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Strichartz type estimates in mixed Besov spaces with application to critical nonlinear Schrödinger equations

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ABSTRACT. We study the Cauchy problem for the nonlinear Schrödinger equation with power nonlinearity in the fractional order Sobolev space $H^s(\mathbb{R}^n)$. We consider the case where the nonlinear term is H^s-critical and the differentiability thereof is less than s. We prove the existence of time global solutions of the Cauchy problem for small initial data. To this end, we prove Strichartz type estimates for linear inhomogeneous Schrödinger equations in mixed Besov spaces.

1. INTRODUCTION

This paper is devoted to the study of nonlinear Schrödinger equation with power nonlinearity:

$$\partial_t u + i\Delta u = f(u), \tag{1.1}$$

$$u(0,\cdot) = u_0, \tag{1.2}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{C}$, and $f(u) = \kappa|u|^{p-1}u$ or $f(u) = \kappa|u|^p$ with $p > 1, \kappa \in \mathbb{C}$.

The solvability of the Cauchy problem (1.1)-(1.2) in the Sobolev space $H^s := H^s(\mathbb{R}^n)$ has been studied in a large amount of literature. In this paper, we consider the case $0 \le s < n/2$ and $p = p(s) := 1 + 4/(n-2s)$. The equation (1.1) is invariant by the scaling $u_\lambda(t,x) = \lambda^{2/(p-1)}u(\lambda^2 t, \lambda x)$ for any $\lambda > 0$, and $\|u_\lambda(0,\cdot)\|_{\dot H^s} := \|(-\Delta)^{s/2}u_\lambda(0,\cdot)\|_{L^2}$ is also invariant if $p = p(s)$. Namely, from the scaling point of view, $p(s)$ is the critical exponent in $H^s(\mathbb{R}^n)$.

Cazenave and Weissler [3] have shown the time local well-posedness, and the existence of the time global solutions for small data of the Cauchy problem (1.1)-(1.2) for $0 \le s < n/2$ and $[s]+1 < p \le p(s)$, where $[s]$ denotes the largest integer less than or equal to s (see also [12, Theorem 6.1] by Kato). The condition $[s]+1 < p$ is the required regularity for $f(u)$, which can be improved to $s < p$ based on the nonlinear estimates by Ginibre, Ozawa and Velo in [6], and the first author and Ozawa in [14] (see Lemma 2.2 in the present paper).

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The well-posedness has been also considered under the conditions $0 \le s < n/2$ and $p_0(s) < p < p(s)$, where $p_0(s)$ is defined by

$$p_0(s) := \begin{cases} 1 & \text{for } s \le 2, \\ s-1 & \text{for } 2 < s < 4, \\ s-2 & \text{for } 4 \le s. \end{cases}$$

The existence of time local solutions of (1.1)-(1.2) under these conditions has been shown by Tsutsumi [21] for $s = 0$, Ginibre and Velo [7, Theorem 3.1] for $s = 1$ (see also [8]), Tsutsumi [20] for $s = 2$ for $f(u) = \lambda|u|^{p-1}u$ with $i\lambda \in \mathbb{R}$ mainly by the use of the $L^p - L^q$ estimate and the regularization technique. Kato [10, 11] used the Strichartz estimate and gave alternative proofs for the cases $s = 0, 1, 2$ both for $f(u) = \lambda|u|^{p-1}u$ and $f(u) = \lambda|u|^p$ with $\lambda \in \mathbb{C}$. Pecher [16] used the fractional Besov space for the time variable and proved the result when s is a real number with $1 < s < n/2$ and $p_0(s) < p < p(s)$. He has also shown the existence of time global solutions for small data.

While the condition $p_0(s) < p$ for $s \le 2$ and $s \ge 4$ is natural since $1 < p$ and the s-derivative of u by the spatial variables requires the $(s-2)$-derivative of $f(u)$ by (1.1), the condition $p_0(s) < p$ for $2 < s < 4$ is discontinuous at $s = 4$, and it should be replaced by $s/2 < p$ since one time derivative corresponds to two spatial derivatives. In this direction, the special case $s = 2$ and $1 < p \le p(2)$ was proved in [3, Theorem 1.4] (see also [2]). Although the other case has been left open for long time, we need fractional order time derivatives of u to implement this procedure. To this end, Pecher [16, Propositions 2.5, 2.6] introduced a modification of the Strichartz estimates by which we can replace the spatial derivative of order s with the fractional order time derivative of order $s/2$ in terms of Besov spaces (see also [5, 22, 23]). The second author and Uchizono improved the condition $p_0(s) < p < p(s)$ to $s/2 < p < p(s)$ for $2 < s < 4$ in [23] (see also [24]) based on the method in [16]. However, the methods in [16] and [23] are not applicable to time global solutions for the critical case $p = p(s)$ by the technical conditions on the Strichartz estimates there. Especially, the interpolation argument to construct the Strichartz estimates prevent us from treating the critical point $p = p(s)$ in its application to the Cauchy problem. In [15], the present authors have considered the critical case $0 \le s < n/2$ and $s/2 < p = p(s)$ by the modification of the Strichartz estimates in [16] and [23] to the scaling invariant estimates. However, in the most difficult case where $s/2 < p(s) < 2$ with $3 \le s < \min\{4; n/2\}$, or equivalently where $11 \le n \le 13$ and

$$3 \le s < \begin{cases} (n-4)/2, & n = 11, 12, \\ (17 - \sqrt{17})/4, & n = 13, \end{cases} \tag{1.3}$$

we further imposed a technical assumption

$$\text{(i) } n = 11; \text{ or (ii) } n = 12 \text{ and } 7 - \sqrt{15} \le s < 5 - \sqrt{3} \tag{1.4}$$

to obtain time global solutions for small data. The reason why we imposed (1.4) in [15] is explained at the end of §4. In this paper, we have succeeded in removing this technical assumption. Furthermore, compared with the proofs in [15] for the existence of the time global solutions to (1.1)-(1.2), our proof has become much simpler by the use of Theorem 1.2 below, especially the mixed space-time inequality in the Besov space given by (1.9).

To state our result, we set the condition for the nonlinear term $f(u)$. For $1 < p < \infty$, we say that f satisfies $N(p)$ if $f \in C^1(\mathbb{C}, \mathbb{C})$ in the sense of the derivatives by z and $\bar{z}$, $f(0) = f'(0) = 0$, and

$$|f'(z_1) - f'(z_2)| \le \begin{cases} C \max\limits_{w=z_1,z_2} |w|^{p-2}|z_1 - z_2| & \text{if } p \ge 2, \\ C|z_1 - z_2|^{p-1} & \text{if } 1 < p < 2 \end{cases} \tag{1.5}$$

for any $z_1,\ z_2 \in \mathbb{C}$. We note that $f(z) = \kappa|z|^{p-1}z$ and $f(z) = \kappa|z|^p$ with $\kappa \in \mathbb{C}$ satisfy $N(p)$ (see [9, Remark 2.3′]). We obtain the following time global solutions:

Theorem 1.1. *Let $1 < s < \min\{4; n/2\}$ and $s/2 < p = p(s) = 1 + 4/(n - 2s)$. Let f satisfy the condition $N(p)$. Then, for any $u_0 \in H^s(\mathbb{R}^n)$ with $\|u_0\|_{H^s}$ sufficiently small, there exists a unique solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$ to* (1.1)-(1.2).

To prove Theorem 1.1, we derive Strichartz type estimates in mixed Besov spaces. For $\theta \in \mathbb{R}$, $1 \le q, \alpha \le \infty$ and a Banach space V, $L^q(\mathbb{R}, V)$ and $B^\theta_{q,\alpha}(\mathbb{R}, V)$ denote the V-valued Lebesgue and Besov spaces on $\mathbb{R}$ respectively. In what follows, we often write $L^q(V) = L^q(\mathbb{R}, V)$ and $B^\theta_{q,\alpha}(V) = B^\theta_{q,\alpha}(\mathbb{R}, V)$ for short. We refer to (2.1) in Section 2 for the definition of the Besov space $B^\theta_{q,\alpha}(V)$. The Chemin-Lerner type space $\ell^2(L^q(L^r)) := \ell^2(\mathbb{Z}_+, L^q(\mathbb{R}, L^r(\mathbb{R}^n)))$, which was introduced in [4], is defined by the totality of all functions on $\mathbb{R}^{1+n}$ with

$$\|u\|_{\ell^2(L^q(L^r))} = \left\{ \|\psi *_x u\|^2_{L^q(L^r)} + \sum_{k=1}^{\infty} \|\varphi_k *_x u\|^2_{L^q(L^r)} \right\}^{1/2} < \infty.$$

Here, ψ and φ_k are Littlewood-Paley functions on $\mathbb{R}^n$ defined in §2. The space $\ell^2(B^\theta_{q,\alpha}(L^r)) := \ell^2(\mathbb{Z}_+, B^\theta_{q,\alpha}(\mathbb{R}, L^r(\mathbb{R}^n)))$ is defined analogously.

Definition. Let $n \ge 1$. A pair of numbers (q, r) is said to be admissible if $2 \le q, r \le \infty$ and $2/q = \delta(r) := n/2 - n/r$ with $(n, q, r) \ne (2, 2, \infty)$.

We obtain the following Strichartz type estimates:

Theorem 1.2. *Let $n \ge 1$, $0 < \theta < 1$, and $0 < \sigma < 2$. Let (q, r) and (γ, ρ) be admissible pairs with $\rho \ne \infty$. Let $1 < \bar{q}_0, \bar{q}_1, \bar{r}_0, \bar{r}_1 < \infty$ satisfy $2/\bar{q}_0 - \delta(\bar{r}_0) = 2(1 - \theta)$, $2/\bar{q}_1 - \delta(\bar{r}_1) = 2 - \sigma$ and $\bar{q}_1 \le q$. Then, the solution u to*

$$\partial_t u + i\Delta u = f, \quad u(0, \cdot) = u_0 \tag{1.6}$$

satisfies the following inequalities:

$$\|u\|_{B^\theta_{q,2}(L^r)} \lesssim \|u_0\|_{H^{2\theta}} + \|f\|_{B^\theta_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{\bar{q}_0}(L^{\bar{r}_0}))}, \tag{1.7}$$

$$\|u\|_{L^q(B^\sigma_{r,2})} \lesssim \|u_0\|_{H^\sigma} + \|f\|_{B^{\sigma/2}_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{\bar{q}_1}(L^{\bar{r}_1}))}. \tag{1.8}$$

We have $u \in C(\mathbb{R}, H^\sigma(\mathbb{R}^n))$ if the right-hand side of (1.8) *is finite. Moreover, if $\theta > \sigma/2$, then u satisfies the following inequality:*

$$\|u\|_{B^{\theta-\sigma/2}_{q,2}(B^\sigma_{r,2})} \lesssim \|u_0\|_{H^{2\theta}} + \|f\|_{B^\theta_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{\bar{q}_0}(L^{\bar{r}_0}))} + \|f\|_{\ell^2(B^{\theta-\sigma/2}_{\bar{q}_1,2}(L^{\bar{r}_1}))}. \tag{1.9}$$

If $\bar{q}_0 \le q$, then we do not need the last term in the right-hand side of (1.9).

The inequality (1.9), which is essentially used to get rid of the technical assumption (1.4), is a new ingredient in this paper; on the other hand (1.7) and (1.8) have already been proved in [15]. As we have mentioned above, these types of estimates were first derived in Pecher [16]. The advantage of Theorem 1.2 is that the homogeneous counterparts of (1.7)-(1.9) are scale-invariant, so that these estimates are applicable to critical nonlinear problem, while the estimates in [16] are not. In [16], estimates like (1.7) and (1.8) are proved, but (q, r) is restricted to (γ, ρ) or $(\infty, 2)$, and more importantly, $l^2(L^{\bar{q}_0}(L^{\bar{r}_0}))$ in (1.7) is replaced with $\cap_{\pm} L^{\bar{q}_\pm}(L^{\bar{r}_\pm})$, where the pair $(\bar{q}_\pm, \bar{r}_\pm)$ is defined by

$$(1/\bar{q}_\pm, 1/\bar{r}_\pm) = (1-\theta \mp \varepsilon)(1/\gamma', 1/\rho') + (\theta \pm \varepsilon)(1/q, 1/r) \tag{1.10}$$

for $\varepsilon > 0$; similarly $l^2(L^{\bar{q}_1}(L^{\bar{r}_1}))$ in (1.8) is also replaced with $\cap_{\pm} L^{\bar{q}_\pm}(L^{\bar{r}_\pm})$, where $(\bar{q}_\pm, \bar{r}_\pm)$ is in this case defined by (1.10) with θ replaced with $\sigma/2$. The space $L^{q_\pm}(\mathbb{R}, L^{r_\pm}(\mathbb{R}^n))$ has different scale from $\dot{B}^\theta_{q,2}(\mathbb{R}, L^r(\mathbb{R}^n))$ because of the presence of ε in (1.10).

The methods to prove Theorems 1.1, 1.2 will be applicable to the analogous equation to (1.1) which has different order of derivatives for time and spatial variables.

2. Preliminaries

We first review the definition of Besov spaces. For the detail, we refer the reader to [1, 17, 19]. Let φ be a function on $\mathbb{R}$ whose Fourier transform $\hat{\varphi}$ is a non-negative, even smooth function which satisfies $\operatorname{supp}\hat{\varphi} \subset \{\tau \in \mathbb{R};\ 1/2 \le |\tau| \le 2\}$ and $\sum_{j=-\infty}^{\infty} \hat{\varphi}(\tau/2^j) = 1$ for $\tau \neq 0$. For $j \in \mathbb{Z}$, we set $\varphi_j = \mathscr{F}^{-1}\hat{\varphi}(\cdot/2^j)$ and $\psi_j = \sum_{k=-\infty}^{j} \varphi_k$. If $j = 0$, we simply write $\psi = \psi_0$. Let $\theta \in \mathbb{R}$ and $1 \le q, \alpha \le \infty$. For a Banach space V, we define the V-valued Besov space $B^\theta_{q,\alpha}(V) := B^\theta_{q,\alpha}(\mathbb{R}, V)$ by the set of all $u \in \mathscr{S}'(\mathbb{R}, V)$ satisfying $\|u\|_{B^\theta_{q,\alpha}(V)} < \infty$, where

$$\|u\|_{B^\theta_{q,\alpha}(V)} = \|\psi *_t u\|_{L^q(V)} + \begin{cases} \Big\{ \sum\limits_{j=1}^{\infty} \big(2^{\theta j} \|\varphi_j *_t u\|_{L^q(V)}\big)^\alpha \Big\}^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup\limits_{j \ge 1} 2^{\theta j} \|\varphi_j *_t u\|_{L^q(V)} & \text{if } \alpha = \infty. \end{cases} \tag{2.1}$$

We also need Littlewood-Paley decomposition on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, we define $\psi_k(x)$ and $\varphi_k(x)$ by

$$\psi_k(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} \hat{\psi}_k(|\xi|)\, d\xi \quad \text{and} \quad \varphi_k(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} \hat{\varphi}_k(|\xi|)\, d\xi$$

respectively. If $n = 1$, then these functions coincide with previous ones. For $s \in \mathbb{R}$ and $1 \le r, \alpha \le \infty$, the Besov space $B^s_{r,\alpha} := B^s_{r,\alpha}(\mathbb{R}^n)$ is defined by the set of all $u \in \mathscr{S}'(\mathbb{R}^n)$ satisfying

$$\|u\|_{B^s_{r,\alpha}} = \|\psi *_x u\|_{L^r} + \Big\{ \sum_{k=1}^{\infty} \big(2^{sk} \|\varphi_k *_x u\|_{L^r}\big)^\alpha \Big\}^{1/\alpha} < \infty,$$

with trivial modification as above if $\alpha = \infty$. In this paper, we use the notation $\varphi_{k/2} = \hat{\varphi}_k(-\Delta) = \mathscr{F}_\xi^{-1}\hat{\varphi}_k(|\xi|^2)$. This is an abuse of symbol, but no confusion is likely to arise. This notation matches the equivalence

$$\|u\|_{B^s_{r,\alpha}} \sim \|\mathscr{F}_x^{-1}\hat{\psi}(|\xi|^2)\hat{u}(\xi)\|_{L^r} + \Big\{ \sum_{k=1}^{\infty} (2^{sk/2} \|\varphi_{k/2} *_x u\|_{L^r})^\alpha \Big\}^{1/\alpha},$$

with trivial modification if $\alpha = \infty$. For the proof, see [15, Lemma 2.3].

We prepare the following estimates. The first result is the interpolation inequality in the Besov space for the time variable.

Lemma 2.1 (see e.g., [15, Lemma 2.2])**.** *Let $s \in \mathbb{R}$, $0 < \theta < 1$, $1 \le q_0, q_1 \le \infty$, $1 \le \alpha \le \infty$. Put $1/q := (1-\theta)/q_0 + \theta/q_1$. Let V, V_0, V_1 be Banach spaces which satisfy $V_0 \cap V_1 \subset V$ and $\|u\|_V \lesssim \|u\|_{V_0}^{1-\theta}\|u\|_{V_1}^{\theta}$ for any $u \in V_0 \cap V_1$. Then the inequality*

$$\|u\|_{B^{\theta s}_{q,\alpha/\theta}(V)} \lesssim \|u\|^{1-\theta}_{B^0_{q_0,\infty}(V_0)} \|u\|^{\theta}_{B^s_{q_1,\alpha}(V_1)}$$

holds for any $u \in B^0_{q_0,\infty}(\mathbb{R}, V_0) \cap B^s_{q_1,\alpha}(\mathbb{R}, V_1)$.

The second result is the estimate for the nonlinear term in the Besov space for the spatial variables.

Lemma 2.2. *Let $1 < p < \infty$, $0 < s < \min\{2; p\}$, $1 \le r_0, r_1, \rho, \alpha \le \infty$. Let f satisfy $N(p)$. Let $1/r_0 = (p-1)/\rho + 1/r_1$. Then we have*

$$\|f(u)\|_{\dot{B}^s_{r_0,\alpha}} \lesssim \|u\|^{p-1}_{L^\rho} \|u\|_{\dot{B}^s_{r_1,\alpha}} \tag{2.2}$$

for any $u \in L^\rho(\mathbb{R}^n) \cap \dot{B}^s_{r_1,\alpha}(\mathbb{R}^n)$. Moreover, if $s < \min\{1; p-1\}$, then the estimate

$$\begin{aligned} &\|f(u) - f(v)\|_{\dot{B}^s_{r_0,\alpha}} \\ &\lesssim \begin{cases} \|u-v\|^{p-1}_{L^\rho}\|u\|_{\dot{B}^s_{r_1,\alpha}} + \|v\|^{p-1}_{L^\rho}\|u-v\|_{\dot{B}^s_{r_1,\alpha}} & \text{if } p \le 2, \\ \left(\|u\|_{L^\rho\cap\dot{B}^s_{r_1,\alpha}} + \|v\|_{L^\rho\cap\dot{B}^s_{r_1,\alpha}}\right)^{p-1} \|u-v\|_{L^\rho\cap\dot{B}^s_{r_1,\alpha}} & \text{if } p > 2 \end{cases} \end{aligned} \tag{2.3}$$

holds for $u, v \in L^\rho(\mathbb{R}^n) \cap \dot{B}^s_{r_1,\alpha}(\mathbb{R}^n)$.

Proof. For the proof of (2.2), see [6, Lemma 3.4] or [15, Lemma 2.1]. We prove (2.3); we only consider the case $p \le 2$, but we can treat the case $p > 2$ similarly. For $u : \mathbb{R}^n \to \mathbb{C}$, $\tau \in \mathbb{R}^n$ and $0 \le \theta \le 1$, we set $\delta_\tau u = u(\cdot + \tau) - u$ and $u_\theta = u + \theta\delta_\tau u$. We use the representation

$$\|u\|_{\dot{B}^s_{r_1,\alpha}} \sim \left(\int_{\mathbb{R}^n} \|\delta_\tau u\|^\alpha_{L^{r_1}} |\tau|^{-\alpha s - n} d\tau\right)^{1/\alpha}$$

for $0 < s < 1$, We have $\delta_\tau f(u) = f(u(\cdot+\tau)) - f(u) = \int_0^1 f'(u_\theta)\delta_\tau u\, d\theta$, so that

$$\delta_\tau(f(u) - f(v)) = \int_0^1 (f'(u_\theta) - f'(v_\theta))\delta_\tau u\, d\theta + \int_0^1 f'(v_\theta)\delta_\tau(u-v)\, d\theta.$$

By (1.5) and the Hölder inequality, we see

$$\begin{aligned} \|\delta_\tau(f(u) - f(v))\|_{L^{r_0}} \lesssim &\int_0^1 \|u_\theta - v_\theta\|^{p-1}_{L^\rho} \|\delta_\tau u\|_{L^{r_1}}\, d\theta \\ &+ \int_0^1 \|v_\theta\|^{p-1}_{L^\rho} \|\delta_\tau(u-v)\|_{L^{r_1}}\, d\theta. \end{aligned}$$

Since $\|u_\theta\|_{L^\rho} \lesssim \|u\|_{L^\rho}$, we can easily show (2.3). □

The third result is the estimate for the nonlinear term in the Besov space for the time variable. Although the result for some special s, γ, ρ, q_0, r_0, q and r has been shown in [15, Claim 4.3, Claim 4.6], we give more general estimate to prove our theorem.

Lemma 2.3. *Let $0 < s < 4$, $s \ne 2$, $\max\{1; s/2\} < p$. Let f satisfy $N(p)$. Let $1 \le \rho, r_0, r, q_0 \le \infty$, and let $1 < \gamma \le \infty$, $1 \le q < \infty$. Let $\gamma, \rho, r_0, r, q_0, q$ satisfy*

$$\frac{1}{\gamma'} = \frac{p-1}{q_0} + \frac{1}{q}, \quad \frac{1}{\rho'} = \frac{p-1}{r_0} + \frac{1}{r}.$$

Let $1 \le \alpha < \infty$. *Then the inequality*

$$\|f(u)\|_{B^{s/2}_{\gamma',\alpha}(L^{\rho'})} \lesssim \|u\|^{p-1}_{L^{q_0}(L^{r_0})} \|u\|_{B^{s/2}_{q,\alpha}(L^r)}$$

holds for any $u \in L^{q_0}(\mathbb{R}, L^{r_0}(\mathbb{R}^n)) \cap B^{s/2}_{q,\alpha}(\mathbb{R}, L^r(\mathbb{R}^n))$.

Proof. First, we consider the case $0 < s < 2$. By $s/2 < 1$, we are able to use the equivalent norm (see [17] and [23, (2.3)])

$$\|f(u)\|_{B^{s/2}_{\gamma',\alpha}(L^{\rho'})} = \|f(u)\|_{L^{\gamma'}(L^{\rho'})} + \left\{ \int_0^\infty \left(\tau^{-s/2} \|f(u(\cdot)) - f(u(\cdot + \tau))\|_{L^{\gamma'}(L^{\rho'})} \right)^\alpha \frac{d\tau}{\tau} \right\}^{1/\alpha}.$$

Here, the difference is taken with respect to t. The first term in the right-hand side is bounded by $\|u\|^{p-1}_{L^{q_0}(L^{r_0})} \|u\|_{L^q(L^r)}$ by the Hölder inequality. The second term is bounded by $\|u\|^{p-1}_{L^{q_0}(L^{r_0})} \|u\|_{B^{s/2}_{q,\alpha}(L^r)}$ by the inequality

$$|f(u(\cdot)) - f(u(\cdot + \tau))| \lesssim (|u(\cdot)| + |u(\cdot + \tau)|)^{p-1} \, |u(\cdot) - u(\cdot + \tau)|.$$

So that, we obtain the required inequality.

Next, we consider the case $2 < s < 4$. Put $\theta := s/2 - 1$. By $\theta < 2$, we are able to use the equivalent norm

$$\begin{aligned} \|f(u)\|_{B^{\theta+1}_{\gamma',\alpha}(L^{\rho'})} &= \|f(u)\|_{L^{\gamma'}(L^{\rho'})} \\ &+ \left\{ \int_0^\infty \left(\tau^{-\theta-1} \|f(u(\cdot)) - 2f(u(\cdot + \tau)) + f(u(\cdot + 2\tau))\|_{L^{\gamma'}(L^{\rho'})} \right)^\alpha \frac{d\tau}{\tau} \right\}^{1/\alpha} \end{aligned} \tag{2.4}$$

(see [17, p. 22, Remark 1, p. 27, Theorem 3]). The first term in the right-hand side is bounded by $\|u\|^{p-1}_{L^{q_0}(L^{r_0})} \|u\|_{L^q(L^r)}$ by the Hölder inequality. To estimate the second term, we put $v(\cdot) := u(\cdot + \tau)$ and $w(\cdot) := u(\cdot + 2\tau)$. We use the inequality

$$\begin{aligned} |f(u) - 2f(v) + f(w)| &\lesssim (|u| + |v| + |w|)^{p-1} |u - 2v + w| \\ &+ \begin{cases} (|u - v| + |v - w|)^{p-1} \, |u - v| & \text{if } p < 2, \\ (|u| + |v| + |w|)^{p-2} (|u - v| + |v - w|)^2 & \text{if } p \ge 2 \end{cases} \end{aligned}$$

to have

$$\begin{aligned} \|f(u) - 2f(v) + f(w)\|_{L^{\gamma'}(L^{\rho'})} &\lesssim \|u\|^{p-1}_{L^{p\gamma'}(L^{p\rho'})} \|u - 2v + w\|_{L^q(L^r)} \\ &+ \begin{cases} \|u - v\|^p_{L^{p\gamma'}(L^{p\rho'})} & \text{if } p < 2, \\ \|u\|^{p-2}_{L^{q_0}(L^{r_0})} \|u - v\|^2_{L^{q_*}(L^{r_*})} & \text{if } p \ge 2, \end{cases} \end{aligned}$$

where we have put $2/q_* := 1/q_0 + 1/q$ and $2/r^* := 1/r_0 + 1/r$. So that, the second term in the right-hand side in (2.4) is bounded by

$$\|u\|^{p-1}_{L^{q_0}(L^{r_0})} \|u\|_{B^{\theta+1}_{q,\alpha}(L^r)} + \begin{cases} \|u\|^p_{B^{(\theta+1)/p}_{p\gamma',p\alpha}(L^{p\rho'})} & \text{if } p < 2, \\ \|u\|^{p-2}_{L^{q_0}(L^{r_0})} \|u\|^2_{B^{(\theta+1)/2}_{q_*,2\alpha}(L^{r_*})} & \text{if } p \ge 2, \end{cases}$$

where we have used $(\theta+1)/p < 1$ and $(\theta+1)/2 < 1$ by $s/2 < p$ and $s < 4$. By Lemma 2.1, we have

$$\|u\|_{B^{(\theta+1)/p}_{p\gamma',p\alpha}(L^{p\rho'})} \lesssim \|u\|^{1-1/p}_{B^0_{q_0,\infty}(L^{r_0})}\|u\|^{1/p}_{B^{\theta+1}_{q,\alpha}(L^r)},$$

$$\|u\|_{B^{(\theta+1)/2}_{q_*,2\alpha}(L^{r_*})} \lesssim \|u\|^{1/2}_{B^0_{q_0,\infty}(L^{r_0})}\|u\|^{1/2}_{B^{\theta+1}_{q,\alpha}(L^r)}.$$

We note that the embedding $L^{q_0}(\mathbb{R},V) \hookrightarrow B^0_{q_0,\infty}(\mathbb{R},V)$ holds for any $1 \le q \le \infty$ and any Banach space V since $\|\varphi_j *_t u\|_{L^q(V)} \le \|\varphi\|_{L^1(V)}\|u\|_{L^q(V)}$ for any $j \ge 1$ holds by the Young inequality in the definition of the Besov space (2.1). Therefore, we obtain the required result by the embedding $L^{q_0}(\mathbb{R},L^{r_0}) \hookrightarrow B^0_{q_0,\infty}(\mathbb{R},L^{r_0})$. □

3. Proof of Theorem 1.2

We have only to prove (1.9) since (1.7) and (1.8) has been proved in [15]. We set $U(t) = \exp(-it\Delta)$. We can estimate homogeneous and inhomogeneous parts independently.

Estimate of homogeneous part. We derive the estimate of $U(t)u_0$. By the formula $\varphi_j *_t e^{it|\xi|^2} = e^{it|\xi|^2}\hat{\varphi}_j(|\xi|^2)$, we have $\varphi_j *_t U(t)u_0 = U(t)\varphi_{j/2} *_x u_0$. From this equality together with the Strichartz estimate [3, 8, 10, 13, 18, 25], we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t \varphi_{k/2} *_x U(t)u_0\|^2_{L^q(L^r)} \\
&\quad\lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_{j/2} *_x \varphi_{k/2} *_x u_0\|^2_{L^2} \\
&\quad\lesssim \sum_{k=1}^{\infty} 2^{2\theta k}\|\varphi_{k/2} *_x u_0\|^2_{L^2} \lesssim \|u_0\|^2_{H^{2\theta}}.
\end{aligned}$$

Here, we have used the fact that $\hat{\varphi}_j(|\xi|^2)\hat{\varphi}_k(|\xi|^2) \neq 0$ implies $|j-k| \le 1$.Since the low frequency parts are easier to treat, we obtain

$$\|U(t)u_0\|_{B^{\theta-\sigma/2}_{q,2}(B^{\sigma}_{r,2})} \lesssim \|u_0\|_{H^{2\theta}}. \qquad \square$$

Estimate of inhomogeneous part. In what follows, $\tilde{f}(\tau,\xi)$ denotes the Fourier transform of f in the space-time, whereas $\hat{f}(t,\xi)$ denotes the Fourier transform with respect to the spatial variables. We put $\hat{\chi}_j(\tau) = \sum_{k=j-2}^{j+2}\hat{\varphi}_k(\tau)$ and $\hat{\chi}_{k/2}(\xi) = \hat{\chi}_k(|\xi|^2)$.

We estimate the solution to (1.6) with $u_0 = 0$. By the Fourier transform, the solution can be expressed as

$$\hat{u}(t,\xi) = \int_0^t e^{i(t-t')|\xi|^2}\hat{f}(t',\xi)dt' = \int_{-\infty}^{\infty}\frac{e^{it\tau}-e^{it|\xi|^2}}{2\pi i(\tau-|\xi|^2)}\tilde{f}(\tau,\xi)\,d\tau.$$

We define v and v_0 by

$$\hat{v}(t,\xi) = \text{p.v.-}\int_{-\infty}^{\infty} e^{it\tau}\frac{\tilde{f}(\tau,\xi)}{2\pi i(\tau-|\xi|^2)}\,d\tau = \frac{1}{2}\int_{-\infty}^{\infty}\operatorname{sign}(t-t')e^{i(t-t')|\xi|^2}\hat{f}(t',\xi)dt'$$

and $\hat{v}_0(\xi) = \hat{v}(0,\xi)$ respectively, so that $\hat{u} = \hat{v} - e^{it|\xi|^2}\hat{v}_0$. Here, we have used the formula p.v.-$\int_{-\infty}^{\infty}\{(\tau-|\xi|^2)\}^{-1}e^{it\tau}\,d\tau = \pi i\,\operatorname{sign}(t)e^{it|\xi|^2}$. Hence we have

$$u(t) = v(t) - U(t)v_0 = Gf(t) - U(t)v_0,$$

where $Gf(t) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(t-t')U(t-t')f(t')dt'$. We have

$$\|u\|_{B^{\theta-\sigma/2}_{q,2}(B^{\sigma}_{r,2})} \lesssim \|u\|_{B^{\theta}_{q,2}(L^r)} + \|u\|_{L^q(B^{\sigma}_{r,2})} + J,$$

where $J = \left\{ \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t \varphi_{k/2} *_x u\|^2_{L^q(L^r)} \right\}^{1/2}$. Since the first two terms can be estimated by (1.7) and (1.8) respectively, it suffices to estimate J. Since $v_0 = v(0)$, like homogeneous part we have

$$J^2 \lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t \varphi_{k/2} *_x v\|^2_{L^q(L^r)} + \sum_{k=1}^{\infty} 2^{2\theta k}\|\varphi_{k/2} *_x v\|^2_{L^\infty(L^2)}.$$

Taking the relation $\hat{\varphi}_k(|\xi|^2) = \hat{\varphi}_k(|\xi|^2)\hat{\chi}(|\xi|^2)$ into account, we decompose

$$\begin{aligned}
\varphi_{k/2} *_x v &= \chi_k *_t \varphi_{k/2} *_x v + (\varphi_{k/2} *_x v - \chi_k *_t \varphi_{k/2} *_x v) \\
&= G(\chi_k *_t \varphi_{k/2} *_x f) + K_k *_{t,x} \chi_{k/2} *_x f,
\end{aligned}$$

where

$$K_l(t,x) = \frac{1}{(2\pi)^{1+n}} \iint_{\mathbb{R}^{1+n}} e^{it\tau + ix\xi}\,\frac{\hat{\varphi}_l(|\xi|^2)(1-\hat{\chi}_l(\tau))}{i(\tau - |\xi|^2)}\, d\tau d\xi = 2^{nl/2}K_0(2^l t, 2^{l/2}x).$$

Hence, we see

$$\begin{aligned}
J &\lesssim \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t G(\chi_k *_t \varphi_{k/2} *_x f)\|^2_{L^q(L^r)} \Big)^{1/2} \\
&\quad + \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|K_k *_{t,x} \varphi_j *_t \chi_{k/2} *_x f\|^2_{L^q(L^r)} \Big)^{1/2} \\
&\quad + \Big(\sum_{k=1}^{\infty} 2^{2\theta k}\|G(\chi_k *_t \varphi_{k/2} *_x f)\|^2_{L^\infty(L^2)} \Big)^{1/2} \\
&\quad + \Big(\sum_{k=1}^{\infty} 2^{2\theta k}\|K_k *_{t,x} \chi_{k/2} *_x f\|^2_{L^\infty(L^2)} \Big)^{1/2} \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Using the Strichartz inequality, we see

$$\begin{aligned}
J_1^2 &\lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t \chi_k *_t f\|^2_{L^{\gamma'}(L^{\rho'})} \\
&\lesssim \sum_{j=1}^{\infty} 2^{2\theta j}\|\varphi_j *_t f\|^2_{L^{\gamma'}(L^{\rho'})} \lesssim \|f\|^2_{B^{\theta}_{\gamma'}(L^{\rho'})}.
\end{aligned}$$

Similarly we can obtain $J_3 \lesssim \|f\|_{B^{\theta}_{\gamma'}(L^{\rho'})}$. Let (q_0, r_0) be a pair satisfying $1 \le q_0 \le \infty$, $1 \le r_0 \le 2$ and $2/q_0 - \delta(r_0) = 2(1-\theta)$. For this pair, we define the pair (q_*, r_*) by $1/q_* = 1 - 1/q_0$ and $1/r_* = 1 + 1/2 - 1/r_0$. By the change of variables, we have

$$\|K_k\|_{L^{q_*,1}(L^{r_*})} = 2^{(1/q_0 - \delta(r_0)/2 - 1)k}\|K_0\|_{L^{q_*}(L^{r_*})} = C2^{-\theta k}.$$

Then, by the Hölder and the Young inequalities, we see

$$\begin{aligned}
\|K_k *_{t,x} \chi_{k/2} *_x f\|_{L^\infty(L^2)} &\lesssim \|K_k\|_{L^{q_*,1}(L^{r_*})}\|\chi_{k/2} *_x f\|_{L^{q_0,\infty}(L^{r_0})} \\
&= C2^{-\theta k}\|\chi_{k/2} *_x f\|_{L^{q_0,\infty}(L^{r_0})},
\end{aligned}$$

which implies $J_4 \lesssim \|f\|_{\ell^2(L^{q_0,\infty}(L^{r_0}))}$. Let (q_1, r_1) be a pair satisfying $1 \le q_1 \le q$, $1 \le r_1 \le r$ and $2/q_1 - \delta(r_1) = 2 - \sigma$. Similarly as above, but using generalized Young inequality instead of the Hölder inequality, we can obtain the following estimates:

$$\|K_k *_{t,x} \varphi_j *_t \chi_{k/2} *_x f\|_{L^q(L^r)} \lesssim \|\varphi_j *_t f\|_{L^{\gamma'}(L^{\rho'})},$$
$$\|K_k *_{t,x} \varphi_j *_t \chi_{k/2} *_x f\|_{L^q(L^r)} \lesssim 2^{-\sigma k/2}\|\varphi_j *_t \chi_{k/2} *_x f\|_{L^{q_1,\infty}(L^{r_1})}.$$

Hence, we see

$$\begin{aligned} J_2 &\lesssim \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{j} 2^{(2\theta-\sigma)j+\sigma k}\|\varphi_j *_t f\|^2_{L^{\gamma'}(L^{\rho'})}\Big)^{1/2} \\ &\quad + \Big(\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} 2^{(2\theta-\sigma)j}\|\varphi_j *_t \chi_{k/2} *_x f\|^2_{L^{q_1,\infty}(L^{r_1})}\Big)^{1/2} \\ &=: J_{2,1} + J_{2,2}. \end{aligned}$$

Since $\sum_{k=1}^{j} 2^{\sigma k} \lesssim 2^{\sigma j}$, we have $J_{2,1} \lesssim \|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})}$. On the other hand, we see

$$\begin{aligned} J_{2,2}^2 &= \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{(2\theta-\sigma)j}\|\varphi_j *_t \chi_{k/2} *_x f\|^2_{L^{q_1,\infty}(L^{r_1})} \\ &\le \sum_{k=1}^{\infty}\|\chi_{k/2} *_x f\|^2_{B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1})} \lesssim \|f\|^2_{\ell^2(B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1}))}. \end{aligned}$$

Collecting these estimates, we obtain

$$J \lesssim \|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{q_0,\infty}(L^{r_0}))} + \|f\|^2_{\ell^2(B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1}))}.$$

Now, if $(q_j, r_j) \ne (\bar{q}_j, \bar{r}_j)$, then we choose (q_j, r_j) as follows: (i) if $\bar{r}_j < \rho'$, then we choose $r_j = \rho'$, so that by [15, Lemma 2.4 (1)], $\|f\|_{\ell^2(L^{q_0,\infty}(L^{r_0}))}$ and $\|f\|_{\ell^2(B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1}))}$ are bounded by $\|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})}$; (ii) if $\bar{r}_j \ge \rho'$, then we choose r_j satisfying $\rho' \le r_j \le \bar{r}_j$, so that by [15, Lemma 2.5], $\|f\|_{\ell^2(L^{q_0,\infty}(L^{r_0}))}$ and $\|f\|_{\ell^2(B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1}))}$ are bounded by $\|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{\bar{q}_0}(L^{\bar{r}_0}))}$ and $\|f\|_{B^{\theta}_{\gamma,2}(L^{\rho'})} + \|f\|_{\ell^2(B^{\theta-\sigma/2}_{q_1,\infty}(L^{r_1}))}$ respectively. Thus we obtain

$$J \lesssim \|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})} + \|f\|_{\ell^2(L^{\bar{q}_0}(L^{\bar{r}_0}))} + \|f\|^2_{\ell^2(B^{\theta-\sigma/2}_{\bar{q}_1,\infty}(L^{\bar{r}_1}))},$$

which proves (1.9).

Finally, we show that we can drop the term $\|f\|_{\ell^2(B^{\theta-\sigma/2}_{\bar{q}_1,\infty}(L^{\bar{r}_1}))}$ if $\bar{q}_0 \le q$. To this end, we estimate $J_{2,2}$ differently. In this case, let (q_1, r_1) be a pair (q_1, r_1) be a pair satisfying $1 \le q_1 \le q$, $1 \le r_1 \le r$ and $2/q_1 - \delta(r_1) = 2(1-\theta)$. Then, by the generalized Young inequality, we have

$$\|K_k *_{t,x} \varphi_j *_t \chi_{k/2} *_x f\|_{L^q(L^r)} \lesssim 2^{-\theta k}\|\varphi_j *_t \chi_{k/2} *_x f\|_{L^{q_1,\infty}(L^{r_1})}.$$

Hence, using [15, Lemma 2.5] if necessary, we have

$$
\begin{aligned}
J_{2,2}^2 &\lesssim \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{(2\theta-\sigma)j-(2\theta-\sigma)k}\|\chi_{k/2} *_x f\|^2_{L^{q_1,\infty}(L^{r_1})}\\
&\lesssim \sum_{k=1}^{\infty}\|\chi_{k/2} *_x f\|^2_{L^{q_1,\infty}(L^{r_1})} \lesssim \|f\|^2_{\ell^2(L^{q_1,\infty}(L^{r_1}))}\\
&\lesssim \|f\|^2_{B^{\theta}_{\gamma',2}(L^{\rho'})} + \|f\|^2_{\ell^2(L^{\bar q_0}(L^{\bar r_0}))}.
\end{aligned}
$$

By this estimate, together with $\|u\|_{L^q(B^{\sigma}_{r,2})} \le \|u\|_{L^q(B^{2\theta}_{r,2})}$, we obtain the desired result. $\square$

4. Proof of Theorem 1.1

Except for the case $3 \le s < 4$, the proof of Theorem 1.1 has been shown in [15]. We only consider the case $3 \le s < 4$. We define r by $1/r := 1/2 - 1/n$ throughout this section.

We use the following corollary of Theorem 1.2.

Corollary 4.1. *Let $n \ge 3$, $3 \le s < 4$. The solution to the Cauchy problem* (1.6) *satisfies the inequality:*

$$
\begin{aligned}
\max&\left\{\|u\|_{L^\infty(H^s)}, \|u\|_{B^{s/2}_{2,2}(L^r)}, \|u\|_{B^{(s-1)/2}_{2,2}(B^1_{r,2})}\right\}\\
&\lesssim \|u_0\|_{H^1} + \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)} + \|f\|_{L^\infty(H^{s-2})}. \qquad (4.1)
\end{aligned}
$$

Here, we put $u_1 := \partial_t u(0,\cdot)$. Moreover, if the right-hand side of (4.1) *is finite with $f \in C(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$, then $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.*

Proof. The continuity of the solution follows directly from Theorem 1.2 and the equation (1.1). By the time-derivative of (1.1), the function $\partial_t u$ satisfies

$$(\partial_t + i\Delta)\partial_t u = \partial_t f.$$

We divide the proof of the required inequality (4.1) into three parts (1), (2) and (3) as follows.

(1) We have

$$\|\partial_t u\|_{L^\infty(H^{s-2})} \lesssim \|u_1\|_{H^{s-2}} + \|\partial_t f\|_{B^{(s-2)/2}_{2,2}(L^{r'})} + \|\partial_t f\|_{\ell^2(L^{2/(4-s)}(L^2))}$$

by (1.8), where we have put $\sigma = s-2$, $q = \infty$, $\gamma = 2$, $\rho = r$, $\bar q_1 = 2/(4-s)$, $\bar r_1 = 2$, and we note that $1 \le \sigma < 2$ and $2/\bar q_1 - \delta(\bar r_1) = 2 - \sigma$ hold. Since we have

$$\|\partial_t f\|_{B^{(s-2)/2}_{2,2}(L^{r'})} \lesssim \|f\|_{B^{s/2}_{2,2}(L^{r'})}, \qquad (4.2)$$

$$\|\partial_t f\|_{\ell^2(L^{2/(4-s)}(L^2))} \lesssim \|\partial_t f\|_{\ell^2(B^{(s-3)/2}_{2,2}(L^2))} \lesssim \|f\|_{B^{(s-1)/2}_{2,2}(L^2)} \qquad (4.3)$$

by the embedding $B^{(s-3)/2}_{2,2}(L^2) \hookrightarrow L^{2/(4-s)}(L^2)$, we obtain

$$\|\partial_t u\|_{L^\infty(H^{s-2})} \lesssim \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)}. \qquad (4.4)$$

We have

$$\|u\|_{L^\infty(L^2)} \lesssim \|u_0\|_{L^2} + \|f\|_{L^2(L^{r'})} \qquad (4.5)$$

by the endpoint Strichartz estimate [13]. We also have

$$\|u\|_{L^\infty(\dot H^s)} = \|\Delta u\|_{L^\infty(\dot H^{s-2})} \le \|\partial_t u\|_{L^\infty(\dot H^{s-2})} + \|f\|_{L^\infty(\dot H^{s-2})} \qquad (4.6)$$

by the equation $\partial_t u + i\Delta u = f$. Thus, we obtain

$$\begin{aligned}\|u\|_{L^\infty(H^s)} &\lesssim \|u\|_{L^\infty(L^2)} + \|u\|_{L^\infty(\dot H^s)}\\ &\lesssim \|u_0\|_{L^2} + \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})}\\ &\qquad + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)} + \|f\|_{L^\infty(H^{s-2})} \end{aligned} \tag{4.7}$$

by (4.4), (4.5) and (4.6), where we have used the embedding $H^{s-2}(\mathbb{R}^n) \hookrightarrow \dot H^{s-2}(\mathbb{R}^n)$.

(2) Similarly to the argument in (1), we have

$$\|\partial_t u\|_{B^{(s-2)/2}_{2,2}(L^r)} \lesssim \|u_1\|_{H^{s-2}} + \|\partial_t f\|_{B^{(s-2)/2}_{2,2}(L^{r'})} + \|\partial_t f\|_{\ell^2(L^{2/(4-s)}(L^2))}$$

by (1.7), where we have put $\theta = (s-2)/2$, $q = 2$, $\gamma = 2$, $\rho = r$, $\bar q_0 = 2/(4-s)$, $\bar r_0 = 2$, and we note that $1/2 \le \theta < 1$ and $2/\bar q_0 - \delta(\bar r_0) = 2(1-\theta)$ hold. By (4.2) and (4.3), we obtain

$$\|\partial_t u\|_{B^{(s-2)/2}_{2,2}(L^r)} \lesssim \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)}. \tag{4.8}$$

We have

$$\|u\|_{L^2(L^r)} \lesssim \|u_0\|_{L^2} + \|f\|_{L^2(L^{r'})} \tag{4.9}$$

by the endpoint Strichartz estimate. We also have

$$\|u\|_{\dot B^{s/2}_{2,2}(L^r)} = \|\partial_t u\|_{\dot B^{(s-2)/2}_{2,2}(L^r)}. \tag{4.10}$$

Thus, we obtain

$$\begin{aligned}\|u\|_{B^{s/2}_{2,2}(L^r)} &\lesssim \|u\|_{L^2(L^r)} + \|u\|_{\dot B^{s/2}_{2,2}(L^r)}\\ &\lesssim \|u_0\|_{L^2} + \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)}\end{aligned} \tag{4.11}$$

by (4.8), (4.9) and (4.10).

(3) Similarly to the arguments in (1) and (2), we have

$$\|u\|_{L^2(B^1_{r,2})} \lesssim \|u_0\|_{H^1} + \|f\|_{B^{1/2}_{2,2}(L^{r'})} + \|f\|_{\ell^2(L^2(L^2))}$$

by (1.8), where we have put $\sigma = 1$, $q = 2$, $\gamma = 2$, $\rho = r$, $\bar q_1 = 2$, $\bar r_1 = 2$, and we note that $2/\bar q_1 - \delta(\bar r_1) = 2 - \sigma$ holds. Since we have $\ell^2(L^2(L^2)) = L^2(B^0_{2,2}) = L^2(L^2)$, we obtain

$$\|u\|_{L^2(B^1_{r,2})} \lesssim \|u_0\|_{H^1} + \|f\|_{B^{1/2}_{2,2}(L^{r'})} + \|f\|_{L^2(L^2)}. \tag{4.12}$$

When $s = 3$, we have

$$\begin{aligned}\|u\|_{\dot B^1_{2,2}(B^1_{r,2})} &\lesssim \|\partial_t u\|_{L^2(B^1_{r,2})}\\ &\lesssim \|u_1\|_{H^1} + \|\partial_t f\|_{B^{1/2}_{2,2}(L^{r'})} + \|\partial_t f\|_{L^2(L^2)}\\ &\lesssim \|u_1\|_{H^1} + \|f\|_{B^{3/2}_{2,2}(L^{r'})} + \|f\|_{B^1_{2,2}(L^2)},\end{aligned} \tag{4.13}$$

where we have used (4.12) with u replaced by $\partial_t u$. When $3 < s < 4$, we have

$$\begin{aligned}\|u\|_{\dot B^{(s-1)/2}_{2,2}(B^1_{r,2})} &\lesssim \|\partial_t u\|_{\dot B^{(s-3)/2}_{2,2}(B^1_{r,2})}\\ &\lesssim \|u_1\|_{H^{s-2}} + \|\partial_t f\|_{B^{(s-2)/2}_{2,2}(L^{r'})}\\ &\qquad + \|\partial_t f\|_{\ell^2(L^{2/(4-s)}(L^2))} + \|\partial_t f\|_{\ell^2(B^{(s-3)/2}_{2,2}(L^2))}\end{aligned} \tag{4.14}$$

by (1.9), where we have put $\theta = (s-2)/2$, $\sigma = 1$, $q = 2$, $\gamma = 2$, $\rho = r$, $\bar{q}_0 = 2/(4-s)$, $\bar{r}_0 = 2$, $\bar{q}_1 = 2$, $\bar{r}_1 = 2$, and we note that $1/2 < \theta < 1$, $2/\bar{q}_0 - \delta(\bar{r}_0) = 2(1-\theta)$ and $2/\bar{q}_1 - \delta(\bar{r}_1) = 2 - \sigma$ hold. Since we have

$$\|\partial_t f\|_{\ell^2(B^{(s-3)/2}_{2,2}(L^2))} \lesssim \|\partial_t f\|_{B^{(s-3)/2}_{2,2}(L^2)} \lesssim \|f\|_{B^{(s-1)/2}_{2,2}(L^2)},$$

we obtain

$$\|u\|_{\dot{B}^{(s-1)/2}_{2,2}(B^1_{r,2})} \lesssim \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)} \tag{4.15}$$

by (4.2) and (4.3). Thus, we obtain

$$\begin{aligned}\|u\|_{B^{(s-1)/2}_{2,2}(B^1_{r,2})} &\lesssim \|u\|_{L^2(B^1_{r,2})} + \|u\|_{\dot{B}^{(s-1)/2}_{2,2}(B^1_{r,2})} \\ &\lesssim \|u_0\|_{H^1} + \|u_1\|_{H^{s-2}} + \|f\|_{B^{s/2}_{2,2}(L^{r'})} + \|f\|_{B^{(s-1)/2}_{2,2}(L^2)} \end{aligned} \tag{4.16}$$

by (4.12) and (4.15).

By (4.7), (4.11) and (4.16), we obtain the required inequality □

Lemma 4.1. *Let $n \geq 5$, $2 < s < \min\{4; n/2\}$. Let f satisfy $N(p)$ with $1 < p \leq p(s)$. Then the following estimates hold.*

(1) $\|f(u)\|_{L^\infty(H^{s-2})} \lesssim \|u\|^p_{L^\infty(H^s)}$ *if $p > \max\{1; s-2\}$;*

(2) $\|f(u)\|_{B^{(s-1)/2}_{2,2}(L^2)} \lesssim \|u\|^{p-1}_{L^\infty(H^s)} \|u\|_{B^{(s-1)/2}_{2,2}(B^1_{r,2})}$ *if $p > \max\{1 + 2/n; (s-1)/2\}$;*

(3) $\|f(u)\|_{B^{s/2}_{2,2}(L^{r'})} \lesssim \|u\|^{p-1}_{L^\infty(H^s)} \|u\|_{B^{s/2}_{2,2}(L^r)}$ *if $p > \max\{1 + 4/n; s/2\}$;*

Proof. (1) By the assumption $1 < p \leq p(s)$, there exists $r_\sharp$ which satisfies $0 < 1/r_\sharp \leq 2/n(p-1)$, $1/2 - s/n \leq 1/r_\sharp \leq 1/2$ and $1/r_\sharp < 1/2(p-1)$. We define $r_{\sharp\sharp}$ by $1/r_{\sharp\sharp} = 1/2 - (p-1)/r_\sharp$. By Lemma 2.2 and the embeddings $H^s(\mathbb{R}^n) \hookrightarrow L^{r_\sharp}(\mathbb{R}^n)$, $H^s(\mathbb{R}^n) \hookrightarrow H^{s-2, r_{\sharp\sharp}}(\mathbb{R}^n)$, we have

$$\|f(u)\|_{H^{s-2}} \lesssim \|u\|^{p-1}_{L^{r_\sharp}} \|u\|_{H^{s-2, r_{\sharp\sharp}}} \lesssim \|u\|^p_{H^s} \tag{4.17}$$

which yields the required estimate.

(2) By the assumption $1 + 2/n \leq p \leq p(s)$, there exists r_* which satisfies $1/n(p-1) \leq 1/r_* \leq 2/n(p-1)$, $1/2 - s/n \leq 1/r_* \leq 1/2$ and $0 < 1/r_* < 1/2(p-1)$. We define r_{**} by $1/r_{**} = 1/2 - (p-1)/r_*$. By Lemma 2.3 and the embeddings $H^s(\mathbb{R}^n) \hookrightarrow L^{r_*}(\mathbb{R}^n)$, $B^1_{r,2}(\mathbb{R}^n) \hookrightarrow L^{r_{**}}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|f(u)\|_{B^{(s-1)/2}_{2,2}(L^2)} &\lesssim \|u\|^{p-1}_{L^\infty(L^{r_*})} \|u\|_{B^{(s-1)/2}_{2,2}(L^{r_{**}})} \\ &\lesssim \|u\|^{p-1}_{L^\infty(H^s)} \|u\|_{B^{(s-1)/2}_{2,2}(B^1_{r,2})}.\end{aligned}$$

(3) We put $r_\star := n(p-1)/2$. By the assumption $1 + 4/n \leq p \leq p(s)$, $r_\star$ satisfies $1/2 - s/n \leq 1/r_\star \leq 1/2$ and $1/r' = (p-1)/r_\star + 1/r$. By Lemma 2.3 and the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{r_\star}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|f(u)\|_{B^{s/2}_{2,2}(L^{r'})} &\lesssim \|u\|^{p-1}_{L^\infty(L^{r_\star})} \|u\|_{B^{s/2}_{2,2}(L^r)} \\ &\lesssim \|u\|^{p-1}_{L^\infty(H^s)} \|u\|_{B^{s/2}_{2,2}(L^r)}\end{aligned}$$

as required. □

Lemma 4.2. *Let $n \geq 5$, $2 < s < \min\{4; n/2\}$. Let f satisfy $N(p)$ with $1 < p \leq p(s)$. Then $u_1 := -i\Delta u_0 + f(u_0)$ satisfies the inequality*

$$\|u_1\|_{H^{s-2}} \lesssim \|u_0\|_{H^s} + \|u_0\|_{H^s}^p.$$

Proof. We obtain the required inequality by (4.17). □

Proof of Theorem 1.1. Let $R > 0$ be a constant. We define a function space X, $X(R)$ and the metric d by

$$X := L^\infty(\mathbb{R}, H^s(\mathbb{R}^n)) \cap B_{2,2}^{s/2}(\mathbb{R}, L^r(\mathbb{R}^n)) \cap B_{2,2}^{(s-1)/2}(\mathbb{R}, B_{r,2}^1(\mathbb{R}^n)),$$
$$X(R) := \left\{u \in X;\ \|u\|_X \leq R\right\}$$

and

$$d(u, v) := \|u - v\|_{L^\infty(L^2) \cap L^2(L^r)}.$$

We regard the solution of the Cauchy problem (1.1) and (1.2) as the fixed point of the integral equation given by

$$u(t) = \Phi(u)(t) := U(t)u_0 + \int_0^t U(t - t')f(u(t'))dt'$$

for $t \in \mathbb{R}$, where $u(t) := u(t, \cdot)$. Let n, s, p satisfy the assumption in the theorem. We show that Φ is a contraction mapping on the metric space $(X(R), d)$ for some $R > 0$.

By Corollary 4.1, Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} \|\Phi(u)\|_X &\lesssim \|u_0\|_{H^1} + \|u_1\|_{H^{s-2}} + \|f\|_{B_{2,2}^{s/2}(L^{r'})} + \|f\|_{B_{2,2}^{(s-1)/2}(L^2)} + \|f\|_{L^\infty(H^{s-2})} \\ &\lesssim \|u_0\|_{H^s} + \|u_0\|_{H^s}^p + \|u\|_X^p \end{aligned} \tag{4.18}$$

for any $u, v \in X$. By the endpoint Strichartz estimate, we have

$$d(\Phi(u), \Phi(v)) \lesssim \|f(u) - f(v)\|_{L^2(L^{r'})} \tag{4.19}$$

for any $u, v \in X$. By $|f(u) - f(v)| \lesssim (|u|^{p-1} + |v|^{p-1})|u - v|$ and the Hölder inequality, we have

$$\begin{aligned} \|f(u) - f(v)\|_{L^2(L^{r'})} &\lesssim \max_{w=u,v} \|w\|_{L^\infty(L^{r_\star})}^{p-1} \|u - v\|_{L^2(L^r)} \\ &\lesssim \max_{w=u,v} \|w\|_{L^\infty(H^s)}^{p-1} \|u - v\|_{L^2(L^r)} \\ &\lesssim \max_{w=u,v} \|w\|_X^{p-1} d(u, v), \end{aligned} \tag{4.20}$$

where we have put $r_\star := n(p - 1)/2$ and we have used the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{r_\star}(\mathbb{R}^n)$. So that, we obtain

$$\|\Phi(u)\|_X \leq C_0(\|u_0\|_{H^s} + \|u_0\|_{H^s}^p) + CR^p, \quad d(\Phi(u), \Phi(v)) \leq CR^{p-1} d(u, v)$$

for any $u, v \in X(R)$, where C_0 and C are some positive constants which are independent of u, v and R. Therefore, Φ is a contraction mapping if R satisfies

$$2C_0(\|u_0\|_{H^s} + \|u_0\|_{H^s}^p) \leq R, \quad CR^{p-1} \leq \frac{1}{2}.$$

Since these conditions are satisfied if the norm of the initial data is sufficiently small, we have a unique solution u of (1.1) and (1.2) in $X(R)$ for some $R > 0$.

We will show that the solution u obtained above belongs to $C(\mathbb{R}, H^s(\mathbb{R}^n))$. This follows from Corollary 4.1 once we prove $f(u) \in C(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$, since the right-hand side of (4.1) is finite by $u \in X(R)$ as we have shown in (4.18). We only consider the case $p \leq 2$, but even if $p \geq 2$, the proof works with a slight modification. Let $0 < \sigma < 1$. We define ρ, ρ_0 and ρ_1 by $1/\rho = 1/2 - (s-2)/n = 1/\rho_0 - \sigma/n = 1/\rho_1 - s/2n$; we have $\rho_1 < 2 < \rho_0 < \rho$ since $3 \le s < 4$. By the Sobolev embedding theorem, we have the inclusions $B^{s/2}_{\rho_1,2}(\mathbb{R}^n) \hookrightarrow H^{s-2}(\mathbb{R}^n) \hookrightarrow B^{\sigma}_{\rho_0,2}(\mathbb{R}^n) \hookrightarrow L^{\rho}(\mathbb{R}^n)$. We note that the homogeneous parts of these spaces have the same scale. Similarly as Lemma 4.1, by the use of Lemma 2.2, we have the inequalities

$$\|f(v)\|_{L^\rho} \lesssim \|f(v)\|_{B^{\sigma}_{\rho_0,2}} \lesssim \|f(v)\|_{B^{s/2}_{\rho_1,2}} \lesssim \|v\|^p_{H^s}, \tag{4.21}$$

$$\|f(v) - f(w)\|_{L^\rho} \lesssim \max\{\|v\|_{H^s}; \|w\|_{H^s}\}^{p-1}\|v - w\|_{H^2_\rho}, \tag{4.22}$$

$$\|f(v) - f(w)\|_{B^{\sigma}_{\rho_0,2}} \lesssim \|v - w\|^{p-1}_{H^2_\rho}\|v\|_{H^s} + \|w\|^{p-1}_{H^s}\|v - w\|_{B^{\sigma+2}_{\rho_0,2}} \tag{4.23}$$

for $v, w \in H^s$. We remark that

$$u \in C^1(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \hookrightarrow C^1(\mathbb{R}, B^{\sigma}_{\rho_0,2}(\mathbb{R}^n)) \hookrightarrow C^1(\mathbb{R}, L^\rho(\mathbb{R}^n)).$$

Let $h \in \mathbb{R}$. Using the equation (1.1) and the inequality (4.22), we have

$$\|\Delta(u(t+h) - u(t))\|_{L^\rho} \le \|\partial_t(u(t+h) - u(t))\|_{L^\rho} + CR^{p-1}\|u(t+h) - u(t)\|_{H^2_\rho}.$$

Since $CR^{p-1} < 1/2$, we have

$$\|u(t+h) - u(t)\|_{H^2_\rho} \lesssim \|\partial_t(u(t+h) - u(t))\|_{L^\rho} + \|u(t+h) - u(t)\|_{L^\rho},$$

hence we obtain $u \in C(\mathbb{R}, H^2_\rho(\mathbb{R}^n))$. Similarly, using the inequality (4.23) instead of (4.22), we see

$$\begin{aligned}\|u(t+h) - u(t)\|_{B^{\sigma+2}_{\rho_0,2}} \lesssim{}& \|\partial_t(u(t+h) - u(t))\|_{B^{\sigma}_{\rho_0,2}} + R\|u(t+h) - u(t)\|^{p-1}_{H^2_\rho} \\ &+ \|u(t+h) - u(t)\|_{B^{\sigma}_{\rho_0,2}},\end{aligned}$$

so that $u \in C(\mathbb{R}, B^{\sigma+2}_{\rho_0,2}(\mathbb{R}^n))$. By interpolation, we see

$$H^{s-2}(\mathbb{R}^n) = (B^{\sigma}_{\rho_0,2}(\mathbb{R}^n), B^{s/2}_{\rho_1,2}(\mathbb{R}^n))_{\theta,2}$$

with $\theta = 2(s - \sigma - 2)/(s - 2\sigma)$, see e.g. [1, Theorem 6.4.5]. Hence, by (4.21) and (4.23), we obtain $f(u) \in C(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$.

Let us prove the uniqueness of the solution in $C(\mathbb{R}, H^s(\mathbb{R}^n))$. Let $u \in X(R) \cap C(\mathbb{R}, H^s(\mathbb{R}^n))$ be the solution obtained by the above argument, and let v be another solution in $C(\mathbb{R}, H^s(\mathbb{R}^n))$ for the same datum u_0. Let t_0 be defined by $t_0 := \sup\{t \ge 0;\ u(\tau) = v(\tau) \text{ for } 0 \le \tau \le t\}$. If $t_0 < \infty$, then we consider an interval $I := [t_0, t_0 + \varepsilon)$ with $\varepsilon > 0$. Since $v(t) \in L^{r_\star}(\mathbb{R}^n)$ for any $t \in \mathbb{R}$ by the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{r_\star}(\mathbb{R}^n)$ and $p = p(s)$, we have

$$\begin{aligned}\|u - v\|_{L^2(I,L^r)} &\lesssim \|f(u) - f(v)\|_{L^2(I,L^{r'})} \\ &\lesssim \max_{w=u,v}\|w\|^{p-1}_{L^\infty(I,H^s)}\|u - v\|_{L^2(I,L^r)}\end{aligned}$$

by the similar argument on (4.19) and (4.20). By $u, v \in C(\mathbb{R}, H^s(\mathbb{R}^n))$ and $u \in X(R)$ with R sufficiently small, we obtain $\|u - v\|_{L^2(I,L^r)} = 0$ for a sufficiently small $\varepsilon > 0$. Thus, $u = v$ on I, which contradicts the definition of t_0. We have shown $t_0 = \infty$, namely, $u = v$ on $[0, \infty)$. Analogously, we are able to show $u = v$ on $(-\infty, 0]$. So that, we obtain $u = v$ on $\mathbb{R}$. □

Remark. In the previous paper [15], we found the solution to $u = \Phi(u)$ as follows. We use the space

$$\begin{aligned}Y = {} & L^\infty(\mathbb{R}, H^s(\mathbb{R}^n)) \cap W^1_\infty(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \\ & \cap B^{s/2}_{q,2}(\mathbb{R}, L^r(\mathbb{R}^n)) \cap L^q(\mathbb{R}, B^s_{r,q}(\mathbb{R}^n)) \cap W^1_q(\mathbb{R}, B^{s-2}_{r,q}(\mathbb{R}^n))\end{aligned}$$

instead of X. By the contraction mapping principle, for sufficiently small u_0, we find a unique fixed point of Φ in $Y(R) = \{u \in Y; \|u\|_Y \le R\}$ equipped with the metric d. Here, (q, r) is an admissible pair to be chosen suitably, and $R > 0$ is sufficiently small number. We first note that in (1.8) we can replace the third indices 2 of Besov spaces with q. We choose admissible pairs (q, r) and (γ, ρ) so that $1/\gamma' = p/q > s/2 - 1$. As in the proof above, we estimate $\partial_t \Phi(u)$ by (1.7) with $\theta = (s-2)/2$, $\bar{r}_0 = \rho'$, and by (1.8) with $\sigma = s - 2$, $\bar{q}_1 = q$. Then, $\|\partial_t f(u)\|_{B^{(s-2)/2}_{\gamma',2}(L^{\rho'})}$ and $\|\partial_t f(u)\|_{l^2(L^{\bar{q}_0}(L^{\bar{r}_0}))}$ are bounded by $\|u\|^{p-1}_{L^q(B^s_{r,q})} \|u\|_{B^{s/2}_{q,2}(L^r)}$ if $p/q > s/2 - 1$. The most delicate point is the estimate of $\|\partial_t f(u)\|_{l^q(L^{\bar{q}_1}(L^{\bar{r}_1}))} \sim \|\partial_t f(u)\|_{L^q(B^0_{\bar{r}_1,q})}$. By the Leibniz rule with $1/\nu_2 = 1/r - (s-2)/n$, we have

$$\begin{aligned}\|\partial_t f(u)\|_{B^0_{\bar{r}_1,q}} & \lesssim \|f'(u)\|_{L^{\nu_0/(p-1)}} \|\partial_t u\|_{B^0_{\nu_2,q}} + \|f'(u)\|_{B^0_{\nu_0/(p-1),q}} \|\partial_t u\|_{L^{\nu_2}} \\ & \lesssim (\|u\|_{H^s} + \|u\|_{B^s_{2,q(p-1)}})^{p-1} \|\partial_t u\|_{B^{s-2}_{r,q\wedge\nu_2}}.\end{aligned}$$

The right-hand side is bounded by $\|u\|^{p-1}_{H^s} \|\partial_t u\|_{B^{s-2}_{r,q}}$ provided that $2/(p-1) \le q \le \nu_2$. It is possible to choose q satisfying this condition together with $p/q > s/2 - 1$ only when (1.4) is satisfied.

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