq 3}, \qquad (3.165) \\ R_2 &= -\nabla \check{B}_1^L (w_a)_{\geq 2} + t^{-1} \nabla B_3^L (w_a)_{\geq 2}, \qquad (3.166) \\ R_3 &= -B_1(w_a)_{\geq 2}, \qquad (3.167) \\ R_4 &= R_{40} + R_{41}, \qquad (3.168) \\ R_5 &= -B_3(w_a)_{\geq 2}, \qquad (3.169) \end{aligned}$$

where

$$\begin{aligned} R_{40} &= tF_2(PB_0|w_a|^2), \qquad (3.170) \\ R_{41} &= -\mathscr{B}_2(w_a, w_a, s_a + B_{1a} + B_{2a})_{\geq 2}. \qquad (3.171) \end{aligned}$$

We now turn to estimate (w_a, s_a, B_a, B_{ea}). We use the spaces $\ddot{H}^k = \dot{H}^1 \cap \dot{H}^k$ and the notation $v \in (X, f)$ to mean that $v \in \mathscr{C}(I, X)$ with $\|v(t); X\| \leq f(t)$ for all $t \in I$, with $I = (0, \tau]$ for some $\tau, 0 \leq \tau \leq 1$, with $\tau = 1$ in the present case.

Lemma 3.10. *Let $\beta > 0$. Let $w_+ \in H^{k_+}, xw_+ \in H^{k_+-1}$ with $k_+ \geq 7 \vee (5 + \beta^{-1})$. Then the components of w_a, s_a, B_a and B_{ea} defined by* (3.159)–(3.162) *satisfy the following properties:*

$$\begin{aligned} w_{a0} &\in (H^{k_+}, 1), \quad xw_{a0} \in (H^{k_+-1}, 1), \qquad (3.172) \\ \partial_t w_{a0} &\in (H^{k_+-2}, 1), \quad \partial_t x w_{a0} \in (H^{k_+-3}, 1), \qquad (3.173) \\ B_{1a0}, B_{3a0} &\in (\ddot{H}^{k_++1}, 1), \quad \check{B}_{1a0} \in (\langle x \rangle \dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^{k_+}, t^{-1}), \qquad (3.174) \\ \partial_t B_{1a0}, \partial_t B_{3a0} &\in (H^{k_+-1}, 1), \qquad (3.175) \\ \partial_t \check{B}_{1a0} &\in (\langle x \rangle \dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^{k_+}, t^{-2}) + (\ddot{H}^{k_+-2}, t^{-1}), \qquad (3.176) \\ s_{a0} &\in (\ddot{H}^{k_+-1}, (1 - \ln t)), \quad \partial_t s_{a0} \in (\ddot{H}^{k_+-1}, t^{-1}), \qquad (3.177) \\ B_{2a1} &\in (H^{k_++1}, t(1 - \ln t)), \quad \check{B}_{2a1} \in (\ddot{H}^{k_++1}, (1 - \ln t)), \qquad (3.178) \\ \partial_t B_{2a1} &\in (H^{k_++1}, (1 - \ln t)) + (H^{k_+-1}, t(1 - \ln t)), \qquad (3.179) \end{aligned}$$

$$\partial_t \check{B}_{2a1} \in (\ddot{H}^{k_++1}, t^{-1}) + (H^{k_+-1}, (1-\ln t)). \tag{3.180}$$

Let in addition $k_+ \geq 2(1+\beta^{-1})$ *and define* $k_1 = (k_+ - 2) \wedge (k_+ - \beta^{-1})$, $k_2 = (k_1 - 2) \wedge (k_1 + 1 - \beta^{-1})$. *Then*

$$w_{a1} \in (H^{k_1}, t(1-\ln t)^2), \quad xw_{a1} \in (H^{k_1}, t(1-\ln t)^2), \tag{3.181}$$

$$\partial_t w_{a1} \in (H^{k_1}, (1-\ln t)^2), \quad \partial_t xw_{a1} \in (H^{k_1}, (1-\ln t)^2), \tag{3.182}$$

$$B_{1a1}, B_{3a1} \in (H^{k_1+1}, t(1-\ln t)^2), \quad \check{B}_{1a1} \in (\ddot{H}^{k_1+1}, (1-\ln t)^2), \tag{3.183}$$

$$\partial_t B_{1a1}, \partial_t B_{3a1} \in (H^{k_1+1}, (1-\ln t)^2), \tag{3.184}$$

$$\partial_t \check{B}_{1a1} \in (\ddot{H}^{k_1+1}, t^{-1}(1-\ln t)^2) + (H^{k_+-2}, (1-\ln t)^2), \tag{3.185}$$

$$s_{a1} \in (H^{k_1}, t(1-\ln t)^2), \quad \partial_t s_{a1} \in (H^{k_1}, (1-\ln t)^2), \tag{3.186}$$

$$B_{2a2} \in (H^{k_1+1}, t^2(1-\ln t)^3), \quad \check{B}_{2a2} \in (H^{k_1+1}, t(1-\ln t)^3), \tag{3.187}$$

$$\partial_t B_{2a2} \in (H^{k_1+1}, t(1-\ln t)^3) + (H^{k_+-2}, t^2(1-\ln t)^2), \tag{3.188}$$

$$\partial_t \check{B}_{2a2} \in (H^{k_1+1}, (1-\ln t)^3) + (H^{k_+-2}, t(1-\ln t)^2), \tag{3.189}$$

$$w_{a2} \in (H^{k_2}, t^2(1-\ln t)^4), \quad xw_{a2} \in (H^{k_2}, t^2(1-\ln t)^4), \tag{3.190}$$

$$\partial_t w_{a2} \in (H^{k_2}, t(1-\ln t)^4), \quad \partial_t xw_{a2} \in (H^{k_2}, t(1-\ln t)^4), \tag{3.191}$$

Proof. We note that $k_1 \geq 5$ and $k_2 \geq 3$ provided that $k_+ \geq 2(1+\beta^{-1})$. The properties (3.172)–(3.189) are proved similarly as Lemma 6.1 in [13]. For the proof of (3.190) and (3.191), we need the following estimates:

$$\|\omega^{k_2} \check{B}^S_{1a0}\|_2 \vee \|\check{B}^S_{1a0}\|_\infty \vee t^{-1}(\|\omega^{k_2} B^S_{3a0}\|_2 \vee \|B^S_{3a0}\|_\infty) \leq C, \tag{3.192}$$

$$\|\omega^{k_2} \check{B}^S_{1a1}\|_2 \vee \|\check{B}^S_{1a1}\|_\infty \vee t^{-1}(\|\omega^{k_2} B^S_{3a1}\|_2 \vee \|B^S_{3a1}\|_\infty) \leq Ct(1-\ln t)^2. \tag{3.193}$$

These estimates are derived by (3.174) and (3.183), together with the Sobolev inequality, (2.32) and (2.33). For example, we have

$$\|\omega^{k_2} \check{B}^S_{1a1}\|_2 \leq t^{\beta(k_1+1-k_2)} \|\omega^{k_1+1} \check{B}_{1a0}\|_2 \leq Ct(1-\ln t)^2,$$

since $\beta(k_1 + 1 - k_2) \geq 1$. Using these estimates together with the Leibniz rule, we obtain

$$\langle x \rangle \sum_{j+l=1} (\check{B}^S_{1aj} - t^{-1} B^S_{3aj}) w_{al} \in (H^{k_2}, t(1-\ln t)^2).$$

Indeed, we estimate for instance

$$\|\omega^{k_2} \check{B}^S_{1a1} w_{a0}\|_2 \leq \|\omega^{k_2} \check{B}^S_{1a1}\|_2 \|w_{a0}\|_\infty + \|\check{B}^S_{1a1}\|_\infty \|\omega^{k_2} w_{a0}\|_2 \leq Ct(1-\ln t)^2.$$

The other terms in $\partial_t w_{a2}$ are estimated in H^{k_2} by (3.172)–(3.189) and the Leibniz rule. The worst term is $s_{a0}^2 w_{a1} = O(t(1-\ln t)^4)$, so that we obtain (3.191). Integrating (3.191) in t, we obtain (3.190). □

We summarize the information on (w_a, s_a, B_a, B_{ea}) which follows from Lemma 3.10 in the following proposition.

Proposition 3.11. *Let $\beta > 0$. Let $w_+ \in H^{k_+}, xw_+ \in H^{k_+-1}$ with $k_+ \geq 7 \vee (5+\beta^{-1}) \vee 2(1+\beta^{-1})$ and let $k_1 = (k_+ - 2) \wedge (k_+ - \beta^{-1})$, $k_2 = (k_1 - 2) \wedge (k_1 + 1 - \beta^{-1})$. Let $(w_a, s_a, B'_a, B'_{ea})$ be defined by* (3.159)–(3.162) *and $B'_a = B_{1a} + B_{2a}, B'_{ea} = B_{3a}$. Then $(w_a, s_a, B'_a, B'_{ea})$ satisfies the following properties:*

$$w_a \in (H^{k_2}, 1), \quad xw_a \in (H^{k_2}, 1), \tag{3.194}$$

$$\partial_t w_a \in (H^{k_2}, (1-\ln t)^2), \quad \partial_t x w_a \in (H^{k_2}, (1-\ln t)^2), \tag{3.195}$$

$$s_a \in (\ddot{H}^{k_1}, (1-\ln t)), \quad \partial_t s_a \in (\ddot{H}^{k_1}, t^{-1}), \tag{3.196}$$

$$\begin{aligned} B'_a, B'_{ea} &\in (\ddot{H}^{k_1+1}, 1), \quad t^{-1}B'_{ea} \in (\ddot{H}^{k_1+1}, t^{-1}), \\ \check{B}'_a &\in (\langle x \rangle \dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^{k_1+1}, t^{-1}), \end{aligned} \tag{3.197}$$

$$\partial_t B'_a, \partial_t B'_{ea} \in (H^{k_1+1}, (1-\ln t)^2), \tag{3.198}$$

$$\begin{aligned} \partial_t \check{B}'_a &\in (\langle x \rangle \dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^{(k_1+1)\wedge(k_+-2)}, t^{-2}), \\ \partial_t(t^{-1}B'_{ea}) &\in (\ddot{H}^{k_1+1}, t^{-2}). \end{aligned} \tag{3.199}$$

Furthermore $B'^S_{ma} \equiv \check{B}^S_{1a} + \check{B}_{2a} - t^{-1}B^S_{3a}$ satisfies the estimate

$$\|B'^S_{ma}\|_\infty + t\|\partial_t B'^S_{ma}\|_\infty \leq C(1-\ln t). \tag{3.200}$$

Proof. The estimate (3.200) is proved similarly as (3.192) and (3.193) in the proof of Lemma 3.10. Unlike (3.193), it suffices to show $\|\check{B}^S_{1a1}\|_\infty \vee t^{-1}\|B^S_{3a1}\|_\infty \leq C$, and this is indeed proved by the Sobolev inequality as before under the condition $k_1 - 1/2 > 0$. The properties and estimates (3.194)–(3.199) follow from Lemma 3.10. □

We now turn to the estimates of the remainders. The final result will be that the remainders satisfy the assumption (A3) of above with

$$h(t) = t^2(1-\ln t)^6.$$

We first consider the part not containing B_0 and B_{e0}, namely R_{11}, R_2, R_3, R_{41} and R_5. The estimates for that part follow from or extend Lemma 3.10. The part containing B_0 and B_{e0} require different arguments and additional assumptions.

Proposition 3.12. *Let $\beta > 0$. Let $w_+ \in H^{k_+}, xw_+ \in H^{k_+-1}$ with $k_+ \geq 7 \vee (5+\beta^{-1}) \vee 2(1+\beta^{-1})$ and let $k_1 = (k_+ - 2) \wedge (k_+ - \beta^{-1}), k_2 = (k_1 - 2) \wedge (k_1 + 1 - \beta^{-1})$. Then the remainders R_2, R_3, R_{41}, R_5 and R_{11} satisfy the following properties:*

$$R_2 \in (H^{k_2}, t(1-\ln t)^4), \tag{3.201}$$

$$R_3, R_5, t\check{R}_3 \in (H^{k_2+1}, t^2(1-\ln t)^4), \tag{3.202}$$

$$\partial_t R_3, t\partial_t \check{R}_3, \partial_t R_5 \in (H^{k_2+1}, t(1-\ln t)^4), \tag{3.203}$$

$$R_{41}, t\check{R}_{41} \in (H^{k_2+1}, t^3(1-\ln t)^5), \tag{3.204}$$

$$\partial_t R_{41}, t\partial_t \check{R}_{41} \in (H^{k_2+1}, t^2(1-\ln t)^5), \tag{3.205}$$

$$R_{11}, xR_{11} \in (H^{k_2-2}, t^2(1-\ln t)^6), \tag{3.206}$$

$$\partial_t R_{11}, x\partial_t R_{11} \in (H^{k_2-2}, t(1-\ln t)^6). \tag{3.207}$$

Proof. We can prove the properties (3.201), (3.202) and (3.203) of R_2, R_3 and R_5 follow from the properties of w_{a0}, w_{a1} and w_{a2} in Lemma 3.10.

We next show the properties (3.204) and (3.205) of R_{41}. Expanding the right-hand side of (3.171) and estimating each term by Lemma 3.10, we see that the worst term is $tF_2(Ps_{a0}|w_{a1}|^2) = O(t^2(1-\ln t)^5)$, so that we obtain the estimate of R_{41}. We can obtain the estimates of $t\check{R}_{41}$, $\partial_t R_{41}$ and $t\partial_t \check{R}_{41}$ similarly.

We proceed to the estimate of R_{11}. We first consider

$$(\check{B}^S_{1a} w_a)_{\geq 2} = \check{B}^S_{1a0} w_{a2} + \check{B}^S_{1a1} w_{a1} + \check{B}^S_{1a1} w_{a2}.$$

We obtain $\check{B}^S_{1a1} w_{a2} \in (H^{k_2}, t^2(1-\ln t)^6)$ by (3.183) and (3.190) together with the Leibniz rule, without using the cut-off by χ^S. On the other hand, the estimates of $\check{B}^S_{1a0} w_{a2}$ and $\check{B}^S_{1a1} w_{a1}$ need the cut-off by χ^S. Indeed, we obtain $\check{B}^S_{1a0} w_{a2}, \check{B}^S_{1a1} w_{a1} \in (H^{k_2}, t^2(1-\ln t)^4)$ by the use of the estimates (3.192) and (3.193). The other terms in R_{11} are estimated in H^{k_2-2}. The term of the lowest regularity is $\Delta w_{a2} \in (H^{k_2-2}, t^2(1-\ln t)^4)$, and the term of the worst decay is $s_{a0}^2 w_{a2} \in (H^{k_2}, t^2(1-\ln t)^6)$. Thus we obtain $R_{11} \in (H^{k_2-2}, t^2(1-\ln t)^6)$. We can estimate xR_{11}, $\partial_t R_{11}$ and $x\partial_t R_{11}$ analogously. □

To obtain the estimates of R_{10} and R_{40}, we shall need the following estimate:

Lemma 3.13. *Let $s \in \mathbb{R}$ and let m be a nonnegative integer. Then, for any $\psi \in H^s$ with $x\psi \in H^{s-1}$, the following estimate holds for $j, l = 0, 1$:*

$$\left\| \langle x \rangle^l \partial_t^j \left\{ U(t)\psi - \sum_{k=0}^{m-1} \frac{(it\Delta)^k}{2^k k!} \psi \right\} ; H^{s-2m-2j-l} \right\| \leq Ct^{m-j}. \tag{3.208}$$

Proof. We first consider the case $j = 0, l = 0$. By the Plancherel theorem together with the Taylor expansion, the square of the left-hand side is bounded by

$$\int d\xi\, (1 + |\xi|^2)^{s-2m} \left| \left\{ e^{-it|\xi|^2/2} - \sum_{k=0}^{m-1} \frac{(-it|\xi|^2)^k}{2^k k!} \right\} \hat{\phi}(\xi) \right|^2 \leq ct^{2m} \int d\xi\, (1 + |\xi|^2)^s |\hat{\phi}(\xi)|^2,$$

so that (3.208) follows. If $(j, k) \neq (0, 0)$, we use the relations

$$[x, U(t)] = -it\nabla,\ [x, \Delta^k] = -2k\Delta^{k-1}\nabla \text{ and } \partial_t U(t) = (it\Delta/2)U(t). \qquad \square$$

We now turn to the estimates of the parts R_{10} and R_{40} of the remainders containing B_0 and B_{e0}. For that purpose, we set

$$\begin{aligned} \mathscr{R}_1(v) &= -iB_0 \cdot \nabla v - (i/2)(\nabla \cdot B_0)v - (B_0(s_a + B'_a) + (1/2)B_0^2 + \check{B}_0 - t^{-1}B_{e0})v, \\ \mathscr{R}_4(f) &= tF_2(PB_0 f), \end{aligned}$$

so that $R_{10} = \mathscr{R}_1(w_a)$, $R_{40} = \mathscr{R}_4(|w_a|^2)$. We also set

$$\begin{aligned} \dot{\mathscr{R}}_1(v) &= -i(\partial_t B_0) \cdot \nabla v - (i/2)(\partial_t \nabla \cdot B_0)v \\ &\quad - \{(\partial_t B_0)(s_a + B_a) + B_0(\partial_t(s_a + B'_a)) + (\partial_t(\check{B}_0 - t^{-1}B_{e0}))\}v, \\ (\nabla\mathscr{R}_1)(v) &= -i(\nabla B_0) \cdot \nabla v - (i/2)(\nabla\nabla \cdot B_0)v \\ &\quad - \{(\nabla B_0)(s_a + B_a) + B_0(\nabla(s_a + B'_a)) + (\nabla(\check{B}_0 - t^{-1}B_{e0}))\}v, \\ (\nabla\dot{\mathscr{R}}_1)(v) &= -i(\nabla\partial_t B_0) \cdot \nabla v - (i/2)(\nabla\partial_t\nabla \cdot B_0)v \\ &\quad - \{(\nabla\partial_t B_0)(s_a + B_a) + (\partial_t B_0)(\nabla(s_a + B_a)) \\ &\qquad + (\nabla B_0)(\partial_t(s_a + B'_a)) + B_0(\nabla\partial_t(s_a + B'_a)) + (\nabla\partial_t(\check{B}_0 - t^{-1}B_{e0}))\}v, \end{aligned}$$

so that $\partial_t\mathscr{R}_1(v) = \dot{\mathscr{R}}_1(v) + \mathscr{R}_1(\partial_t v)$,

$$\nabla\partial_t\mathscr{R}_1(v) = (\nabla\dot{\mathscr{R}}_1)(v) + (\nabla\mathscr{R}_1)(\partial_t v) + \dot{\mathscr{R}}_1(\nabla v) + \mathscr{R}_1(\nabla\partial_t v).$$

To obtain desired estimates for R_{10} and R_{40}, we need additional assumptions for w_+, which will be clearly understood in terms of an approximation w_b of w_a. Let w_+ satisfy the assumption of Lemma 3.10. We set $w_b = w_{b0} + w_{b1}$ with

$$\begin{aligned} w_{b0} &:= w_+ + \frac{1}{2}it\Delta w_+, \\ i\partial_t w_{b1} &:= i(s_{a0} + B_{a0}) \cdot \nabla w_+ + \frac{i}{2}(\nabla \cdot (s_{a0} + B_{1a0}))w_+ + \frac{1}{2}(s_{a0} + B_{1a0})^2 w_+ \\ &\quad + (\check{B}^S_{1a0} + \check{B}_{2a1} - t^{-1}B^S_{3a0})w_+, \quad w_{b1}(0) = 0. \end{aligned}$$

Then, it follows that

$$\langle x\rangle^l w_b \in (H^{5-l}, 1), \quad \langle x\rangle^l \partial_t w_b \in (H^{5-l}, (1-\ln t)^2) \tag{3.209}$$

for $l = 0, 1$. On the other hand, from Lemma 3.13, we see

$$\|\langle x\rangle^l \partial_t^j (w_{a0} - w_+); H^{5-l}\| \le Ct^{1-j}, \quad \|\langle x\rangle^l \partial_t^j (w_{a0} - w_{b0}); H^{3-l}\| \le Ct^{2-j}.$$

Using the first estimate together with Lemma 3.10, we can show

$$\|\langle x\rangle^l \partial_t (w_{a1} - w_{b1}); H^{5-l}\| \le Ct(1-\ln t)^2.$$

From these estimates together with (3.190)–(3.191), for $j, l = 0, 1$, we obtain

$$\|\langle x\rangle^l \partial_t^j (w_a - w_+); H^{3-l}\| \le Ct^{1-j}(1-\ln t)^2, \tag{3.210}$$

$$\|\langle x\rangle^l \partial_t^j (w_a - w_b); H^{3-l}\| \le Ct^{2-j}(1-\ln t)^4. \tag{3.211}$$

Now we shall prove the following estimate of R_{10}.

Lemma 3.14. *Let w_+ satisfy the assumption of Proposition 3.11. Let B_0 and B_{e0} satisfy (2.31) for $2 \le r \le \infty$ and $0 \le j, k \le 1$. Let w_b, B_0 and B_{e0} satisfy in addition*

$$\begin{aligned}&\|\langle x\rangle^l ((t\partial_t)^p \nabla^q B_0)(t\partial_t)^r \nabla^s w_b\|_2 \vee \|\langle x\rangle^l ((t\partial_t)^r \nabla^s (\check{B}_0 - t^{-1}B_{e0}))w_b\|_2 \\ &\quad \le Ct^{1/2+l}(1-\ln t)^4\end{aligned} \tag{3.212}$$

for $l, p, r, s = 0, 1$, $q = 0, 1, 2$ with $l + q + s \le 2$, $p + r \le 1$. Then

$$\|\langle x\rangle \partial_t R_{10}\|_2 \vee t\|\nabla \partial_t R_{10}\|_2 \le Ct^{1/2}(1-\ln t)^5. \tag{3.213}$$

Proof. We first let v be an arbitrary function defined on the space-time, and derive estimates of $\mathscr{R}_1(v), \dot{\mathscr{R}}_1(v), (\nabla\mathscr{R}_1)(v)$ and $(\nabla\dot{\mathscr{R}}_1)(v)$. From Proposition 3.11 and (2.31), we estimate

$$\begin{aligned}\|\langle x\rangle^l \mathscr{R}_1(v)\|_2 &\le \|\langle x\rangle^l B_0 \cdot \nabla v\|_2 + \|\langle x\rangle^l (\nabla \cdot B_0) v\|_2 \\ &\qquad + \{\|B_0\|_2(\|s_a + B_a'\|_\infty + \|B_0\|_\infty) + \|\check{B}_0 - t^{-1}B_{e0}\|_2\}\|\langle x\rangle^l v\|_\infty \\ &\le \|\langle x\rangle^l B_0 \cdot \nabla v\|_2 + \|\langle x\rangle^l (\nabla \cdot B_0) v\|_2 + Ct^{1/2}(1-\ln t)\|\langle x\rangle^l v\|_\infty \\ &\le Ct^{1/3}\|\langle x\rangle^l \nabla v\|_6 + Ct^{-1/2}\|\langle x\rangle^l v\|_\infty\end{aligned} \tag{3.214}$$

for $l = 0, 1$. If $l = 0$, we may replace the first term with $Ct^{1/2}\|\nabla v\|_2$. Similarly we have

$$\begin{aligned}\|\langle x\rangle^l \dot{\mathscr{R}}_1(v)\|_2 &\le \|\langle x\rangle^l (\partial_t B_0) \cdot \nabla v\|_2 + \|\langle x\rangle^l (\partial_t \nabla \cdot B_0) v\|_2 + C(1-\ln t)\|\langle x\rangle^l (\partial_t B_0) v\|_2 \\ &\qquad + Ct^{-1}\|\langle x\rangle^l B_0 v\|_2 + \|\langle x\rangle^l (\partial_t (\check{B}_0 - t^{-1}B_{e0}))v\|_2 \\ &\le Ct^{-2/3}\|\langle x\rangle^l \nabla v\|_6 + Ct^{-3/2}\|\langle x\rangle^l v\|_\infty\end{aligned} \tag{3.215}$$

for $l = 0, 1$. If $l = 0$, we may replace the first term with $Ct^{-1/2}\|\nabla v\|_\infty$. Furthermore, we have

$$\begin{aligned}\|(\nabla\mathscr{R}_1)(v)\|_2 &\le \|(\nabla B_0)\cdot\nabla v\|_2 + \|(\nabla\nabla\cdot B_0)v\|_2 + Ct^{-1/2}(1-\ln t)\|v\|_\infty \\ &\le Ct^{-1/2}\|\nabla v\|_\infty + Ct^{-3/2}\|v\|_\infty. \qquad (3.216)\\ \|(\nabla\dot{\mathscr{R}}_1)(v)\|_2 &\le \|(\nabla\partial_t B_0)\cdot\nabla v\|_2 + \|(\nabla\partial_t\nabla\cdot B_0)v\|_2 + (1-\ln t)\|(\nabla\partial_t B_0)v\|_2 \\ &\quad + t^{-1}\|(\partial_t B_0)v\|_2 + t^{-1}\|(\nabla B_0)v\|_2 \\ &\quad + t^{-1}\|B_0 v\|_2 + \|(\nabla\partial_t(\check{B}_0 - t^{-1}B_{e0}))v\|_2 \\ &\le Ct^{-3/2}\|\nabla v\|_\infty + Ct^{-5/2}\|v\|_\infty. \qquad (3.217)\end{aligned}$$

It follows from (3.211) and (3.214)–(3.217) that

$$\|\langle x\rangle\partial_t\mathscr{R}_1(w_a - w_b)\|_2 \vee t\|\nabla\partial_t\mathscr{R}_1(w_a - w_b)\|_2 \le Ct^{1/2}(1-\ln t)^5. \qquad (3.218)$$

For example, applying (3.214) with $v = \partial_t(w_a - w_b)$ and (3.215) with $v = w_a - w_b$, we obtain the estimate of $\|\langle x\rangle\partial_t\mathscr{R}_1(w_a - w_b)\|_2$. We can obtain the estimate of $\|\nabla\partial_t\mathscr{R}_1(w_a - w_b)\|_2$ analogously.

On the other hand, using the estimates (3.214)–(3.217) together with the additional assumption (3.212) and the property (3.209) of w_b, we see

$$\|\langle x\rangle\partial_t\mathscr{R}_1(w_b)\|_2 \vee t\|\nabla\partial_t\mathscr{R}_1(w_b)\|_2 \le Ct^{1/2}(1-\ln t)^5. \qquad \square$$

Lemma 3.15. *Let w_+ satisfy the assumption of Proposition 3.11. Let B_0 and B_{e0} satisfy (2.31) for $r = 2$ and $0 \le j, k \le 1$. Let B_0 satisfy in addition*

$$\|(\nabla^k\partial_t^j B_0)w_+\|_2 \le Ct^{3/2-j-k} \qquad (3.219)$$

for $0 \le j, k, j+k \le 1$. Then the following estimate holds for $0 \le j, k, j+k \le 1$:

$$\|\nabla^{k+1}\partial_t^j R_{40}\|_2 \vee t\|\nabla^{k+1}\partial_t^j \check{R}_{40}\|_2 \le Ct^{5/2-j-k}(1-\ln t)^4. \qquad (3.220)$$

Proof. We define $v_\pm = w_a \pm w_+$, so that $\mathscr{R}_4(|w_a|^2 - |w_+|^2) = \mathscr{R}_4(\operatorname{Re}\overline{v}_+ v_-)$. From Lemma 3.10 and (3.210), we have the estimates

$$\|\langle x\rangle^l\partial_t^j v_+; H^{3-l}\| \le C(1-\ln t)^{2j}, \quad \|\langle x\rangle^l\partial_t^j v_-; H^{3-l}\| \le Ct^{1-j}(1-\ln t)^2 \qquad (3.221)$$

with $j, l = 0, 1$. Using (2.40) and (2.41), we estimate

$$\begin{aligned}\|\nabla^{k+1}\partial_t^j\mathscr{R}_4(\operatorname{Re}\overline{v}_+ v_-)\|_2 &\le tI_{j+k+1}(\|\nabla^k\partial_t^j B_0\overline{v}_+ v_-\|_2) + \delta_{j1}I_1(\|B_0\overline{v}_+ v_-\|_2),\\ \|\nabla^{k+1}\partial_t^j\check{\mathscr{R}}_4(\operatorname{Re}\overline{v}_+ v_-)\|_2 &\le I_{j+k}(\|\langle x\rangle\nabla^k\partial_t^j B_0\overline{v}_+ v_-\|_2).\end{aligned}$$

Using these estimates, (2.31) with $r = 2$ and (3.221), we have

$$\begin{aligned}&\|\nabla^{k+1}\partial_t^j \mathscr{R}_4(|w_a|^2 - |w_+|^2)\|_2 \vee t\|\nabla^{k+1}\partial_t^j \check{\mathscr{R}}_4(|w_a|^2 - |w_+|^2)\|_2\\ &\quad\le Ct^{5/2-j-k}(1-\ln t)^4. \end{aligned}\tag{3.222}$$

We next estimate $\mathscr{R}_4(|w_+|^2)$. Using again (2.40), (2.41), we have

$$\begin{aligned}&\|\nabla^{k+1}\partial_t^j \mathscr{R}_4(|w_+|^2)\|_2 \le tI_{j+k+1}(\|\nabla^k\partial_t^j B_0|w_+|^2\|_2) + \delta_{j1}I_1(\|B_0|w_+|^2\|_2),\\ &\|\nabla^{k+1}\partial_t^j \check{\mathscr{R}}_4(|w_+|^2)\|_2 \le I_{j+k}(\|\langle x\rangle\nabla^k\partial_t^j B_0|w_+|^2\|_2)\end{aligned}$$

for the relevant values of j and k. From (3.219) and (2.31), we have

$$\begin{aligned}\|\langle x\rangle^l(\nabla^k\partial_t^j B_0)|w_+|^2\|_2 &\le \|(\nabla^k\partial_t^j B_0)w_+\|_2\|\langle x\rangle w_+\|_\infty \le Ct^{3/2-j-k},\\ \|\langle x\rangle^l B_0\nabla|w_+|^2\|_2 &\le 2\|B_0\|_2\|\nabla w_+\|_\infty\|\langle x\rangle w_+\|_\infty \le Ct^{1/2}\end{aligned}$$

for $l = 0, 1$. Substituting the last two estimates into the previous ones, we obtain

$$\begin{aligned}&\|\nabla^{k+1}\partial_t^j \mathscr{R}_4(|w_+|^2)\|_2 \vee t\|\nabla^{k+1}\partial_t^j \check{\mathscr{R}}_4(|w_+|^2)\|_2\\ &\quad\le Ct^{5/2-j-k}.\end{aligned}\tag{3.223}$$

The estimate (3.220) follows from (3.222) and (3.223). □

In Lemmas 3.14 and 3.15, we have assumed the estimates (3.212) and (3.219). These estimates are stronger than those we can expect from Lemma 2.1, or more precisely the estimates for solutions to the free wave equations (2.31), so that we impose an additional assumption for the support of w_+. Let w_+ satisfy the assumption of Proposition 3.11, and let χ_0 be the characteristic function of the support of w_+. Then a sufficient condition to ensure (3.212) and (3.219) is that

$$\|\chi_0\nabla^k\partial_t^j B_0\|_2 \vee \|\chi_0\nabla^k\partial_t^j(\check{B}_0 - t^{-1}B_{e0})\|_2 \le Ct^{5/2-j-k}\tag{3.224}$$

for $j = 0, 1, k = 0, 1, 2$. We note that we can only obtain the bound $Ct^{1/2-j-k}$ unless χ_0 is multiplied. To obtain (3.224), we assume in addition

$$\operatorname{supp} w_+ \subset \{x : ||x| - 1| \ge \eta\}\tag{3.225}$$

for some $\eta, 0 < \eta < 1$, which is the same condition that occurs in [6,28]. Under this condition, it is easy to see that (3.224) holds for compactly supported $(A_+, \dot{A}_+)$ and $(A_{e+}, \dot{A}_{e+})$. In fact, if

$$\operatorname{supp}(A_+, \dot{A}_+), \quad \operatorname{supp}(A_{e+}, \dot{A}_{e+}) \subset \{x : |x| \le R\},$$

then by the Huygens principle

$$\mathrm{supp}\, A_0 \cup \mathrm{supp}\,(x \cdot A_0 - tA_{e0}) \subset \{(x,t) : ||x| - t| \leq R\},$$

so that

$$\mathrm{supp}\, B_0 \cup \mathrm{supp}\,(\check{B}_0 - t^{-1} B_{e0}) \subset \{(x,t) : ||x| - 1| \leq tR\},$$

and the left-hand side of (3.224) vanishes for $t \leq \eta/R$. More general assumptions on $(A_+, \dot{A}_+)$ and $(A_{e+}, \dot{A}_{e+})$ are given in the following Lemma.

Lemma 3.16. *Let w_+ satisfy the support condition (3.225) for some η with $0 < \eta < 1$. Let χ_R be the characteristic function of the set $\{x : |x| \geq R\}$. Let $(A_+, \dot{A}_+)$ and $(A_{e+}, \dot{A}_{e+})$ satisfy (2.7) together with the estimates*

$$\begin{cases} \|\chi_R \nabla^k (x \cdot \nabla)^j A_+\|_2 \vee \|\chi_R \nabla^k (x \cdot \nabla)^j (x \cdot A_+)\|_2 \leq CR^{-2}, \\ \|\chi_R (x \cdot \nabla)^j \dot{A}_+ ; L^2 \cap L^{6/5}\| \vee \|\chi_R (x \cdot \nabla)^j A_{e+} ; L^2 \cap L^{6/5}\| \leq CR^{-2} \end{cases} \tag{3.226}$$

for $0 \leq j \leq 1, 0 \leq k \leq 2$ and for all $R \geq R_0$ for some $R_0 > 0$. Then (3.224) holds for $0 \leq j \leq 1, 0 \leq k \leq 2$ and for all $t \in (0,1]$.

Proof. For the proof, see Lemma 5.2, part (2) of [6] and Lemma 6.6 of [13]. □

We finally collect the results of this part to show that the asymptotic functions constructed here satisfy the assumptions (A1), (A2) and (A3) of Section 3.

Proposition 3.17. *Let $\beta > 0$. Let $w_+ \in H^{k_+}, xw_+ \in H^{k_+-1}$ with $k_+ \geq 7 \vee (5 + \beta^{-1}) \vee 2(1 + \beta^{-1})$. Let B_0, B_{e0} satisfy the condition (2.31) for $2 \leq r \leq \infty$ and $0 \leq j, k \leq 3$. Then*

(1) The asymptotic functions (w_a, s_a, B_a, B_{ea}) defined by (3.159)–(3.162) satisfy the assumptions (A1) and (A2).

(2) Let in addition B_0, B_{e0} and w_+ satisfy the condition (3.224) for $0 \leq j, l \leq 1, 0 \leq k, m \leq 2$ and $k + l + m \leq 2$. Then the remainders R_j, $1 \leq j \leq 5$, defined by (3.14) satisfy the assumption (A3) with

$$h(t) = t^2(1 - \ln t)^6.$$

(3) The same result as in Part (2) holds under the assumptions of Lemma 3.16.

Proof. Part (1) follows from Proposition 3.11 and from (2.31). Part (2) follows from Proposition 3.12 and Lemmas 3.14 and 3.15. Part (3) follows from Part (2), from (3.224) and from Lemma 3.16. □

4. Modified Wave Operators for the Original System

In this section we construct the wave operators for the original system (1.2). For that purpose we first state the main result on the Cauchy problem at $t = 0$ for the auxiliary system (2.22).

Proposition 4.1. *Let $\beta = 1/2$. Let $X(\cdot)$ be defined by* (3.20) *with $h(t) = t^2(1 - \ln t)^6$. Let u_+ be such that $w_+ \equiv \overline{Fu_+} \in H^7, xw_+ \in H^6$. Let B_0 and B_{e0} satisfy the conditions* (2.31) *with $2 \leq r \leq \infty$ and* (3.224)*, for $0 \leq j \leq 1$ and $0 \leq k \leq 2$. Define (w_a, s_a, B_a, B_{ea}) by* (3.1) *and* (3.159)–(3.162)*. Then there exists τ, $0 < \tau \leq 1$ such that the auxiliary system* (2.22) *has a unique solution (w, s, B_2) such that $\sigma \equiv s - s_a$ satisfies $\sigma(0) = 0$ and such that $(q, G_2) \equiv (w - w_a, B_2 - B_{2a}) \in X((0, \tau])$. In particular the following estimates hold for all $t \in (0, \tau]$:*

$$\|\nabla^k \partial_t^j \langle x \rangle^l q\|_2 \leq Ct^{2-j-k/2}(1 - \ln t)^6 \tag{4.1}$$

for $0 \leq j, l \leq 1$ and $0 \leq 2j + l + k \leq 3$,

$$\|\nabla^{k+1} \partial_t^j G_2\|_2 \vee t\|\nabla^{k+1} \partial_t^j \check{G}_2\|_2 \leq Ct^{5/2-j-k}(1 - \ln t)^6 \tag{4.2}$$

for $0 \leq j, k, j + k \leq 1$.

In addition, the following estimates hold for all $t \in (0, \tau]$:

$$\|\partial_t^j G_2\|_2 \vee t\|\partial_t^j \check{G}_2\|_2 \leq Ct^{5/2-j}(1 - \ln t)^6, \tag{4.3}$$

$$\|\nabla^{k+1} \partial_t^j G_1\|_2 \vee t\|\nabla^{k+1} \partial_t^j \check{G}_1\|_2 \vee \|\nabla^{k+1} \partial_t^j G_3\|_2 \leq Ct^{2-j-k/2}(1 - \ln t)^6 \tag{4.4}$$

for $0 \leq j, k, j + k \leq 1$,

$$\|\partial_t^j G_1\|_2 \vee t\|\partial_t^j \check{G}_1\|_2 \vee \|\partial_t^j G_3\|_2 \leq Ct^{2-j}(1 - \ln t)^6, \tag{4.5}$$

$$\|\nabla^k \partial_t^j \sigma\|_2 \leq Ct^{2-j-k/2}(1 - \ln t)^6 \tag{4.6}$$

for $j = 0, 1$ and $0 \leq k \leq 2$.

The solution is actually unique under the conditions on (q, G_2) stated in Proposition 3.9.

Proof. The existence of (q, G_2) and the estimates (4.1), (4.2) thereof follow from Propositions 3.9 and 3.17. The estimates (4.4) and (4.6) follow from Lemma 3.6, especially the estimates (3.95)–(3.102). The estimates (4.3) and (4.5) are derived by (2.40) and (2.41), similarly as in the proof of Lemma 3.1. □

The result for the original system (1.2) for (u, A, A_e) is obtained by translating Proposition 4.1 for the auxiliary system (2.22). For that purpose, we need to reconstruct the phase ϕ satisfying $\nabla\phi = s$. Let $w_a = w_{a0} + w_{a1} + w_{a2}$ and $s_a = s_{a0} + s_{a1}$ be defined by (3.159)–(3.162), and let (w, s, B_2) be the solution to (2.22) obtained by Proposition 4.1. We define

$$\begin{aligned}\phi_a = &\int_1^t dt' \ (\check{B}_1^L(w_{a0}(t')) - t'^{-1} B_3^L(w_{a0}(t')))\\ &+ 2\int_0^t dt' \ (\check{B}_1^L(w_{a0}(t'), w_{a1}(t')) - t'^{-1} B_3^L(w_{a0}(t'), w_{a1}(t'))),\end{aligned} \tag{4.7}$$

so that $s_a = \nabla\phi_a$. We shall also need a special term of ϕ_a, namely

$$\phi_b = \int_1^t dt' \ \check{B}_1(w_+) - \int_1^t dt' \ t'^{-1} B_3(w_+) = (\ln t) x \cdot B_1(w_+) - (\ln t) B_3(w_+). \tag{4.8}$$

Furthermore, we define

$$\psi = \int_0^t dt' \ \{\check{B}_1^L(q, 2w_a + q) + \check{B}_1^L(w_a)_{\geq 2} - t'^{-1}(B_3^L(q, 2w_a + q) + B_3^L(w_a)_{\geq 2})\}(t'), \tag{4.9}$$

so that $\nabla\psi = \sigma$ by (3.8), (3.11) and (3.166). Finally we define $\phi = \phi_a + \psi$, so that $\nabla\phi = s$ and ϕ satisfy (2.21). The phases ϕ_a, ϕ_b and ϕ satisfy the following properties. We use again the notation (1.6) as in Lemma 3.10.

Lemma 4.2. *Let $\beta = 1/2$ and $w_+ \in H^7, xw_+ \in H^6$. We define ϕ_a, ϕ_b by (4.7) and (4.8) respectively. Then the following properties hold for ϕ_a and ϕ_b:*

$$\langle x\rangle^{-1}\partial_t\phi_b \in (\ddot{H}^8, t^{-1}), \tag{4.10}$$

$$\partial_t\phi_b \in (\langle x\rangle\dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^7, t^{-1}), \tag{4.11}$$

$$\langle x\rangle^{-1}\phi_b \in (\ddot{H}^8, (1 - \ln t)), \tag{4.12}$$

$$\phi_b \in (\langle x\rangle\dot{H}^1 \cap \dot{H}^2 \cap \dot{H}^7, (1 - \ln t)), \tag{4.13}$$

$$\partial_t(\phi_a - \phi_b) \in (\ddot{H}^5, (1 - \ln t)^2), \tag{4.14}$$

$$\phi_a - \phi_b \in (\ddot{H}^5, t(1 - \ln t)^2). \tag{4.15}$$

Furthermore, we define ϕ by (4.9) with w_a defined by (3.159)–(3.162) and q obtained by Proposition 4.1. Then, we have $\psi \in \mathscr{C}((0,\tau], \ddot{H}^3)$ with the estimate

$$\|\nabla^{k+1}\partial_t^j\psi\|_2 \leq Ct^{2-j-k/2}(1 - \ln t)^6 \tag{4.16}$$

for $j = 0, 1$ and $0 \leq k \leq 2$.

Proof. We can estimate $\check{B}_1(w_+)$ and $B_3(w_+)$ in the same way as in the proof of Lemma 2.3, especially the inequalities (2.61)–(2.63). Then we obtain

$$\|\check{B}_1(w_+); \dot{H}^2 \cap \dot{H}^7\| \le Ct^{-1}, \quad \|B_1(w_+); \ddot{H}^8\| \vee \|B_3(w_+); \ddot{H}^8\| \le C.$$

These estimates yield (4.10)–(4.13). We proceed to the estimate of $\phi_a - \phi_b$. From Lemma 3.10, especially from the property (3.183), the contributions of $\check{B}_{1a1}$ and $t^{-1}B_{3a1}$ are estimated by $C(1-\ln t)^2$. On the other hand, by the estimates (2.32) and (2.33), we obtain

$$\|\check{B}_1^S(w_{a0}); \ddot{H}^5\| \vee t^{-1}\|B_3^S(w_{a0}); \ddot{H}^5\| \le C.$$

Finally, by the estimate $\|\langle x\rangle^l(w_{a0} - w_+); H^{5-l}\| \le Ct$ with $l = 0, 1$, we have

$$\|\check{B}_1(w_{a0}) - \check{B}_1(w_+); \ddot{H}^5\| \vee t^{-1}\|B_3(w_{a0}) - B_3(w_+); \ddot{H}^6\| \le C.$$

Collecting these estimates, we obtain (4.14) and (4.15). Finally, we can obtain (4.16) from Proposition 4.1, especially from the estimate (4.6). □

We can now define the modified wave operator for the MS system in the form (1.2). We start from the asymptotic data $(u_+, A_+, \dot{A}_+, A_{e+}, \dot{A}_{e+})$ for (u, A, A_e). We define $w_+ = \overline{Fu_+}$, we define B_0 and B_{e0} by (2.3), (1.3) and (1.4), namely

$$A_0(t) = (\cos\omega t)A_+ + \omega^{-1}(\sin\omega t)\dot{A}_+ = -t^{-1}D_0(t)B_0(1/t), \tag{4.17}$$

$$A_{e0}(t) = (\cos\omega t)A_{e+} + \omega^{-1}(\sin\omega t)\dot{A}_{e+} = -t^{-1}D_0(t)B_{e0}(1/t). \tag{4.18}$$

We define (w_a, s_a, B_a, B_{ea}) by (3.1) and (3.159)–(3.162). We solve the auxiliary system (2.22) by Proposition 4.1. We reconstruct the phase $\phi = \phi_a + \psi$ as explained above. We set $B = B_0 + B_1 + B_2$ and $B_e = B_{e0} + B_3$, where $B_1 = B_1(w, w)$ and $B_3 = B_3(w, w)$ are defined by (2.17). We finally substitute (w, ϕ, B) into (2.2), (2.3) and obtain a solution (u, A, A_e) of the system (1.2) defined for large time. The modified wave operator is the mapping $\Omega : (u_+, A_+, \dot{A}_+, A_{e+}, \dot{A}_{e+}) \to (u, A, A_e)$ thereby obtained.

We now turn to the study of the asymptotic properties of (u, A, A_e) and in particular of its convergence to its asymptotic form (u_a, A_a, A_{ea}) defined in analogy with (2.2) and (2.3) by

$$u_a(t) = M(t)D(t)\exp(i\phi_a(1/t))\overline{w}_a(1/t), \tag{4.19}$$

$$A_a(t) = -t^{-1}D_0(t)B_a(1/t) = A_0 - t^{-1}D_0(t)(B_{1a} + B_{2a})(1/t), \tag{4.20}$$

$$A_{ea}(t) = -t^{-1}D_0(t)B_{ea}(1/t) = A_{e0} - t^{-1}D_0(t)B_{3a}(1/t). \tag{4.21}$$

The properties of u are best expressed in terms of $\tilde{u}$ and $\tilde{u}_a$ defined by

$$\tilde{u}(t) = U(-t)u(t), \qquad \tilde{u}_a(t) = U(-t)u_a(t), \tag{4.22}$$

so that

$$\tilde{u}(t) = M(t)^* F^* \exp(i\phi(1/t))\overline{w}(1/t), \tag{4.23}$$

$$\tilde{u}_a(t) = M(t)^* F^* \exp(i\phi_a(1/t))\overline{w}_a(1/t). \tag{4.24}$$

In order to translate the properties of B and B_e into properties of A and A_e, we need the following commutation relations

$$\nabla^k Q^j A(t) = (-1)^{j+1} t^{-1-k} D_0(t)(\nabla^k (t\partial_t)^j B)(1/t), \tag{4.25}$$

$$\nabla^k Q^j A_e(t) = (-1)^{j+1} t^{-1-k} D_0(t)(\nabla^k (t\partial_t)^j B_e)(1/t), \tag{4.26}$$

where $Q = t\partial_t + x \cdot \nabla + \mathbb{1}$ (see Section 2, especially (2.28) and (2.29)).

We can now state the main result for the original system (1.2).

Proposition 4.3. *Let $\beta = 1/2$. Let u_+ be such that $w_+ = \overline{Fu_+} \in H^7, xw_+ \in H^6$ and such that w_+ satisfies the support condition* (3.225)*. Let $A_+, \dot{A}_+, A_{e+}, \dot{A}_{e+}$ satisfy the assumptions in Lemmas 2.1 and 3.16 for $0 \le j \le 1$, $0 \le k \le 2$. Define (w_a, s_a, B_a, B_{ea}) by* (3.1) *and* (3.159)*–*(3.162)*, and $(\phi_a, u_a, A_a, A_{ea})$ by* (4.7) *and* (4.19)*–*(4.21)*. Let (w, s, B_2) be the solution of the auxiliary system* (2.22) *obtained in Proposition 4.1, let $\phi = \phi_a + \psi$ with ψ defined by* (4.9)*, let $B = B_0 + B_1 + B_2, B_e = B_{e0} + B_3$, let (u, A, A_e) be defined by* (2.2)*,* (2.3) *and let $\tilde{u}$ be defined by* (4.22)*. Let $T = \tau^{-1}$ and $I = [T, \infty)$. Then*

(1) (u, A, A_e) satisfies the system (1.2) *in $I, x^k \partial_t^j \nabla^l \tilde{u} \in \mathscr{C}(I, L^2)$ for $0 \le j, l, j + l \le 1$ and $0 \le 2j + k + l \le 3$, and $\tilde{u}$ satisfies the following estimates for the same values of j, k, l and for all $t \in I$:*

$$\|x^k \partial_t^j \nabla^l (\tilde{u} - \tilde{u}_a)\|_2 \le Ct^{-2-j+k/2}(1 + \ln t)^6. \tag{4.27}$$

Furthermore $\partial_t \nabla U(-t) \exp(-i\phi_b(1/t, x/t))u(t) \in \mathscr{C}(I, L^2)$ and the following estimate holds for all $t \in I$:

$$\|\partial_t \nabla U(-t) \exp(-i\phi_b(1/t, x/t))(u(t) - u_a(t))\|_2 \le Ct^{-3}(1 + \ln t)^6. \tag{4.28}$$

Finally the following estimate holds

$$\|x^l (u - u_a)\|_r \le Ct^{-2+l-\delta(r)/2}(1 + \ln t)^6 \tag{4.29}$$

for $l = 0, 1$, for $2 \le r \le \infty$ and for all $t \in I$, with $\delta(r) = 3/2 - 3/r$.

(2) $A, A_e \in \mathscr{C}(I, \dot{H}^1 \cap \dot{H}^2), x \cdot A - tA_e \in \mathscr{C}(I, \dot{H}^2), QA, QA_e \in \mathscr{C}(I, H^1)$ *and* $Q(x \cdot A - tA_e) \in \mathscr{C}(I, \langle x \rangle \dot{H}^1)$ *where* $Q = t\partial_t + x \cdot \nabla + \mathbb{1}$*. Furthermore* $A - A_a, A_e - A_{ea}, x \cdot (A - A_a) - t(A_e - A_{ea}) \in \mathscr{C}(I, H^2)$ *and* $Q(A - A_a), Q(A_e - A_{ea}), Q(x \cdot (A - A_a) - t(A_e - A_{ea})) \in \mathscr{C}(I, H^1)$ *and the following estimates hold for all* $t \in I$*:*

$$\|Q^j(A - A_a)\|_2 \vee \|Q^j(A_e - A_{ea})\|_2 \vee t^{-1}\|Q^j(x \cdot (A - A_a) - t(A_e - A_{ea}))\|_2 \\ \leq Ct^{-3/2}(1 + \ln t)^6, \tag{4.30}$$

$$\|\nabla^{k+1} Q^j(A - A_a)\|_2 \vee \|\nabla^{k+1} Q^j(A_e - A_{ea})\|_2 \\ \vee t^{-1}\|\nabla^{k+1} Q^j(x \cdot (A - A_a) - t(A_e - A_{ea}))\|_2 \leq Ct^{-5/2-k/2}(1 + \ln t)^6 \tag{4.31}$$

for $0 \leq j, k, j + k \leq 1$*.*

Proof. Once we obtain Proposition 4.1 and Lemma 4.2, we can prove the proposition similarly as Proposition 7.2 in [13]. □

Proof of Theorem 1.1. We prove the theorem under the same assumptions on the scattering state $(u_+, A_+, \dot{A}_+, A_{e+}, \dot{A}_{e+})$ as those in Proposition 4.3. We first note that $\tilde{J}$ and $\tilde{J}_e$ satisfy the identity $\nabla \cdot \tilde{J} + \partial_t \tilde{J}_e = 0$. Hence, integrating by parts, we obtain

$$\tilde{A}_1(t) = -\int_t^\infty dt' \ \omega^{-1} \sin(\omega(t - t'))P\tilde{J}(t') - \int_t^\infty dt' \ \nabla\omega^{-2} \cos(\omega(t - t'))\tilde{J}_e(t').$$

By definition, it follows that $\square \tilde{A}_1 = \tilde{J}$ and $\square \tilde{A}_{e1} = \tilde{J}_e$. Moreover, we have

$$\|\tilde{A}_1; \dot{H}^1\| + \|\partial_t \tilde{A}_1\|_2 + \|\tilde{A}_{e1}; \dot{H}^1\| + \|\partial_t \tilde{A}_{e1}\|_2 \to 0$$

as $t \to \infty$. Like the translation from (A', A'_e) to (B', B'_e) by (2.9)–(2.12), we obtain

$$\tilde{A}_1(t) = -t^{-1}D_0(t)B_1(w_+), \quad \tilde{A}_{e1}(t) = -t^{-1}D_0(t)B_3(w_+).$$

These relations show that $S(t) = -\phi_b(1/t)$. We define

$$u_b(t) = M(t)D(t)\exp(i\phi_b(1/t))\overline{w}_{a0}(1/t),$$
$$\tilde{u}_b(t) = U(-t)u_b(t) = M(t)^*F^*\exp(i\phi_b(1/t))\overline{w}_{a0}(1/t),$$

in accordance with (4.19) and (4.22)–(4.24). We note that $u_b(t) = \exp(-iS(t, x/t))U(t)u_+$. Then we have

$$\tilde{u}_a(t) - \tilde{u}_b(t) = M(t)^*F^*\exp(i\phi_a(1/t))(\overline{w}_a(1/t) - \overline{w}_{a0}(1/t)) \\ + M(t)^*F^*(\exp(i\phi_a(1/t)) - \exp(i\phi_b(1/t)))\overline{w}_{a0}(1/t).$$

From Lemma 3.10, we have

$$\|w_a(1/t) - w_{a0}(1/t)\|_2 \le \|w_{a1}(1/t)\|_2 + \|w_{a2}(1/t)\|_2 \le Ct^{-1}(1+\ln t)^2.$$

On the other hand, from Lemma 4.2, we have $\|\phi_a(1/t) - \phi_b(1/t)\|_\infty \le Ct^{-1}(1+\ln t)^2$. Hence, we obtain

$$\|\tilde{u}_a(t) - \tilde{u}_b(t)\|_2 \le \|w_a(1/t) - w_{a0}(1/t)\|_2 + \|\phi_a(1/t) - \phi_b(1/t)\|_\infty \le Ct^{-1}(1+\ln t)^2.$$

This estimate and (4.27) yield $\|\tilde{u}(t) - \tilde{u}_b(t)\|_2 \le Ct^{-1}(1+\ln t)^2$. On the other hand, from Propositions 3.11 and 4.1, we have

$$\|B(t) - B_0(t); \dot{H}^1\| \le \|B'_a; \dot{H}^1\| + \|G_1; \dot{H}^1\| + \|G_2; \dot{H}^1\| \le C,$$

so that $\|A(t) - A_0(t); \dot{H}^1\| \le Ct^{-1/2}$. Similarly we have $\|A_e(t) - A_{e0}(t); \dot{H}^1\| \le Ct^{-1/2}$. Thus we have proved (1.5). □

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