A REMARK ON LOCAL WELL-POSEDNESS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POWER NONLINEARITY —AN ALTERNATIVE APPROACH

TAKESHI WADA

Abstract. We study the nonlinear Schrödinger equation (NLS)

$$\partial_t u + i\Delta u = i\lambda|u|^{p-1}u$$

in $\boldsymbol{R}^{1+n}$, where $n \geq 3$, $p > 1$, and $\lambda \in \boldsymbol{C}$. We prove that (NLS) is locally well-posed in H^s if $1 < s < \min\{4; n/2\}$ and $\max\{1; s/2\} < p < 1+4/(n-2s)$. To obtain a good lower bound for p, we use fractional order Besov spaces for the time variable. The use of such spaces together with time cut-off makes it difficult to derive positive powers of time length from nonlinear estimates, so that it is difficult to apply the contraction mapping principle. For the proof we improve Pecher's inequality (1997), which is a modification of the Strichartz estimate, and apply this inequality to the nonlinear problem together with paraproduct formula.

1. Introduction

In this paper we study the Cauchy Problem for the following nonlinear Schrödinger equation

$$\partial_t u + i\Delta u = f(u), \tag{1.1}$$

$$u(0, \cdot) = u_0, \tag{1.2}$$

where $u : \boldsymbol{R}^{1+n} \to \boldsymbol{C}$, and $f(u) = i\lambda|u|^{p-1}u$ with $p > 1, \lambda \in \boldsymbol{C}$. The solvability of (1.1)-(1.2) in the Sobolev space $H^s = H^s(\boldsymbol{R}^n)$ has been studied in a large amount of literature. Let $0 \leq s < n/2$. It is well-known that the Cauchy problem (1.1)-(1.2) is time locally well-posed in H^s if $s < p \leq p^*(s)$, where $p^*(s) = 1 + 4/(n-2s)$, see e.g. [4, 6–8, 11–13, 15, 21]. On the one hand, the condition $p \leq p^*(s)$ comes from scaling; the upper bound $p^*(s)$ is the critical exponent in H^s from the scaling point of view. On the other hand, the condition $s < p$ comes from the regularity of the nonlinear term. When we solve (1.1)-(1.2) in H^s, we usually take spatial derivatives of order s of the equation. Namely this lower bound is the condition for the nonlinear term to be differentiable at least s times. However, this lower bound for p is not necessarily optimal. For example, (1.1)-(1.2) is time locally well-posed in H^2 if $1 < p \leq p^*(2) = 1 + 4/(n-4)$. (For simplicity we only consider the case $n \geq 5$.) This result was first proved by Tsutsumi [20] in the case where $1 < p < 1+4/(n-2)$ with $\lambda \in \boldsymbol{R}$, generalized by Kato [11,12] in the subcritical case $1 < p < 1 + 4/(n-4)$ with $\lambda \in \boldsymbol{C}$, and recently settled by Cazenave-Fang-Han [3] in the critical case $p = 1 + 4/(n-4)$. The point is that we can first evaluate $\partial_t u$

2010 *Mathematics Subject Classification.* 35Q55, 35Q41.
Key words and phrases. Nonlinear Schrödinger equations, well-posedness, Besov spaces.
Supported in part by JSPS, Grant-in-Aid for Scientific Research (C) #25400176.

instead of Δu, since the Schrödinger equation is second order in x and first order in t. Once we obtain the estimate of $\partial_t u$, then using the equation itself we can recover spatial regularity.

For $1 < s < 2$, Pecher [17] treated similar problem and showed that (1.1)-(1.2) is time locally well-posed in H^s if $1 < s < p^*(s)$ (see also Fang-Han [5]). One of main ingredients in his result is a modification of Strichartz estimates by which we can replace fractional order spatial derivatives with half the numbers of time derivatives in terms of Besov spaces. The result in [17] was extended to the case where $2 < s < 4$ and $s/2 < p < p^*(s)$ by Uchizono-Wada [23].

For a Banach space V, we define the V-valued Besov space $B^\theta_{q,\alpha}(\boldsymbol{R};V)$ by

$$B^\theta_{q,\alpha}(\boldsymbol{R};V) = \{u \in \mathscr{S}'(\boldsymbol{R};V);\ \|u\|_{B^\theta_{q,\alpha}(\boldsymbol{R};V)} < \infty\},$$

where $\theta \in \boldsymbol{R}$, $1 \le q \le \infty$, $1 \le \alpha < \infty$ and

$$\|u\|_{B^\theta_{q,\alpha}(\boldsymbol{R};V)} = \|\psi *_t u\|_{L^q(\boldsymbol{R};V)} + \Big\{\sum_{j\ge1} \big(2^{\theta j}\|\varphi_j *_t u\|_{L^q(\boldsymbol{R};V)}\big)^\alpha\Big\}^{1/\alpha}.$$

If $\alpha = \infty$, then we replace the second term with $\sup_{j\ge1} 2^{\theta j}\|\varphi_j *_t u\|_{L^q(\boldsymbol{R};V)}$. Here, ψ and φ_j are Littlewood-Paley functions (see §2).

We also need the definition of admissible pairs.

Definition. Let $n \ge 1$. A pair of numbers (q, r) is said to be admissible if $2 \le q, r \le \infty$ and $2/q = \delta(r) := n/2 - n/r$ with $(n, q, r) \ne (2, 2, \infty)$.

Now we can state modified Strichartz estimates by Pecher [17] as follows. The statement includes a slight improvement by Uchizono-Wada [22, 23].

Theorem A. *Let $0 < \theta_- < \theta < \theta_+ < 1$, and let (q, r) be an admissible pair with $2 < q, r < \infty$. Then the solution u to the equation*

$$\partial_t u + i\Delta u = f, \quad u(0,\cdot) = u_0 \tag{1.3}$$

satisfies the following estimates:

$$\|u\|_{L^\infty(\boldsymbol{R};H^{2\theta})} \lesssim \|u_0\|_{H^{2\theta}} + \|f\|_{B^\theta_{q',2}(\boldsymbol{R};L^{r'})} + \max_\pm \|f\|_{L^{q_*(\theta_\pm)}(\boldsymbol{R};L^{r_*(\theta_\pm)})}, \tag{1.4}$$

where $1/q_*(\theta) = (1-\theta)/q'$ and $1/r_*(\theta) = (1-\theta)/r' + \theta/2$;

$$\begin{aligned}&\|u\|_{B^\theta_{q,2}(\boldsymbol{R};L^r)\cap L^q(\boldsymbol{R};B^{2\theta}_{r,q})}\\ &\quad\lesssim \|u_0\|_{H^{2\theta}} + \|f\|_{B^\theta_{q',2}(\boldsymbol{R};L^{r'})} + \max_\pm \|f\|_{L^{\bar q(\theta_\pm)}(\boldsymbol{R};L^{\bar r(\theta_\pm)})},\end{aligned} \tag{1.5}$$

where $1/\bar q(\theta) = (1-\theta)/q' + \theta/q$ and $1/\bar r(\theta) = (1-\theta)/r' + \theta/r$.

There are several equivalent definitions of the Besov space. For simplicity, let $1 \le p < \infty$, $1 \le \alpha \le \infty$ and $0 < \theta < 1$. Firstly, we can define the Besov space $B^\theta_{q,\alpha}(\boldsymbol{R};V)$ by the Littlewood-Paley decomposition as above; we denote this space by $B_1(\boldsymbol{R};V)$ in the introduction. Secondly, we can define the Besov space by real interpolation; namely we define

$$B_2(\boldsymbol{R};V) = (L^q(\boldsymbol{R};V), W^1_q(\boldsymbol{R};V))_{\theta,\alpha}.$$

Thirdly, we can define the Besov space by finite difference; namely we define

$$B_3(\boldsymbol{R};V) = \{u \in L^q(\boldsymbol{R};V); \|u\|_{B_3(\boldsymbol{R};V)} < \infty\},$$

where $\|u\|_{B_3(\boldsymbol{R};V)} = \|u\|_{L^q(\boldsymbol{R};V)} + \left(\int_0^\infty \|u(\cdot+h)-u\|_{L^q(\boldsymbol{R};V)}^\alpha h^{-\alpha\theta-1}dh\right)^{1/\alpha}$. We have $B_1(\boldsymbol{R};V) = B_2(\boldsymbol{R};V) = B_3(\boldsymbol{R};V)$ and the norms of these spaces are mutually equivalent (see [18]).

To consider the time local theory, we need Besov spaces on intervals. Let $I \subset \boldsymbol{R}$ be an interval. We can define the Besov space $B^\theta_{q,\alpha}(I;V)$ in several ways. Firstly, we can define this space by restriction, namely we define $B_1(I;V) = B_1(\boldsymbol{R};V)/\sim$, where $u \sim v$ means $u = v$ a.e. on I. The norm on $B_1(I;V)$ is defined by

$$\|u\|_{B_1(I;V)} = \inf_{v|_I=u} \|v\|_{B_1(\boldsymbol{R};V)}.$$

Secondly, we can define $B_2(I;V) = (L^q(I;V), W^1_q(I;V))_{\theta,\alpha}$. Thirdly, we can define

$$B_3(I;V) = \{u \in L^q(I;V); \|u\|_{B_3(I;V)} < \infty\},$$

where $\|u\|_{B_3(I;V)} = \|u\|_{L^q(I;V)} + \left(\int_0^\infty \|u(\cdot+h)-u\|_{L^q(I_h;V)}^\alpha h^{-\alpha\theta-1}dh\right)^{1/\alpha}$ with $I_h = \{t \in \boldsymbol{R}; t, t+h \in I\}$. For fixed I, we again have $B_1(I;V) = B_2(I;V) = B_3(I;V)$ with equivalence of the norms (see [19]). However, it is not clear whether the norms on these spaces are *uniformly* equivalent with respect to $|I|$, namely the length of I.

To prove the time local well-posedness of (1.1)-(1.2) for large data, we should take $|I|$ small enough so that the contraction mapping principle works. Therefore it is important to observe how various constants in both linear and nonlinear estimates depend on $|I|$. In the preceding works [5, 17, 22, 23], the proof of Theorem A is based on the the restriction method and real interpolation, on the other hand the nonlinear estimates are based on the finite difference. Hence it is important to ensure the uniform equivalence of the norms in $B_i(I;V)$, $i = 1,2,3$, with respect to $|I|$.

Alternatively, we can only use restriction method, but if we take this approach, we should multiply time cut-off by the nonlinear term, so that negative powers of $|I|$ appear from time derivatives of the cut-off function, which makes it difficult for the contraction mapping principle to work.

However, in the preceding works do not seem to take this point into account. Therefore, in the present paper, we will give an alternative proof of Theorem 1.1 below, which has already appeared in [5, 17, 23], in order to ensure the time local well-posedness really holds:

Theorem 1.1. *Let $n \geq 3, 1 < s < \min\{4; n/2\}$ and $\max\{1; s/2\} < p < 1 + 4/(n-2s)$. Then for any $u_0 \in H^s$, there exists $T = T(\|u_0\|_{H^s})$ such that (1.1)-(1.2) has a unique solution u in $C([-T,T];H^s)$.*

This paper is organized as follows. In §2, we first introduce the definition of the Besov space and summarize basic properties thereof. Next we introduce Lemma 2.4, which is the key estimate in the proof of Theorem 1.1. This lemma is essentially obtained in the preceding work [16], but we modify it so that we can directly apply this estimate to our problem. In §3, we prove Theorem 1.1 when $1 < s < 2$. The proof for $2 < s < 4$ is given in §4.

2. Preliminaries

We first review the definition of Besov spaces. For the detail, we refer the reader to [2, 19]. We need Littlewood-Paley functions $\{\phi_j\}_{j=-\infty}^\infty$ on $\boldsymbol{R}$; namely, let ϕ be a function whose Fourier transform $\hat{\phi}$ is a non-negative even function which satisfies $\operatorname{supp}\hat{\phi} \subset \{\tau \in \boldsymbol{R};\, 1/2 \le |\tau| \le 2\}$ and $\sum_{j=-\infty}^{\infty} \hat{\phi}(\tau/2^j) = 1$ for $\tau \neq 0$. For $j \in \boldsymbol{Z}$, we set $\hat{\phi}_j(\cdot) = \hat{\phi}(\cdot/2^j)$ and $\hat{\psi}_j = \sum_{k=-\infty}^{j} \hat{\phi}_k$. If $j = 0$, we simply write $\psi = \psi_0$. We also need Littlewood-Paley functions on $\boldsymbol{R}^n$. For $x \in \boldsymbol{R}^n$, we define $\psi_j(x)$ and $\phi_j(x)$ by

$$\psi_j(x) = \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{ix\xi} \hat{\psi}_j(|\xi|)\, d\xi \quad \text{and} \quad \phi_j(x) = \frac{1}{(2\pi)^n} \int_{\boldsymbol{R}^n} e^{ix\xi} \hat{\phi}_j(|\xi|)\, d\xi$$

respectively. If $n = 1$, then these functions coincide with previous ones. For $s \in \boldsymbol{R}$ and $1 \le r, \alpha \le \infty$, the Besov space $B^s_{r,\alpha}(\boldsymbol{R}^n)$ is defined by

$$B^s_{r,\alpha}(\boldsymbol{R}^n) = \{u \in \mathscr{S}'(\boldsymbol{R}^n);\, \|u\|_{B^s_{r,\alpha}(\boldsymbol{R}^n)} < \infty\},$$

where $\mathscr{S}'(\boldsymbol{R}^n)$ is the space of tempered distributions on $\boldsymbol{R}^n$, and

$$\|u\|_{B^s_{r,\alpha}(\boldsymbol{R}^n)} = \|\psi *_x u\|_{L^r(\boldsymbol{R}^n)} + \begin{cases} \Big\{ \sum\limits_{j \ge 1} \big(2^{sj} \|\phi_j *_x u\|_{L^r(\boldsymbol{R}^n)}\big)^\alpha \Big\}^{1/\alpha}, & \alpha < \infty, \\ \sup\limits_{j \ge 1} 2^{sj} \|\phi_j *_x u\|_{L^r(\boldsymbol{R}^n)}, & \alpha = \infty. \end{cases}$$

Here $*_x$ denotes the convolution with respect to the variables in $\boldsymbol{R}^n$. We next prepare the Besov space of vector-valued functions. Let $\theta \in \boldsymbol{R}$, $1 \le q, \alpha \le \infty$ and V a Banach space. We put

$$B^\theta_{q,\alpha}(\boldsymbol{R}; V) = \Big\{ u \in \mathscr{S}'(\boldsymbol{R}; V);\, \|u\|_{B^\theta_{q,\alpha}(\boldsymbol{R}; V)} < \infty \Big\},$$

where

$$\|u\|_{B^\theta_{q,\alpha}(\boldsymbol{R}; V)} = \|\psi *_t u\|_{L^q(\boldsymbol{R}; V)} + \Big\{ \sum_{j \ge 1} \big(2^{\theta j} \|\varphi_j *_t u\|_{L^q(\boldsymbol{R}; V)}\big)^\alpha \Big\}^{1/\alpha} \tag{2.1}$$

with trivial modification as above if $\alpha = \infty$. Here $*_t$ denotes the convolution in $\boldsymbol{R}$. In most cases, V is a function spaces on $\boldsymbol{R}^n$ like $L^r(\boldsymbol{R}^n)$, so that $B^\theta_{q,\alpha}(\boldsymbol{R}; V) = B^\theta_{q,\alpha}(\boldsymbol{R}; L^r(\boldsymbol{R}^n))$, whose elements are regarded as functions defined on the space-time with variables $(t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n$. This is why we use the symbols $*_t$ and $*_x$. For vector-valued Besov spaces, see [1, 18].

Lemma 2.1. *Let $\theta \in \boldsymbol{R}$, $1 \le q_0, q_1 \le \infty$, $1 \le \alpha \le \infty$. Let $1/q = (1-\beta)/q_0 + \beta/q_1$ with $0 < \beta < 1$. Let V, V_0, V_1 be Banach spaces which satisfy $V_0 \cap V_1 \subset V$ and $\|u\|_V \lesssim \|u\|_{V_0}^{1-\beta} \|u\|_{V_1}^{\beta}$ for any $u \in V_0 \cap V_1$. Then we have $B^0_{q_0,\infty}(\boldsymbol{R}; V_0) \cap B^\theta_{q_1,\alpha}(\boldsymbol{R}; V_1) \subset B^{\beta\theta}_{q,\alpha/\beta}(\boldsymbol{R}; V)$ with the inequality*

$$\|u\|_{B^{\beta\theta}_{q,\alpha/\beta}(\boldsymbol{R}; V)} \lesssim \|u\|^{1-\beta}_{B^0_{q_0,\infty}(\boldsymbol{R}; V_0)} \|u\|^{\beta}_{B^\theta_{q_1,\alpha}(\boldsymbol{R}; V_1)}.$$

Proof. See Lemma 2.2 in [16]. □

Lemma 2.2. (i) *Let V be a Banach space. Let $1 \le q < p < \infty$, and $\theta = 1/q - 1/p$. Let $1 \le \alpha \le \infty$. Then $B^\theta_{q,\alpha}(\boldsymbol{R}; V) \subset L^{p,\alpha}(\boldsymbol{R}; V)$ holds with the inequality $\|u\|_{L^{p,\alpha}(\boldsymbol{R}; V)} \lesssim \|u\|_{B^\theta_{q,\alpha}(\boldsymbol{R}; V)}$. Here, $L^{p,\alpha}(\boldsymbol{R}; V)$ is the V-valued Lorentz space.*

(ii) *Let $n \ge 1$, $1 \le r < r_0 < \infty$, and $s = n/r - n/r_0$. Let $1 \le \alpha \le \infty$. Then $B^s_{r,\alpha}(\boldsymbol{R}^n) \subset L^{r_0,\alpha}(\boldsymbol{R}^n)$ holds with the inequality $\|u\|_{L^{r_0,\alpha}(\boldsymbol{R}^n)} \lesssim \|u\|_{B^s_{r,\alpha}(\boldsymbol{R}^n)}$. Especially, if $\alpha \le r_0$, then $B^s_{r,\alpha}(\boldsymbol{R}^n) \subset L^{r_0}(\boldsymbol{R}^n)$ holds with the inequality $\|u\|_{L^{r_0}(\boldsymbol{R}^n)} \lesssim \|u\|_{B^s_{r,\alpha}(\boldsymbol{R}^n)}$.*

Proof. See Lemma 2.4 in [16]. □

In what follows, we write $L^q(I;L^r) = L^q(I;L^r(\boldsymbol{R}^n))$ etc. for short. Especially, if $I = \boldsymbol{R}$, then we simply write $L^q(L^r) = L^q(\boldsymbol{R};L^r)$.

Lemma 2.3. *Let $f = i\lambda|u|^{p-1}u$ with $\lambda \in \boldsymbol{C}$, $p > 1$. Let $q, q_0, q_1, r, r_0, r_1, \alpha \in [1,\infty]$ with $1/q = (p-1)/q_0 + 1/q_1$, $1/r = (p-1)/r_0 + 1/r_1$. Let $0 < \theta < p$. Then $f(u) \in B^\theta_{q,\alpha}(L^r)$ for any $u \in L^{q_0}(L^{r_0}) \cap B^\theta_{q_1,\alpha}(L^{r_1})$ with the following estimate:*

$$\|f(u)\|_{B^\theta_{q,\alpha}(L^r)} \le C\|u\|^{p-1}_{L^{q_0}(L^{r_0})}\|u\|_{B^\theta_{q_1,\alpha}(L^{r_1})}.$$

Proof. The proof is essentially the same as that of Claim 4.3 in [16]. □

Lemma 2.4. *Let $0 < \theta < 1$. Let (q, r) and (γ, ρ) be admissible pairs. Let $(\bar{q}, \bar{r})$ be a pair satisfying $1 < \bar{q}, \bar{r} < \infty$ and $0 < 2/\bar{q} - \delta(\bar{r}) \le 2(1-\theta)$. Then the solution u to (1.3) satisfies the estimate*

$$\|u\|_{L^\infty(H^{2\theta})\cap B^\theta_{q,2}(L^r)} \le C\|u_0\|_{H^{2\theta}} + C\|f\|_{B^\theta_{\gamma',2}(L^{\rho'})} + C\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))}. \tag{2.2}$$

Here, $\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))} = \|\psi *_x f\|_{L^{\bar{q}}(L^{\bar{r}})} + \{\sum_{k=1}^\infty \|\phi_k *_x f\|^2_{L^{\bar{q}}(L^{\bar{r}})}\}^{1/2}$. Moreover, in the right-hand side of (2.2),

(i) *$\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))}$ can be removed if $\delta(\rho) < 2(1-\theta)$;*
(ii) *$\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))}$ can be replaced with $\|f\|_{B^0_{\bar{q},\infty}(L^{\bar{r}})}$ if $0 < 2/\bar{q} - \delta(\bar{r}) < 2(1-\theta)$;*
(iii) *$\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))}$ can be replaced with $\|f\|_{l^2(L^{\bar{q},\infty}(L^{\bar{r}}))}$ if $\bar{r} \le 2$;*
(iv) *$\|f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))}$ can be replaced with $\|f\|_{l^2(B^0_{\bar{q},\infty}(L^{\bar{r}}))}$ if $\rho' < \bar{r}$.*

Remark. (i) Actually, we can show that $u \in C(\boldsymbol{R};H^{2\theta})$. To prove this, let $\{f_k\}_{k=1}^\infty \subset \mathscr{S}(\boldsymbol{R}^{1+n})$ with $\|f_k - f\|_{B^\theta_{\gamma',2}(L^{\rho'})} + \|f_k - f\|_{l^2(L^{\bar{q}}(L^{\bar{r}}))} \to 0$. Let u_k be the solution to (1.3) with f replaced by f_k. Then $u_k \in C(\boldsymbol{R};H^{2\theta})$. By (2.2), we see that $\|u_k - u\|_{L^\infty(\boldsymbol{R};H^{2\theta})} \to 0$. Therefore $u \in C(\boldsymbol{R};H^{2\theta})$.

(ii) If $2/\bar{q} - \delta(\bar{r}) = 2(1-\theta)$, then the homogeneous counterpart of (2.2) holds. Namely, we have

$$\|u\|_{L^\infty(\dot{H}^{2\theta})\cap \dot{B}^\theta_{q,2}(L^r)} \le C\|u_0\|_{\dot{H}^{2\theta}} + C\|f\|_{\dot{B}^\theta_{\gamma',2}(L^{\rho'})} + C\|f\|_{\dot{l}^2(L^{\bar{q}}(L^{\bar{r}}))}, \tag{2.3}$$

where $\|u\|_{\dot{H}^{2\theta}} = \|(-\Delta)^\theta u\|_{L^2}$, $\|u\|_{\dot{B}^\theta_{q,2}(L^r)} = \Big(\sum_{j=-\infty}^\infty \|\phi_j *_t u\|^2_{L^q(L^r)}\Big)^{1/2}$ and

$$\|f\|_{\dot{l}^2(L^{\bar{q}}(L^{\bar{r}}))} = \Big(\sum_{k=-\infty}^{\infty} \|\phi_k *_x f\|^2_{L^{\bar{q}}(L^{\bar{r}})}\Big)^{1/2}.$$

Moreover, the modifications (i), (iii) and (iv) hold with l^2 and $B^0_{\bar{q},\infty}$ replaced by $\dot{l}^2$ and $\dot{B}^0_{\bar{q},\infty}$ respectively.

Proof. Since the homogeneous estimate has already been proved by Pecher [17], we may assume $u_0 = 0$. The inequality (2.2) has been proved in [16] when $2/\bar{q} - \delta(\bar{r}) = 2(1-\theta)$, so we only consider the case (i)-(iv) in the statement of the lemma. By the Fourier transform,

$$\hat{u}(t,\xi) = \int_0^t e^{i(t-t')|\xi|^2}\hat{f}(t',\xi)dt' = \int_{-\infty}^\infty \frac{e^{it\tau} - e^{it|\xi|^2}}{2\pi i(\tau - |\xi|^2)}\tilde{f}(\tau,\xi)\,d\tau.$$

Here, $\tilde{f}(\tau,\xi)$ denotes the Fourier transform of f in the space-time, whereas $\hat{f}(t,\xi)$ is the Fourier transform with respect to the spatial variables. We define v and v_0 by

$$\hat{v}(t,\xi) = \text{p.v.-}\int_{-\infty}^{\infty} e^{it\tau} \frac{\tilde{f}(\tau,\xi)}{2\pi i(\tau-|\xi|^2)}\, d\tau \tag{2.4}$$

and $\hat{v}_0(\xi) = \hat{v}(0,\xi)$ respectively, so that $\hat{u} = \hat{v} - e^{it|\xi|^2}\hat{v}_0$. Using the formula p.v.-$\int_{-\infty}^{\infty}\{(\tau-|\xi|^2)\}^{-1}e^{it\tau}\,d\tau = \pi i\,\mathrm{sign}(t)e^{it|\xi|^2}$, we can write

$$\hat{v}(t,\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(t-t')e^{i(t-t')|\xi|^2}\hat{f}(t',\xi)dt',$$

and hence

$$u(t) = v(t) - U(t)v_0 = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(t-t')U(t-t')f(t')dt' - U(t)v_0.$$

By the formula $\phi_j *_t e^{it|\xi|^2} = e^{it|\xi|^2}\hat{\phi}_j(|\xi|^2)$, we have $\phi_j *_t U(t)v_0 = U(t)\phi_{j/2} *_x v_0$. Here $\phi_{j/2}*_x = \hat{\phi}_j(-\Delta) = \mathscr{F}_\xi^{-1}\hat{\phi}_j(|\xi|^2)\mathscr{F}_x$. This is an abuse of symbol, but no confusion is likely to arise. This notation matches the equivalence $\|u\|_{B^s_{r,2}} \sim \|\hat{\psi}(-\Delta)u\|_{L^r} + \big(\sum_{j=1}^{\infty}(2^{sj/2}\|\phi_{j/2}*_x u\|_{L^r})^2\big)^{1/2}$, see Lemma 2.3 in [16]. The Strichartz estimate shows

$$\begin{aligned}\|\phi_j *_t U(t)v_0\|_{L^q(L^r)} &= \|U(t)\phi_{j/2} *_x v_0\|_{L^q(L^r)} \lesssim \|\phi_{j/2} *_x v_0\|_{L^2} \\ &\le \|\phi_{j/2} *_x v\|_{L^\infty(L^2)}.\end{aligned}$$

Here, in the last inequality we have used $v_0 = v(0)$. Therefore we obtain

$$\|\phi_j *_t u\|_{L^q(L^r)} + \|\phi_{j/2} *_x u\|_{L^\infty(L^2)} \lesssim \|\phi_j *_t v\|_{L^q(L^r)} + \|\phi_{j/2} *_x v\|_{L^\infty(L^2)}. \tag{2.5}$$

By the Strichartz estimate together with the commutative law for the convolution, we obtain $\|\phi_j *_t v\|_{L^q(L^r)} \lesssim \|\phi_j *_t f\|_{L^{\gamma'}(L^{\rho'})}$. We next estimate $\|\phi_{j/2} *_x v\|_{L^\infty(L^2)}$. We put $\chi_j = \sum_{k=j-2}^{j+2}\phi_j$ and

$$\begin{aligned}K_j(t,x) &= \frac{1}{(2\pi)^{1+n}}\iint_{\boldsymbol{R}^{1+n}} e^{it\tau+ix\xi}\,\frac{\hat{\phi}_j(|\xi|^2)(1-\hat{\chi}_j(\tau))}{i(\tau-|\xi|^2)}\,d\tau d\xi \\ &= 2^{nj/2}K_0(2^j t, 2^{j/2}x).\end{aligned}$$

The second equality is easily shown by the change of variable. For any positive integer m, we can show the estimate $|K_0(t,x)| \lesssim (1+|t|+|x|)^{-m}$, which will be shown at the end of the proof. We multiply $\hat{\phi}_j(|\xi|^2) = \hat{\chi}_j(|\xi|^2)\hat{\phi}_j(|\xi|^2)$ by the both-sides of (2.4) and decompose as

$$\begin{aligned}\hat{\phi}_j(|\xi|^2)\hat{v}(t,\xi) = \text{p.v.-}&\int_{-\infty}^{\infty} e^{it\tau}\frac{\hat{\phi}_j(|\xi|^2)\hat{\chi}_j(\tau)}{i(\tau-|\xi|^2)}\tilde{f}(\tau,\xi)d\tau \\ &+ \int_{-\infty}^{\infty} e^{it\tau}\frac{\hat{\chi}_j(|\xi|^2)\hat{\phi}_j(|\xi|^2)(1-\hat{\chi}_j(\tau))}{2\pi i(\tau-|\xi|^2)}\tilde{f}(\tau,\xi)d\tau,\end{aligned}$$

or equivalently

$$\begin{aligned}\phi_{j/2} *_x v(t) = \frac{1}{2}&\int_{-\infty}^{\infty} \mathrm{sign}(t-t')U(t-t')\phi_{j/2} *_x \chi_j *_t f(t')dt' \\ &+ (K_j *_{t,x} \chi_{j/2} *_x f)(t).\end{aligned}$$

By the Strichartz estimate, we have

$$\|\phi_{j/2} *_x v\|_{L^\infty(L^2)} \lesssim \|\chi_j *_t f\|_{L^{\gamma'}(L^{\rho'})} + \|K_j *_{t,x} \chi_{j/2} *_x f\|_{L^\infty(L^2)}. \tag{2.6}$$

For the low frequency parts $\psi *_t u$ and $\hat{\psi}(-\Delta)u$, we have the trivial estimate

$$\|\psi *_t u\|_{L^q(L^r)} + \|\hat{\psi}(-\Delta)u\|_{L^q(L^r)} \lesssim \|u\|_{L^q(L^r)} \lesssim \|f\|_{L^{\gamma'}(L^{\rho'})}.$$

Hence, we obtain from (2.5)-(2.6) that

$$\|u\|_{L^\infty(H^{2\theta})\cap B^\theta_{q,2}(L^r)} \lesssim \|f\|_{B^\theta_{\gamma',2}(L^{\rho'})} + J, \tag{2.7}$$

where $J = \Big(\sum_{j=1}^\infty (2^{\theta j}\|K_j *_{t,x} \chi_{j/2} *_x f\|_{L^\infty(L^2)})^2\Big)^{1/2}$. Therefore, it suffices to estimate J in the following cases (i)-(iv).

Case (i). Let $\delta(\rho) < 2(1-\theta)$. We define (q_0, r_0) by $1/q_0 = 1/\gamma' - \theta$, $r_0 = \rho'$. By assumption, $2/q_0 = 2(1-\theta) - \delta(\rho) > 0$. We also define $(\tilde{q}_0, \tilde{r}_0)$ by $\tilde{q}_0 = q_0'$, $1/\tilde{r}_0 = 3/2 - 1/r_0$. Then by the change of variables, we see

$$\|K_j\|_{L^{\tilde{q}_0,1}(L^{\tilde{r}_0})} = 2^{(n/2-1/\tilde{q}_0-n/2\tilde{r}_0)j}\|K_0\|_{L^{\tilde{q}_0,1}(L^{\tilde{r}_0})} \lesssim 2^{-\theta j},$$

since $n/2 - 1/\tilde{q}_0 - n/2\tilde{r}_0 = 1/q_0 - \delta(r_0)/2 - 1 = -\theta$. By the Young inequality and the Hölder type inequality for the Lorentz space together with Lemma 2.2, we have

$$\begin{aligned} J &\lesssim \Big(\sum_{j=1}^\infty (2^{\theta j}\|K_j\|_{L^{\tilde{q}_0,1}(L^{\tilde{r}_0})}\|\chi_{j/2} *_x f\|_{L^{q_0,\infty}(L^{r_0})})^2\Big)^{1/2} \\ &\lesssim \Big(\sum_{j=1}^\infty \|\chi_{j/2} *_x f\|^2_{L^{q_0,\infty}(L^{r_0})}\Big)^{1/2} \lesssim \Big(\sum_{j=1}^\infty \|\chi_{j/2} *_x f\|^2_{B^\theta_{\gamma',2}(L^{\rho'})}\Big)^{1/2} \\ &\lesssim \|\psi *_x f\|_{L^{\gamma'}(L^{\rho'})} + \Big(\sum_{j=1}^\infty \|\phi_j *_x f\|^2_{B^\theta_{\gamma',2}(L^{\rho'})}\Big)^{1/2} = \|f\|_{l^2(B^\theta_{\gamma',2}(L^{\rho'}))}. \end{aligned}$$

Since $\gamma', \rho' \le 2$, we have $l^2(B^\theta_{\gamma',2}(L^{\rho'})) \supset B^\theta_{\gamma',2}(l^2(L^{\rho'})) = B^\theta_{\gamma',2}(B^0_{\rho',2}) \supset B^\theta_{\gamma',2}(L^{\rho'})$. Therefore, we obtain $J \lesssim \|f\|_{B^\theta_{\gamma',2}(L^{\rho'})}$, and hence we can drop J from the right-hand side of (2.7).

In what follows we assume $\delta(\rho) \ge 2(1-\theta)$; if not, the desired (actually better) result follows from Case (i).

Case (ii). Let $0 < 2/\bar{q} - \delta(\bar{r}) < 2(1-\theta)$. Then we see $\delta(\rho') \le -2(1-\theta) < \delta(\bar{r})$. For $0 < \beta < 1$, we define (q_0, r_0) by $1/r_0 = (1-\beta)/\bar{r} + \beta/\rho'$ and $1/q_0 = (1-\beta)/\bar{q} + \beta(1/\gamma' - \theta)$, or equivalently $\delta(r_0) = (1-\beta)\delta(\bar{r}) + \beta\delta(\rho')$ and $2/q_0 = (1-\beta)(2/\bar{q} - \delta(\bar{r})) + 2\beta(1-\theta) + \delta(r_0)$. We can choose β satisfying $-2(1-\theta) < \delta(r_0) < \min\{0, \delta(\bar{r})\}$ and $0 < 1/q_0 < 1$. Indeed, if $\bar{r} \le 2$, then we choose $\beta \sim 0$ so that $\delta(r_0) \sim \delta(\bar{r}) > 2(1-\theta)$ and that $1/q_0 \sim 1/\bar{q} > 0$; if $\bar{r} > 2$, then we choose β so that $r_0 \sim 2$, and for such β we have $2/q_0 \sim (1-\beta)(2/\bar{q} - \delta(\bar{r})) + 2\beta(1-\theta) > 0$. In both cases, we see $1/q_0 < (1-\beta)/\bar{q} + \beta/\gamma' < 1$. Moreover we can easily check that $0 < 2/q_0 - \delta(r_0) < 2(1-\theta)$. We define $(\tilde{q}_0, \tilde{r}_0)$ by $\tilde{q}_0 = q_0'$, $1/\tilde{r}_0 = 3/2 - 1/r_0$ as in the previous case. Since $\|K_j\|_{L^{\tilde{q}_0,1}(L^{\tilde{r}_0})} = C\,2^{-(\theta+\epsilon)j}$ with

$$-\epsilon = n/2 - 1/\tilde{q}_0 - n/2\tilde{r}_0 + \theta = 1/q_0 - \delta(r_0)/2 - 1 + \theta < 0,$$

we obtain

$$
\begin{aligned}
J &\lesssim \Big(\sum_{j=1}^{\infty}(2^{\theta j}\|K_j\|_{L^{\tilde q_0,1}(L^{\tilde r_0})}\|\chi_{j/2}*_x f\|_{L^{q_0,\infty}(L^{r_0})})^2\Big)^{1/2}\\
&\lesssim \Big(\sum_{j=1}^{\infty}2^{-\epsilon j}\|\chi_{j/2}*_x f\|^2_{L^{q_0,\infty}(L^{r_0})}\Big)^{1/2} \lesssim \sup_{j\ge 1}\|\chi_{j/2}*_x f\|_{L^{q_0,\infty}(L^{r_0})}.
\end{aligned}
$$

Let $1/\bar\gamma = 1/q_0 + \beta\theta = (1-\beta)/\bar q + \beta/\gamma'$. By Lemmas 2.1 and 2.2, we have $L^{q_0,\infty}(L^{r_0}) \supset B^{\beta\theta}_{\bar\gamma,\infty}(L^{r_0}) \supset B^0_{\bar q,\infty}(L^{\bar r}) \cap B^{\theta}_{\gamma',2}(L^{\rho'})$. From this we obtain the following estimates, thereby obtaining the claim:

$$
\begin{aligned}
J &\lesssim \sup_{j\ge 1}\{\|\chi_{j/2}*_x f\|_{B^0_{\bar q,\infty}(L^{\bar r})} + \|\chi_{j/2}*_x f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})}\}\\
&\lesssim \|f\|_{B^0_{\bar q,\infty}(L^{\bar r})} + \|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})}.
\end{aligned}
\tag{2.8}
$$

Case (iii). Let $\bar r \le 2$. We choose $(q_0, r_0) = (\bar q, \bar r)$ and define $(\tilde q_0, \tilde r_0)$ by $\tilde q_0 = q_0'$, $1/\tilde r_0 = 3/2 - 1/r_0$. Then, similarly as in Case (i), we see $J \lesssim \|f\|_{l^2(L^{q_0,\infty}(L^{r_0}))}$ since $n/2 - 1/\tilde q_0 - n/2\tilde r_0 = 1/\bar q - \delta(\bar r)/2 - 1 \le -\theta$.

Case (iv). Let $\rho' < \bar r$. We may assume $2/\bar q - \delta(\bar r) = 2(1-\theta)$, since (2.8) gives a better estimate if $2/\bar q - \delta(\bar r) < 2(1-\theta)$. For $0 < \beta < 1$, we define (q_0, r_0), $(\tilde q_0, \tilde r_0)$ and $\bar\gamma$ as in Case (ii). We can choose β such that $1 < q_0 < \infty$ and $\rho' < r_0 < \min\{2; \bar r\}$. Then as in Case (ii), we obtain $J \lesssim \|f\|_{l^2(L^{q_0,\infty}(L^{r_0}))}$. By Lemmas 2.1 and 2.2, we have $L^{q_0,\infty}(L^{r_0}) \supset B^{\beta\theta}_{\bar\gamma,\infty}(L^{r_0}) \supset B^0_{\bar q,\infty}(L^{\bar r}) \cap B^{\theta}_{\gamma',2}(L^{\rho'})$. This implies $J \lesssim \|f\|_{l^2(B^0_{\bar q,\infty}(L^{\bar r}))} + \|f\|_{B^{\theta}_{\gamma',2}(L^{\rho'})}$, since $l^2(B^{\theta}_{\gamma',2}(L^{\rho'})) \supset B^{\theta}_{\gamma',2}(L^{\rho'})$.

Finally we show $K_0(t,x) \lesssim (1+|t|+|x|)^{-m}$. Indeed, on the support of the integrand of K_0, we have $|\tau| \notin [1/4, 4]$ and $|\xi|^2 \in [1/2, 2]$, so that $|\tau - |\xi|^2| \ge 1/4$. Therefore $\int_{|\tau-|\xi|^2|\le 6} e^{it\tau}\,\hat\phi_0(|\xi|^2)(1-\hat\chi_0(\tau))(\tau-|\xi|^2)^{-1}\,d\tau$ is bounded. On the other hand, we see $\hat\chi_0(\tau) = 0$ when $|\tau - |\xi|^2| \ge 6$, and hence

$$
\begin{aligned}
\int_{|\tau-|\xi|^2|\ge 6} e^{it\tau}\,\frac{\hat\phi_0(|\xi|^2)(1-\hat\chi_0(\tau))}{i(\tau-|\xi|^2)}\,d\tau &= \int_{|\tau-|\xi|^2|\ge 6} e^{it\tau}\,\frac{\hat\phi_0(|\xi|^2)}{i(\tau-|\xi|^2)}\,d\tau\\
&= 2\,\mathrm{sign}(t)e^{it|\xi|^2}\,\hat\phi_0(|\xi|^2)\int_{6/|t|}^{\infty}\frac{\sin\tau}{\tau}\,d\tau.
\end{aligned}
$$

This is also bounded, and hence we have proved the boundedness of $K_0(t,x)$. Moreover, for $1 \le l \le n$, the integration by parts shows

$$
\begin{aligned}
x_l K_0(t,x) &= \frac{1}{(2\pi)^{1+n}}\iint_{\boldsymbol{R}^{1+n}} e^{it\tau+ix\xi}\,\frac{\partial}{\partial\xi_l}\,\frac{\hat\phi_0(|\xi|^2)(1-\hat\chi_0(\tau))}{\tau-|\xi|^2}\,d\tau d\xi\\
&= \frac{1}{(2\pi)^{1+n}}\iint_{\boldsymbol{R}^{1+n}} e^{it\tau+ix\xi}\,2\xi_l\Big\{\frac{\hat\phi_0'(|\xi|^2)(1-\hat\chi_0(\tau))}{\tau-|\xi|^2}\\
&\qquad + \frac{\hat\phi_0(|\xi|^2)(1-\hat\chi_0(\tau))}{(\tau-|\xi|^2)^2}\Big\}\,d\tau d\xi,
\end{aligned}
$$

and the right-hand side is bounded as above. Repeating this, we can obtain the desired estimate. □

3. Proof of Theorem 1.1—case $s < 2$

We divide the proof of the main theorem into the cases (i) $1 < s < 2$ and (ii) $2 < s < 4$. The case $s = 2$ has already been treated by Tsutsumi [20], Kato [11,12], and Cazenave-Weissler [4]. We may assume $1 < p \le s$ because the case $p > s$ has already treated by Cazenave-Weissler [4].

In this section we consider the case where $1 < s < 2$. We put $\kappa = 1-(n-2s)(p-1)/4$. By assumption, we have $0 < \kappa < 1$. Let (q, r) and (γ, ρ) be admissible pairs which satisfy $\max\{1 - 2\kappa; 0\} < \delta(r) < \min\{2(1 - \kappa); 1\}$ and $\delta(r) + \delta(\rho) = 2(1 - \kappa)$. Let $1/q_0 = 1/\gamma' - 1/q = \kappa$. We put $X^s = L^\infty(\boldsymbol{R}; H^s) \cap B^{s/2}_{q,2}(\boldsymbol{R}; L^r)$. For an interval $I \subset \boldsymbol{R}$, we set $X^s(I) = X^s/\sim$, where $u \sim v$ means $u = v$ a.e. on I. For each equivalence class $[u] \in X^s(I)$, its norm is defined by $\|[u]\|_{X^s(I)} = \inf\{\|v\|_{X^s}; v \sim u\}$. For $R > 0$, we put $B_R = \{[u] \in X^s(I); \|[u]\|_{X^s(I)} \le R\}$. For $[u], [v] \in B_R$, we define the metric $d([u], [v]) = \|u - v\|_{L^\infty(I;L^2)\cap L^q(I;L^r)}$.

Lemma 3.1. *(B_R, d) is a complete metric space.*

Proof. ices to show that B_R is closed in $L^\infty(I; L^2) \cap L^q(I; L^r)$. To this end, we shall show that if $\{[u_k]\}_{k=1}^\infty$ is a sequence in B_R with $d([u_k], [u_\infty]) \to 0$, then $[u_\infty] \in B_R$. Let ϵ be an arbitrary positive number. We may assume that $\|u_k\|_{X^s} \le R+\epsilon$, so that there is a subsequence $\{u_{k(l)}\}_{l=1}^\infty$ which converges $*$-weakly in $L^\infty(\boldsymbol{R}; H^s)$. We put $u_* = \text{w}^*\text{-}\lim_{l\to\infty} u_{k(l)} \in L^\infty(\boldsymbol{R}; H^s)$. Since the sequence $\{u_{k(l)}\}_{l=1}^\infty$ is bounded in $B^{s/2}_{q,2}(\boldsymbol{R}; L^r)$ and converges to u_* in $\mathscr{S}'(\boldsymbol{R}^{1+n})$, we can easily show that $\{u_{k(l)}\}_{l=1}^\infty$ weakly converges to u_* in $B^{s/2}_{q,2}(\boldsymbol{R}; L^r)$. Hence we have $u_* \in X^s$ and

$$\|u_*\|_{X^s} \le \varliminf_{l\to\infty} \|u_{k(l)}\|_{X^s} \le R + \epsilon.$$

On the other hand, $\{u_{k(l)}\}_{l=1}^\infty$ converges to u_∞ in $L^\infty(I; L^2) \cap L^q(I; L^r)$. Therefore, $u_*(t)$ must coincide with $u_\infty(t)$ a.e. t on I, which implies $[u_*] = [u_\infty]$, so that $\|[u_\infty]\|_{X^s(I)} \le \|u_*\|_{X^s} \le R + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\|[u_\infty]\|_{X^s(I)} \le R$, namely $[u_\infty] \in B_R$. □

We take $0 < T \le 1$ to be determined later and put $I = [-T, T]$. Let $\zeta \in C_0^\infty(\boldsymbol{R})$ be a function which satisfies $\zeta(t) = 1$ if $|t| \le 1$ and $\zeta(t) = 0$ if $|t| \ge 2$. We put $\zeta_T(t) = \zeta(t/T)$. We define the operator Φ by

$$\{\Phi(u)\}(t) = U(t)u_0 + \{U \otimes f(u)\}(t),$$

where $(U \otimes g)(t) = \int_0^t U(t - t')g(t')\,dt'$. With suitable choices of R and T, we show that Φ is a contraction mapping on (B_R, d). We similarly define Φ_T by

$$\{\Phi_T(u)\}(t) = U(t)u_0 + \{U \otimes \zeta_T f(u)\}(t).$$

Clearly, $\{\Phi_T(u)\}(t) = \{\Phi(u)\}(t)$ for $t \in I$, and hence $\|[\Phi(u)]\|_{X^s(I)} \le \|\Phi_T(u)\|_{X^s}$. By the paraproduct formula,

$$\begin{aligned}\zeta_T f &= (\psi_4 *_t \zeta_T)\cdot(\psi_2 *_t f) + \sum_{k=3}^\infty (\psi_{k-3} *_t \zeta_T)\cdot(\phi_k *_t f)\\ &\quad + \sum_{k=5}^\infty (\phi_k *_t \zeta_T)\cdot(\psi_{k-3} *_t f) + \sum_{k=3}^\infty (\chi_k *_t \zeta_T)\cdot(\phi_k *_t f)\\ &\equiv (\zeta_T f)_{\rm LL} + (\zeta_T f)_{\rm LH} + (\zeta_T f)_{\rm HL} + (\zeta_T f)_{\rm HH}.\end{aligned} \tag{3.1}$$

Here, we recall the notation $\chi_k = \sum_{j=k-2}^{k+2} \phi_j$. In what follows, $\hat{f}$ denotes the Fourier transform of f with respect to t. Clearly, we have

$$\mathscr{F}_t\{\phi_j *_t (\zeta_T f)_{\mathrm{LH}}\}(\tau) = \sum_{k=3}^{\infty} \int_{-\infty}^{\infty} \hat{\phi}_j(\tau)\hat{\psi}_{k-3}(\tau-\tau')\hat{\phi}_k(\tau')\hat{\zeta}_T(\tau-\tau')\hat{f}(\tau')\,d\tau'.$$

If $\hat{\phi}_j(\tau)\hat{\psi}_{k-3}(\tau-\tau')\hat{\phi}_k(\tau') \neq 0$, then roughly speaking we have $|\tau| \sim |\tau'| \gtrsim |\tau-\tau'|$. More precisely, we have $|\tau-\tau'| < 2^{k-2}$, $2^{k-1} < |\tau'| < 2^{k+1}$ and $2^{j-1} < |\tau| < 2^{j+1}$. From the first two inequalities, we have $2^{k-2} < |\tau| < 2^{k+2}$. This inequality, combined with the third one, implies $j-2 \leq k \leq j+2$. Therefore we obtain

$$\phi_j *_t (\zeta_T f)_{\mathrm{LH}} = \sum_{k=(j-2)\vee 3}^{j+2} \phi_j *_t \{(\psi_{k-3} *_t \zeta_T)\cdot(\phi_k *_t f)\}.$$

We can similarly obtain

$$\phi_j *_t (\zeta_T f)_{\mathrm{HL}} = \sum_{k=(j-2)\vee 5}^{j+2} \phi_j *_t \{(\phi_k *_t \zeta_T)\cdot(\psi_{k-3} *_t f)\},$$

$$\phi_j *_t (\zeta_T f)_{\mathrm{HH}} = \sum_{k=(j-4)\vee 3}^{\infty} \phi_j *_t \{(\chi_k *_t \zeta_T)\cdot(\phi_k *_t f)\}.$$

We also have $\psi *_t (\zeta_T f)_{\mathrm{LH}} = \psi *_t (\zeta_T f)_{\mathrm{HL}} = 0$.

Let (γ_0, ρ_0) be another admissible pair with $\max\{2(1-\kappa)-s; 0\} < \delta(\rho_0) < \min\{2(1-\kappa); 2-s\}$. We put $\nu_0 = 2n/(n-2s)$, so that we have the embedding $L^{\nu_0} \supset H^s$ by the Sobolev inequality. We also put $\bar{r} = \nu_0/p$, so that $\delta(\bar{r}) = s-2(1-\kappa)$. We choose $\bar{q}$ such that $1/\bar{q} = \kappa - \epsilon$ with sufficiently small $\epsilon > 0$, so that $0 < 2/\bar{q} - \delta(\bar{\delta}) < 2-s$. Therefore, from Lemma 2.4 (i)-(ii), we obtain

$$\begin{aligned}\|\Phi_T(u)\|_{X^s} \lesssim{}& \|u_0\|_{H^s} + \|(\zeta_T f)_{\mathrm{LH}}\|_{B^{s/2}_{\gamma',2}(L^{\rho'})} + \|(\zeta_T f)_{\mathrm{LH}}\|_{B^0_{\bar{q},\infty}(L^{\bar{r}})} \\ &+ \|(\zeta_T f)_{\mathrm{LL}} + (\zeta_T f)_{\mathrm{HL}} + (\zeta_T f)_{\mathrm{HH}}\|_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})}.\end{aligned}$$

Apart from LH part, for LL, HL and HH parts we do not need additional space like $B^0_{\bar{q},\infty}(L^{\bar{r}})$, since $\delta(\rho_0) < 2-s$. We begin with the estimate of $(\zeta_T f)_{\mathrm{LH}}$. By the Hölder and the Young inequalities, we have

$$\|\phi_j *_t (\zeta_T f)_{\mathrm{LH}}\|_{L^{\gamma'}(L^{\rho'})} \leq \sum_{k=(j-2)\vee 3}^{j+2} \|\zeta_T\|_{L^{q_0}} \|\phi_k *_t f\|_{L^q(L^{\rho'})},$$

and hence

$$\|(\zeta_T f)_{\mathrm{LH}}\|_{B^{s/2}_{\gamma',2}(L^{\rho'})} \lesssim \|\zeta_T\|_{L^{q_0}} \|f\|_{B^{s/2}_{q,2}(L^{\rho'})} \lesssim T^\kappa \|f\|_{B^{s/2}_{q,2}(L^{\rho'})}.$$

Since $1/\rho' = (p-1)/\nu_0 + 1/r$, Lemma 2.3 shows

$$\|(\zeta_T f)_{\mathrm{LH}}\|_{B^{s/2}_{\gamma',2}(L^{\rho'})} \lesssim T^\kappa \|u\|^{p-1}_{L^\infty(L^{\nu_0})} \|u\|_{B^{s/2}_{q,2}(L^r)} \lesssim T^\kappa \|u\|^{p-1}_{L^\infty(H^s)} \|u\|_{B^{s/2}_{q,2}(L^r)}. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}\|(\zeta_T f)_{\mathrm{LH}}\|_{B^0_{\bar{q},\infty}(L^{\bar{r}})} &\lesssim \|\zeta_T\|_{L^{\bar{q}}} \|f\|_{L^\infty(L^{\bar{r}})} = |\lambda| \|\zeta_T\|_{L^{\bar{q}}} \|u\|^p_{L^\infty(L^{\nu_0})} \\ &\lesssim T^{\kappa-\epsilon} \|u\|^p_{L^\infty(H^s)}\end{aligned} \tag{3.3}$$

by the equality $\|\zeta_T\|_{L^{\bar q}} = CT^{\kappa-\epsilon}$ and the inclusion $L^{\nu_0} \supset H^s$. In the same way, we obtain

$$\|(\zeta_T f)_{\mathrm{HL}}\|_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} \lesssim \|\zeta_T\|_{B^{s/2}_{\gamma_0',2}} \|f\|_{L^\infty(L^{\rho_0'})},$$
$$\|(\zeta_T f)_{\mathrm{LL}}\|_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} \lesssim \|\zeta_T\|_{L^{\gamma_0'}} \|f\|_{L^\infty(L^{\rho_0'})}.$$

For HH part, we have

$$\begin{aligned}\|(\zeta_T f)_{\mathrm{HH}}\|^2_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} &\lesssim \sum_{l=1}^{\infty} 2^{sl} \Big(\sum_{k=l}^{\infty} \|(\chi_k *_t \zeta_T)\cdot(\phi_k *_t f)\|_{L^{\gamma_0'}(L^{\rho_0'})}\Big)^2 \\ &\lesssim \|f\|^2_{L^\infty(L^{\rho_0'})} \sum_{l=1}^{\infty} 2^{sl} \Big(\sum_{k=l}^{\infty} \|\chi_k *_t \zeta_T\|_{L^{\gamma_0'}}\Big)^2.\end{aligned}$$

By the Schwarz inequality, we see

$$\begin{aligned}\Big(\sum_{k=l}^{\infty} \|\chi_k *_t \zeta_T\|_{L^{\gamma_0'}}\Big)^2 &\le \sum_{k=l}^{\infty} 2^{-\epsilon k} \sum_{k=l}^{\infty} 2^{\epsilon k} \|\chi_k *_t \zeta_T\|^2_{L^{\gamma_0'}} \\ &= C 2^{-\epsilon l} \sum_{k=l}^{\infty} 2^{\epsilon k} \|\chi_k *_t \zeta_T\|^2_{L^{\gamma_0'}},\end{aligned}$$

so that

$$\begin{aligned}\|(\zeta_T f)_{\mathrm{HH}}\|^2_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} &\lesssim \|f\|^2_{L^\infty(L^{\rho_0'})} \sum_{l=1}^{\infty} 2^{(s-\epsilon)l} \sum_{k=l}^{\infty} 2^{\epsilon k} \|\chi_k *_t \zeta_T\|^2_{L^{\gamma_0'}} \\ &= \|f\|^2_{L^\infty(L^{\rho_0'})} \sum_{k=1}^{\infty} 2^{\epsilon k} \|\chi_k *_t \zeta_T\|^2_{L^{\gamma_0'}} \sum_{l=1}^{k} 2^{(s-\epsilon)l} \\ &= C \|f\|^2_{L^\infty(L^{\rho_0'})} \sum_{k=1}^{\infty} 2^{sk} \|\chi_k *_t \zeta_T\|^2_{L^{\gamma_0'}} \\ &\lesssim \|\zeta_T\|^2_{B^{s/2}_{\gamma_0',2}} \|f\|^2_{L^\infty(L^{\rho_0'})}.\end{aligned}$$

Collecting these estimates, we obtain

$$\|(\zeta_T f)_{\mathrm{LL}} + (\zeta_T f)_{\mathrm{HL}} + (\zeta_T f)_{\mathrm{HH}}\|_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} \lesssim \|\zeta_T\|_{B^{s/2}_{\gamma_0',2}} \|f\|_{L^\infty(L^{\rho_0'})}.$$

We put $\kappa_0 = 1/\gamma_0' - s/2 = (2 - \delta(\rho_0) - s)/2$. By assumption, we have $0 < \kappa_0 < \kappa$. By the definition of the Besov space, for $0 < T \le 1$, we can show $\|\zeta_T\|_{B^{s/2}_{\gamma_0',2}} \lesssim T^{\kappa_0}$. On the other hand, we define ν_1 by $1/\rho_0' = (p-1)/\nu_0 + 1/\nu_1$, or equivalently $\delta(\nu_1) = 2(1-\kappa) - \delta(\rho_0)$. Since $0 < \delta(\nu_1) < s$, we have the embedding $L^{\nu_1} \supset H^s$. Therefore, we see $\|f\|_{L^{\rho_0'}} \lesssim \|u\|^{p-1}_{L^{\nu_0}} \|u\|_{L^{\nu_1}} \lesssim \|u\|^p_{H^s}$, and consequently

$$\|(\zeta_T f)_{\mathrm{LL}} + (\zeta_T f)_{\mathrm{HL}} + (\zeta_T f)_{\mathrm{HH}}\|_{B^{s/2}_{\gamma_0',2}(L^{\rho_0'})} \lesssim T^{\kappa_0} \|u\|^p_{H^s}. \tag{3.4}$$

Thus, from (3.2)-(3.4) we obtain $\|\Phi_T(u)\|_{X^s} \le C\|u_0\|_{H^s} + CT^{\kappa_0}\|u\|^p_{X^s}$, so that

$$\|[\Phi(u)]\|_{X^s(I)} \le C\|u_0\|_{H^s} + CT^{\kappa_0} R^p$$

for $[u] \in B_R$. If we choose R and T such that $C\|u_0\|_{H^s} \le R/2$ and $CT^{\kappa_0}R^{p-1} \le 1/2$, then we see that Φ maps B_R into itself.

We can estimate the difference $\Phi(u)-\Phi(v)$ more easily. Let $[u],[v]\in B_R$. By the Strichartz estimate together with the inequality

$$|f(u)-f(v)|\le C\max\{|u|;|v|\}^{p-1}|u-v|, \tag{3.5}$$

we obtain the following:

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^\infty(I;L^2)\cap L^q(I;L^r)} &\lesssim \|f(u)-f(v)\|_{L^{\gamma'}(I;L^{\rho'})}\\ &\lesssim T^\kappa \max\{\|u\|_{L^\infty(I;H^s)};\|v\|_{L^\infty(I;H^s)}\}^{p-1}\|u-v\|_{L^q(I;L^r)}\\ &\lesssim T^\kappa R^{p-1}\|u-v\|_{L^\infty(I;L^2)\cap L^q(I;L^r)}.\end{aligned} \tag{3.6}$$

Therefore, if T is sufficiently small, then Φ is a contraction mapping on (B_R,d). By the contraction mapping principle, there exists a unique fixed point of Φ in B_R. Therefore we have proved the existence of the solution to (1.1)-(1.2) in $X^s(I)$. The uniqueness of the solution in $C(I;H^s)$ was proved in [13]. □

4. Proof of Theorem 1.1—case $2<s<4$

In this section we consider the case where $2<s<4$. As in §3, we put $\kappa=1-(n-2s)(p-1)/4$. Let (q,r) and (γ,ρ) be admissible pairs as in §3, namely they satisfy $\max\{1-2\kappa;0\}<\delta(r)<\min\{2(1-\kappa);1\}$ and $\delta(r)+\delta(\rho)=2(1-\kappa)$. We define r_0 by $1/r_0=1/2-(s-2)/n$, or equivalently $\delta(r_0)=s-2$. We put

$$Y^s=L^\infty(\boldsymbol{R};H^2)\cap W^1_\infty(\boldsymbol{R};L^2)\cap L^\infty(\boldsymbol{R};H^2_{r_0})\cap W^1_\infty(\boldsymbol{R};L^{r_0})\cap B^{s/2}_{q,2}(\boldsymbol{R};L^r).$$

For $I=[-T,T]$ with $0<T\le 1$, we set $Y^s(I)=Y^s/\sim$, where $u\sim v$ means $u=v$ on I, and $\|[u]\|_{Y^s(I)}=\inf\{\|v\|_{Y^s};v\sim u\}$. Similarly as Lemma 3.1, we can show that $\tilde B_R=\{[u]\in Y^s(I);\|[u]\|_{Y^s(I)}\le R,u(0)=u_0\}$ is a complete metric space with metric $d([u],[v])=\|u-v\|_{L^\infty(I;L^2)\cap L^q(I;L^r)}$. We define the operator Ψ_T by

$$\{\Psi_T(u)\}(t)=U(t)u_0+\{U\otimes(\zeta_2F_T+\zeta_2 f(u_0))\}(t). \tag{4.1}$$

Here, $F_T(t)=\int_0^t\zeta_T(t')\partial_{t'}f(u(t'))\,dt'$ and $\zeta_T(t)=\zeta(t/T)$ is the same as in §3; especially $\zeta_2(t)=\zeta(t/2)$, so that $\zeta_2\zeta_T=\zeta_T$. If $t\in I$, then $F_T(t)=f(u(t))-f(u_0)$, so that $\{\Psi_T(u)\}(t)=\{\Phi(u)\}(t)$. With suitable choices of R and T, we show that Φ is a contraction mapping on $(\tilde B_R,d)$. Since $\Psi_T(u)$ satisfies the equation

$$(\partial_t+i\Delta)\Psi_T(u)=\zeta_2F_T+\zeta_2f(u_0),$$

it suffices to estimate

$$\|\partial_t\Psi_T(u)\|_{X^{s-2}},\quad \|\Psi_T(u)\|_{L^\infty(L^2)\cap L^q(L^r)},\quad \text{and}\quad \|\zeta_2F_T+\zeta_2f(u_0)\|_Z$$

instead of $\|\Psi_T(u)\|_{Y^s}$. Here, we recall that $X^{s-2}=L^\infty(H^{s-2})\cap B^{s/2-1}_{q,2}(L^r)$, and we set $Z=L^\infty(L^2)\cap L^\infty(L^{r_0})$. We should distinguish the cases $2<s\le 3$ and $3<s<4$, since we have to estimate $\|\partial_t\Psi_T(u)\|_{X^{s-2}}$ differently.

We begin with the case where $2<s\le 3$. Taking the time derivative of (4.1), we obtain

$$\partial_t\Psi_T(u)=U(\cdot)\dot u_0+U\otimes(\zeta_T\partial_tf(u)+\dot\zeta_2F_T+\dot\zeta_2f(u_0)),$$

where $\dot u_0=-i\Delta u_0+f(u_0)$. From Lemma 2.4 we have

$$\begin{aligned}\|\partial_t\Psi_T(u)\|_{X^{s-2}}\lesssim{}& \|\dot u_0\|_{H^{s-2}}+\|\zeta_T\partial_tf\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})}\\ &+\|\dot\zeta_2F_T\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})}+\|\dot\zeta_2f(u_0)\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})}.\end{aligned} \tag{4.2}$$

We shall estimate each term of the right-hand side. We recall that $\nu_0 = 2n/(n-2s)$. We define ν_2 by $1/2 = (p-1)/\nu_0 + 1/\nu_2$, or equivalently $\delta(\nu_2) = 2(1-\kappa)$. Then we have the inclusion $H^{s-2}_{\nu_2} \supset H^{s-2k} \supset H^s$, thereby obtaining $\|f(u_0)\|_{H^{s-2}} \lesssim \|u\|^{p-1}_{L^{\nu_0}} \|u\|_{H^{s-2}_{\nu_2}} \lesssim \|u\|^p_{H^s}$. Therefore

$$\|\dot{u}_0\|_{H^{s-2}} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1}. \tag{4.3}$$

We recall that $1/q_0 = 1/\gamma' - 1/q = \kappa$. We choose a number $\mu > 0$ such that $0 < \min\{1/q; s/2-1\} - \mu \ll 1$. We put $1/q_1 = 1/\gamma' - 1/q + \mu$ and $\kappa_1 = 1/q_1 - s/2 + 1$. Then, we see

$$\kappa_1 = 2 - s/2 - 1/\gamma - 1/q + \mu = \kappa + \mu - s/2 + 1,$$

so that $0 < \kappa_1 < \kappa$. By the Leibniz rule together with Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\|\zeta_T \partial_t f\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})} &\lesssim \|\zeta_T\|_{L^{q_0}} \|\partial_t f\|_{B^{s/2-1}_{q,2}(L^{\rho'})} + \|\zeta_T\|_{B^{s/2-1}_{q_1,2}} \|\partial_t f\|_{L^{q/(1-q\mu)}(L^{\rho'})} \\
&\lesssim T^\kappa \|u\|^{p-1}_{L^\infty(H^2_{r_0})} \|u\|_{B^{s/2}_{q,2}(L^r)} + T^{\kappa_1} \|u\|^{p-1}_{L^\infty(H^2_{r_0})} \|u\|_{B^{\mu+1}_{q,2}(L^r)} \\
&\lesssim T^{\kappa_1} \|u\|^{p-1}_{L^\infty(H^2_{r_0})} \|u\|_{B^{s/2}_{q,2}(L^r)}.
\end{aligned} \tag{4.4}$$

We proceed to the estimate of $\|\dot{\zeta}_2 F_T\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})}$. By the inclusion

$$B^{s/2-1}_{\gamma',2}(L^{\rho'}) \supset W^1_{\gamma'}(L^{\rho'})$$

and the relation $\dot{\zeta}_2 \zeta_T = 0$ together with $1/q' > \kappa$, we see

$$\begin{aligned}
\|\dot{\zeta}_2 F_T\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})} &\lesssim \|\ddot{\zeta}_2 F_T\|_{L^{\gamma'}(L^{\rho'})} + \|\dot{\zeta}_2 F_T\|_{L^{\gamma'}(L^{\rho'})} \\
&\lesssim \|\zeta_T\|_{L^{q'}} \|\partial_t f\|_{L^q(L^{\rho'})} \\
&\lesssim T^\kappa \|u\|^{p-1}_{L^\infty(H^2_{r_0})} \|\partial_t u\|_{L^q(L^r)},
\end{aligned} \tag{4.5}$$

$$\|\dot{\zeta}_2 f(u_0)\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})} \lesssim \|f(u_0)\|_{L^{\rho'}} \lesssim \|u_0\|^{p-1}_{L^{\nu_0}} \|u_0\|_{L^r} \lesssim \|u_0\|^p_{H^s}. \tag{4.6}$$

From (4.2)-(4.6), we obtain

$$\|\partial_t \Psi_T(u)\|_{X^{s-2}} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1} + T^{\kappa_1} \|u\|^p_{Y^s}. \tag{4.7}$$

Similarly (actually more easily), we have

$$\begin{aligned}
&\|\Psi_T(u)\|_{L^\infty(L^2) \cap L^q(L^r)} \lesssim \|u_0\|_{L^2} + \|\zeta_2 F_T\|_{L^{\gamma'}(L^{\rho'})} + \|\zeta_2 f(u_0)\|_{L^{\gamma_1'}(L^{\rho_1'})} \\
&\quad \lesssim \|u_0\|_{L^2} + T^\kappa \|u\|^{p-1}_{L^\infty(H^2_{r_0})} \|\partial_t u\|_{L^q(L^r)} + \|u_0\|^p_{H^s} \\
&\quad \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1} + T^\kappa \|u\|^p_{Y^s}.
\end{aligned} \tag{4.8}$$

We next estimate $\|\zeta_2 F_T\|_Z$. Using the integration by parts, we have

$$\begin{aligned}
F_T(t) &= \int_0^t \zeta_T(t') \partial_{t'} \{f(u(t')) - f(u_0)\}\, dt' \\
&= \zeta_T(t)\{f(u(t)) - f(u_0)\} - \int_0^t \dot{\zeta}_T(t')\{f(u(t')) - f(u_0)\}\, dt'.
\end{aligned} \tag{4.9}$$

Therefore, if we show

$$\|f(u(t)) - f(u_0)\|_{L^2 \cap L^{r_0}} \lesssim |t|^\kappa \|u\|^p_{Y^s}, \tag{4.10}$$

then we easily obtain $\|F_T\|_Z \lesssim T^\kappa \|u\|_{Y^s}^p$. We prove (4.10). For simplicity we assume that $t > 0$. We see

$$\|u(t) - u_0\|_{L^2} \le \int_0^t \|\partial_{t'} u(t')\|_{L^2} dt' \le t\|u\|_{W^1_\infty(L^2)} \le t\|u\|_{Y^s},$$

and hence $\|u(t)-u_0\|_{L^{\nu_2}} \lesssim \|u(t)-u_0\|_{L^2}^\kappa \|u(t)-u_0\|_{H^2}^{1-\kappa} \lesssim t^\kappa \|u\|_{Y^s}$ by interpolation. Therefore, by the inequality (3.5), we have

$$\|f(u(t)) - f(u_0)\|_{L^2} \lesssim \|u\|_{L^\infty(L^{\nu_0})}^{p-1} \|u(t) - u_0\|_{L^{\nu_2}} \lesssim t^\kappa \|u\|_{Y^s}^p.$$

In the same way we can obtain $\|f(u(t)) - f(u_0)\|_{L^{r_0}} \lesssim t^\kappa \|u\|_{Y^s}^p$, and hence we obtain (4.10). On the other hand, like (4.3), we have $\|\zeta_2 f(u_0)\|_Z \lesssim \|f(u_0)\|_{H^{s-2}} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1}$, since $L^{r_0} \supset H^{s-2}$. Therefore, we obtain

$$\|\zeta_2 F_T + \zeta_2 f(u_0)\|_Z \lesssim T^\kappa \|u\|_{Y^s}^p + \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1}. \tag{4.11}$$

Consequently, from the estimates (4.7), (4.8) and (4.11) we obtain

$$\|\Psi_T(u)\|_{Y^s} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1} + T^{\kappa_1} \|u\|_{Y^s}^p,$$

so that

$$\|[\Phi(u)]\|_{Y^s(I)} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s})^{p-1} + T^{\kappa_1} \|[u]\|_{Y^s(I)}^p.$$

The estimate (3.6) still holds for $s > 2$. These estimates show that if $R > 0$ is large and $T > 0$ is small, then Φ defines a contraction mapping on $(\tilde{B}_R, d)$. Therefore, by the contraction mapping principle, there exists a unique fixed point of Φ in $Y^s(I)$.

We next consider the case $3 < s < 4$. Let (γ_1, ρ_1) be another admissible pair satisfying $\max\{4 - 2\kappa - s; 0\} < \delta(\rho_1) < \min\{2(1 - \kappa); 1; 4 - s\}$. From Lemma 2.4 together with paraproduct formula, we have

$$\begin{aligned} \|\partial_t \Psi_T(u)\|_{X^{s-2}} \lesssim{}& \|\dot{u}_0\|_{H^{s-2}} + \|(\zeta_T \partial_t f)_{\mathrm{LH}}\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})} \\ &+ \|(\zeta_T \partial_t f)_{\mathrm{LH}}\|_{B^0_{\bar{q}_1,\infty}(L^{\bar{r}_1})} \\ &+ \|(\zeta_T \partial_t f)_{\mathrm{LL}} + (\zeta_T \partial_t f)_{\mathrm{HL}} + (\zeta_T \partial_t f)_{\mathrm{HH}}\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})} \\ &+ \|\dot{\zeta}_2 F_T\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})} + \|\dot{\zeta}_2 f(u_0)\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})}. \end{aligned} \tag{4.12}$$

Like (3.1), the subscripts L and H mean hi and low-frequency parts. The exponent $\bar{r}_1$ is defined by $1/\bar{r}_1 = (p-1)/\nu_0 + 1/r_0$, or equivalently $\delta(\bar{r}_1) = s - 4 + 2\kappa$, and $\bar{q}_1$ is so chosen that $1/\bar{q}_1 = \kappa - \epsilon$ with sufficiently small $\epsilon > 0$. We shall estimate each term of the right-hand side. We define ν_3 by $1/\rho_1' = (p-1)/\nu_0 + 1/\nu_3$, or equivalently $\delta(\nu_3) = 2(1-\kappa) - \delta(\rho_1)$. By the assumption on ρ_1, we have $2 < \nu_3 < r_0$, and hence we have the inclusion $L^{\nu_3} \supset L^2 \cap L^{r_0} \supset H^{s-2}$. We can show the following estimates (4.13)-(4.15) similarly as (3.2)-(3.4) respectively:

$$\|(\zeta_T\partial_t f)_{\rm LH}\|_{B^{s/2-1}_{\gamma',2}(L^{\rho'})} \lesssim \|\zeta_T\|_{L^{q_0}}\|\partial_t f\|_{B^{s/2-1}_{q,2}(L^{\rho'})}$$

$$\lesssim T^{\kappa}\|u\|^{p-1}_{L^\infty(H^2_{r_0})}\|u\|_{B^{s/2}_{q,2}(L^r)}, \tag{4.13}$$

$$\|(\zeta_T\partial_t f)_{\rm LH}\|_{B^0_{\bar q_1,\infty}(L^{\bar r_1})} \lesssim \|\zeta_T\|_{L^{\bar q_1}}\|\partial_t f\|_{L^\infty(L^{\bar r_1})}$$

$$\lesssim T^{\kappa-\epsilon}\|u\|^{p-1}_{L^\infty(H^2_{r_0})}\|\partial_t u\|_{L^\infty(L^{r_0})}, \tag{4.14}$$

$$\|(\zeta_T\partial_t f)_{\rm LL}+(\zeta_T\partial_t f)_{\rm HL}+(\zeta_T\partial_t f)_{\rm HH}\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})}$$

$$\lesssim \|\zeta_T\|_{B^{s/2-1}_{\gamma_1',2}}\|\partial_t f\|_{L^\infty(L^{\rho_1'})} \lesssim T^{\kappa_2}\|u\|^{p-1}_{L^\infty(H^2_{r_0})}\|\partial_t u\|_{L^\infty(L^2\cap L^{\nu_0})}. \tag{4.15}$$

Here, $\kappa_2 = 1/\gamma_1' - s/2 + 1$, which satisfies $0 < \kappa_2 < \kappa$ by assumption. Similarly as in the previous case, we have

$$\|\dot\zeta_2 F_T\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})} \lesssim T\|u\|^{p-1}_{L^\infty(H^2_{r_0})}\|\partial_t u\|_{L^\infty(L^2\cap L^{\nu_0})}, \tag{4.16}$$

$$\|\dot\zeta_2 f(u_0)\|_{B^{s/2-1}_{\gamma_1',2}(L^{\rho_1'})} \lesssim \|u_0\|^p_{H^s}. \tag{4.17}$$

From (4.12)-(4.17), we obtain

$$\|\partial_t\Psi_T(u)\|_{X^{s-2}} \lesssim \|u_0\|_{H^s}(1+\|u_0\|_{H^s})^{p-1} + T^{\kappa_2}\|u\|^p_{Y^s}. \tag{4.18}$$

We can estimate $\|\Psi_T(u)\|_{L^\infty(L^2)\cap L^q(L^r)}$ and $\|\zeta_2 F_T + \zeta_2 f(u_0)\|_Z$ in the same way as the previous case, so that we can obtain

$$\|[\Phi(u)]\|_{Y^s(I)} \lesssim \|u_0\|_{H^s}(1+\|u_0\|_{H^s})^{p-1} + T^{\kappa_2}\|[u]\|^p_{Y^s(I)}.$$

Therefore, for suitable R and T, Φ is a contraction mapping in $(\tilde B_R, d)$. By the contraction mapping principle, there exists a unique fixed point of Φ in $Y^s(I)$.

We have thus proved that there exists a unique solution $u \in Y^s(I)$ to (1.1)-(1.2). Finally, we show that $u \in C(I; H^s)$. To this end, we prove $\Phi(Y^s(I)) \subset C(I;H^s)\cap C^1(I;H^{s-2})$. It suffices to show that $f(u) \in C(\boldsymbol{R}; H^{s-2})$ for each $u \in Y^s$. Let $0 < \epsilon \le \min\{4-s; 2(1-\kappa)\}$ and $t_0 \in \boldsymbol{R}$. Similarly as above, we have $\|f(u)\|_{H^{s-2+\epsilon}} \lesssim \|u\|^{p-1}_{\nu_0}\|u\|_{H^{s-2+\epsilon}_{\nu_2}} \lesssim \|u\|^p_{Y^s}$ by the inclusion $L^{\nu_0}\cap H^{s-2+\epsilon}_{\nu_2} \supset H^2_{r_0}$. Using this estimate together with (4.10), we see

$$\|f(u(t)) - f(u(t_0))\|_{H^{s-2}} \lesssim |t-t_0|^{2\kappa\epsilon/(s-2+\epsilon)}\|u\|^p_{Y^s} \to 0$$

as $t \to 0$. Therefore we obtain that $f(u) \in C(\boldsymbol{R}; H^{s-2})$. □

Acknowledgements. The author would like to thank the unknown referee for his/her careful reading and helpful suggestions.

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Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: wada@riko.shimane-u.ac.jp