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Enhanced $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ as a signature of minimal renormalizable SUSY $SO(10)$ GUT

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The ratio of the partial widths of some dimension-5 proton decay modes can be predicted without detailed knowledge of supersymmetric (SUSY) particle masses, and this allows us to experimentally test various SUSY grand unified theory (GUT) models without discovering SUSY particles. In this paper, we study the ratio of the partial widths of the $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$ decays in the minimal renormalizable SUSY $SO(10)$ GUT, under only a plausible assumption that the 1st- and 2nd-generation left-handed squarks are mass-degenerate. In the model, we expect that the Wilson coefficients of dimension-5 operators responsible for these modes are on the same order and that the ratio of $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$ partial widths is $O(0.1)$. Hence, we may be able to detect both $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$ decays at Hyper-Kamiokande, thereby gaining a hint for the minimal renormalizable SUSY $SO(10)$ GUT. Moreover, since this partial width ratio is quite suppressed in the minimal $SU(5)$ GUT, it allows us to distinguish the minimal renormalizable SUSY $SO(10)$ GUT from the minimal $SU(5)$ GUT. In the main body of the paper, we perform a fitting of the quark and lepton masses and flavor mixings with the Yukawa couplings of the minimal renormalizable $SO(10)$ GUT, and derive a concrete prediction for the partial width ratio based on the fitting results. We find that the partial width ratio generally varies in the range 0.05–0.6, confirming the above expectation.

Subject Index B40, B42

1. Introduction

The $SO(10)$ grand unified theory (GUT) [1,2] is a well-motivated scenario beyond the Standard Model (SM), since it unifies the SM gauge groups into an anomaly-free group, it unifies the SM matter fields and right-handed neutrino of each generation into one **16** representation and it includes the seesaw mechanism [3–6] for the tiny neutrino mass. The minimal renormalizable $SO(10)$ GUT [7], where the electroweak-symmetry-breaking-Higgs field stems from $\mathbf{10} + \overline{\mathbf{126}}$ fields and the SM Yukawa couplings come solely from renormalizable terms $\tilde{Y}_{10}\,\mathbf{16}\,\mathbf{10}\,\mathbf{16} + \tilde{Y}_{126}\,\mathbf{16}\,\overline{\mathbf{126}}\,\mathbf{16}$, is even more appealing because the mass and flavor mixings of quarks and leptons are derived from a restricted set of parameters. Specifically, the up-type quark, down-type quark, charged lepton and neutrino Dirac Yukawa matrices are derived as $Y_u = Y_{10} + r_2 Y_{126}$, $Y_d = r_1(Y_{10} + Y_{126})$, $Y_e = r_1(Y_{10} - 3Y_{126})$, $Y_D = Y_{10} - 3r_2 Y_{126}$, with $Y_{10} \propto \tilde{Y}_{10}$, $Y_{126} \propto \tilde{Y}_{126}$ and r_1, r_2 being numbers. Also, the Majorana mass for right-handed neutrinos and the type-2 seesaw contribution to the tiny neutrino mass are proportional to Y_{126}.

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The direct experimental signature of the minimal renormalizable $SO(10)$ GUT is, like other GUT models, proton decay. In supersymmetric (SUSY) GUT, proton decay through dimension-5 operators induced by colored Higgsino exchange [8,9] can be within the reach of the Hyper-Kamiokande experiment [10] and is crucial to phenomenology.[1] Regrettably, SUSY particles have not been discovered at the Large Hadron Collider and hence no concrete prediction is available for the partial widths of dimension-5 proton decays, since they are inversely proportional to the soft SUSY breaking scale squared. In this situation, the ratio of the partial widths of different decay modes, which is independent of the soft SUSY breaking scale, allows us to test various SUSY GUT models including the minimal renormalizable SUSY $SO(10)$ GUT.[2]

In this paper, we focus on the ratio of the partial widths of the $p \to K^0\mu^+$ and the $p \to K^+\bar{\nu}_\mu$ decays in the minimal renormalizable SUSY $SO(10)$ GUT. We make only one natural assumption on the SUSY particle mass spectrum, which is that the 1st- and 2nd-generation left-handed squarks are mass-degenerate. In the model with the above assumption, the ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is predicted to be $O(0.1)$. Hence, we may be able to discover both $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$ decays at Hyper-Kamiokande [10], thereby gaining a hint for the model. Moreover, this ratio is predicted to be suppressed by a factor of 0.002 in the minimal $SU(5)$ GUT compared to the minimal renormalizable SUSY $SO(10)$ GUT, and thus observation of both $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$ decays allows us to distinguish the latter from the former.[3]

In the main body of the paper, we numerically confirm that $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is $O(0.1)$ in the minimal renormalizable SUSY $SO(10)$ GUT. To this end, we determine the fundamental Yukawa couplings Y_{10}, Y_{126} through a fitting of the quark and lepton Yukawa couplings and neutrino data, as has been performed in Refs. [12]-[30], and calculate the partial width ratio based on the fitting results.

Previously, enhancement of partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ in $SO(10)$ GUT models compared to the minimal $SU(5)$ GUT is claimed in Refs. [31,32], but only based on a qualitative argument. Our paper is the first study where this ratio is predicted concretely and quantitatively in the minimal renormalizable SUSY $SO(10)$ GUT, with the fundamental Yukawa couplings Y_{10}, Y_{126} determined through a numerical fitting.

The basic reason that $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is $O(0.1)$ in the minimal renormalizable SUSY $SO(10)$ GUT is understood as follows. In the model, the ratio of the Wilson coefficients of dimension-5 operators responsible for the $p \to K^0\mu^+$ decay and those for the $p \to K^+\bar{\nu}_\mu$ decay is proportional to $(Y_{10})_{u_L j}/(Y_{10})_{d_L j}$ or $(Y_{126})_{u_L j}/(Y_{126})_{d_L j}$. Here $(Y_{10})_{u_L j}$ denotes the $(1, j)$-component of Y_{10} in the flavor basis where, when we write the Yukawa coupling as $\psi_i(Y_{10})_{ij}\psi_j$, the left-handed up-type quark component of ψ_i has the diagonalized up-type quark Yukawa coupling. $(Y_{10})_{d_L j}, (Y_{126})_{u_L j}, (Y_{126})_{d_L j}$ are defined in the same way. Y_{10}, Y_{126} are linear combinations of the down-type and up-type quark Yukawa matrices Y_d, Y_u, due to the relations $Y_u = Y_{10} + r_2 Y_{126}$, $Y_d = r_1(Y_{10} + Y_{126})$. Moreover, these linear combinations are generic, because situations where $Y_{10} \propto Y_u, Y_{126} \propto Y_d$ or $Y_{10} \propto Y_d, Y_{126} \propto Y_u$ would not reproduce the correct charged

[1] If $\mathbf{45} + \mathbf{16} + \overline{\mathbf{16}}$ fields are responsible for breaking $SO(10)$ gauge group, then proton decay through dimension-6 operators induced by GUT gauge boson exchange can also be within the reach of Hyper-Kamiokande [11].

[2] If the ratio involves a decay mode that receives contributions from both left-handed dimension-5 operators $QQQL$ and right-handed ones $EUUD$, we need information about the ratio of Wino mass and the μ-term to predict the ratio.

[3] The origin of the suppression factor 0.002 is explained in Sect. 3.1.

lepton Yukawa matrix Y_e. Therefore, considering the large hierarchy $y_u/y_t \ll y_d/y_b$, we expect that the components $(Y_{10})_{u_L j}, (Y_{10})_{d_L j}, (Y_{126})_{u_L j}, (Y_{126})_{d_L j}$ are *all* on the order of the down quark Yukawa coupling y_d times the mixing angle between the right-handed down quark and a state with flavor index j, and are *not* proportional to the up-quark Yukawa coupling y_u. Hence, both $(Y_{10})_{u_L j}/(Y_{10})_{d_L j}$ and $(Y_{126})_{u_L j}/(Y_{126})_{d_L j}$ are $O(1)$ and so is the ratio of the Wilson coefficients of dimension-5 operators for the $p \to K^0\mu^+$ and the $p \to K^+\bar{\nu}_\mu$ decays. The Wino-dressing diagrams give almost the same contribution for the two modes, if the 1st- and 2nd-generation left-handed squarks are mass-degenerate. As a result, the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is determined by the ratio of baryon chiral Lagrangian parameters, which lies in the range $(1 - D + F)^2/(1 + D + F)^2 = 0.085$ to $(1 - D + F)^2/(1 - D/3 + F)^2 = 0.30$, and thus the partial width ratio is $O(0.1)$.

This paper is organized as follows. In Sect. 2, we describe the minimal renormalizable SUSY $SO(10)$ GUT and present formulas for the partial widths of the $p \to K^+\bar{\nu}_\mu$ and $p \to K^0\mu^+$ decays. In Sect. 3, we roughly estimate the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ in the minimal renormalizable SUSY $SO(10)$ GUT without numerically determining the fundamental Yukawa couplings Y_{10}, Y_{126}, and compare it to the partial width ratio in the minimal $SU(5)$ GUT. In Sect. 4, we numerically determine Y_{10}, Y_{126} through a fitting of the quark and charged lepton Yukawa couplings and neutrino mass matrix, and calculate $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ based on the fitting results. Section 5 summarizes the paper.

2. Minimal renormalizable SUSY *SO*(10) GUT

We consider a SUSY $SO(10)$ GUT model that contains chiral superfields H, Δ and $\overline{\Delta}$ in **10**, **126**, $\overline{\mathbf{126}}$ representation, and three matter fields Ψ_i in **16** representation ($i = 1, 2, 3$ denotes flavor index) [7]. The model also contains chiral superfields responsible for breaking $SU(5)$ subgroup of $SO(10)$, but we do not specify them in this paper. The most general renormalizable Yukawa couplings are given by

$$W_{\text{Yukawa}} = (\tilde{Y}_{10})_{ij}\, \Psi_i H \Psi_j + (\tilde{Y}_{126})_{ij}\, \Psi_i \overline{\Delta} \Psi_j, \tag{1}$$

where $(\tilde{Y}_{10})_{ij}$ and $(\tilde{Y}_{126})_{ij}$ are 3×3 complex symmetric matrices. The Higgs fields of the minimal SUSY Standard Model (MSSM), H_u, H_d, are linear combinations of $(\mathbf{1}, \mathbf{2}, \pm 1/2)$ components of H, $\overline{\Delta}$ and other fields. Accordingly, the MSSM Yukawa coupling for up-type quarks, Y_u, that for down-type quarks, Y_d, and that for charged leptons, Y_e, and the Dirac Yukawa coupling for neutrinos, Y_D, are derived from W_{Yukawa} as

$$W_{\text{Yukawa}} \supset (Y_u)_{ij}\, Q_i H_u U_i^c + (Y_d)_{ij}\, Q_i H_d D_i^c + (Y_e)_{ij}\, L_i H_d E_i^c + (Y_D)_{ij}\, L_i H_u N_i^c, \tag{2}$$

where $Y_u,\ Y_d,\ Y_e,\ Y_D$ are given by

$$Y_u = Y_{10} + r_2\, Y_{126}, \tag{3}$$

$$Y_d = r_1\, (Y_{10} + Y_{126})\,, \tag{4}$$

$$Y_e = r_1\, (Y_{10} - 3Y_{126})\,, \tag{5}$$

$$Y_D = Y_{10} - 3r_2\, Y_{126}, \tag{6}$$

at a $SO(10)$ breaking scale. Here $Y_{10} \propto \tilde{Y}_{10}$, $Y_{126} \propto \tilde{Y}_{126}$, and r_1, r_2 are numbers. By a phase redefinition, we take r_1 to be real positive. In principle, r_1, r_2 are determined from the mass matrix for $(\mathbf{1}, \mathbf{2}, \pm 1/2)$ components [33]–[38], but in this paper we treat them as independent parameters.

Majorana mass for the right-handed neutrinos is proportional to $(Y_{126})_{ij}\, v_R\, N_i^c N_j^c$ where v_R denotes $\overline{\Delta}$'s vacuum expectation value. Integrating out N_i^c yields an effective operator $L_i H_u L_j H_u$, which we call the Type-1 seesaw contribution. Additionally, if the $(\mathbf{1}, \mathbf{3}, 1)$ component of $\overline{\Delta}$ mixes with that of **54** representation field, after integrating out these components, we get an effective operator $L_i H_u L_j H_u$, which we call the Type-2 seesaw contribution.

H, $\bar{\Delta}$ and other fields contain pairs of $(\mathbf{3}, \mathbf{1}, -1/3)$, $(\bar{\mathbf{3}}, \mathbf{1}, 1/3)$ components, which we call "colored Higgs fields" and denote by H_C^A, $\overline{H}_C^B$ (A, B are labels), respectively. Exchange of $H_C^A, \overline{H}_C^B$ gives rise to dimension-5 operators inducing proton decay. Those couplings of $H_C^A, \overline{H}_C^B$ which contribute to such operators are

$$W_{\text{Yukawa}} \supset \sum_A \left[\frac{1}{2}(Y_L^A)_{ij}\, Q_i H_C^A Q_j + (\overline{Y}_L^A)_{ij}\, Q_i \overline{H}_C^A L_j + (Y_R^A)_{ij}\, E_i^c H_C^A U_j^c + (\overline{Y}_R^A)_{ij}\, U_i^c \overline{H}_C^A D_j^c \right], \quad (7)$$

where Y_L^A, $\overline{Y}_L^A$, Y_R^A, $\overline{Y}_R^A$ are linear combinations of Y_{10}, Y_{126}. After integrating out $H_C^A, \overline{H}_C^B$, we get dimension-5 operators contributing to proton decay,

$$-W_5 = \frac{1}{2} C_{5L}^{ijkl}\, (Q_k Q_l)(Q_i L_j) + C_{5R}^{ijkl}\, E_k^c U_l^c U_i^c D_j^c \quad (8)$$

(in the first term, isospin indices are summed in each bracket), where

$$C_{5L}^{ijkl}(\mu \sim M_{H_C}) = \sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \left\{ (Y_L^A)_{kl} (\overline{Y}_L^B)_{ij} - \frac{1}{2}(Y_L^A)_{li} (\overline{Y}_L^B)_{kj} - \frac{1}{2}(Y_L^A)_{ik} (\overline{Y}_L^B)_{lj} \right\} \Bigg|_{\mu \sim M_{H_C}}, \quad (9)$$

$$C_{5R}^{ijkl}(\mu \sim M_{H_C}) = \sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \left\{ (Y_R^A)_{kl} (\overline{Y}_R^B)_{ij} - (Y_R^A)_{ki} (\overline{Y}_R^B)_{lj} \right\} \Bigg|_{\mu \sim M_{H_C}}, \quad (10)$$

$\mathcal{M}_{H_C}$ denotes the mass matrix of $H_C^A, \overline{H}_C^B$ fields and M_{H_C} represents a typical value of the eigenvalues of $\mathcal{M}_{H_C}$.

We concentrate on the contribution of the $(Q_k Q_l)(Q_i L_j)$ operators to the $p \to K^+ \bar{\nu}_\mu$ and $p \to K^0 \mu^+$ decays, and calculate the ratio of their partial widths,

$$\frac{\Gamma(p \to K^0 \mu^+)}{\Gamma(p \to K^+ \bar{\nu}_\mu)}, \quad (11)$$

in the minimal renormalizable SUSY $SO(10)$ GUT. It should be noted that the $(Q_k Q_l)(Q_i L_j)$ and the $E_k^c U_l^c U_i^c D_j^c$ operators contribute to the $p \to K^+ \bar{\nu}_\tau$ decay, which is experimentally indistinguishable from the $p \to K^+ \bar{\nu}_\mu$ decay. Hence, our prediction on $\Gamma(p \to K^0 \mu^+)/\Gamma(p \to K^+ \bar{\nu}_\mu)$ should be regarded as the maximum of the following measurable quantity:

$$\frac{\Gamma(p \to K^0 \mu^+)}{\sum_{i=e,\mu,\tau} \Gamma(p \to K^+ \bar{\nu}_i)}. \quad (12)$$

The maximum is attained if the $(Q_k Q_l)(Q_i L_j)$ operators' contribution and the $E_k^c U_l^c U_i^c D_j^c$ operators' contribution to the $p \to K^+ \bar{\nu}_\tau$ decay cancel each other. This cancellation is always possible by adjusting the ratio of the Wino mass and the μ-term.

As stated in the Introduction, for the SUSY particle mass spectrum, we assume that the 1st- and 2nd-generation left-handed squarks are mass-degenerate. To be quantitative, we assume that the 1st- and 2nd-generation left-handed squark masses in the up-quark-Yukawa-diagonal basis satisfy

$$|m^2_{\tilde{c}_L} - m^2_{\tilde{u}_L}| \; < \; 10^{-3} \, m^2_{\tilde{c}_L}. \tag{13}$$

This is a natural assumption at the quantum level, since the 1st- and 2nd-generation quark Yukawa couplings are tiny. To see this, note that the difference in the renormalization group corrections is given in the leading-log approximation by

$$\Delta m^2_{\tilde{c}_L} - \Delta m^2_{\tilde{u}_L} \; \simeq \; -\frac{3}{16\pi^2} \log\left(\frac{\Lambda^2}{m^2}\right) \left\{ y_c^2 - y_u^2 + (Y_d Y_d^\dagger)_{c_L c_L} - (Y_d Y_d^\dagger)_{u_L u_L} \right\} m^2, \tag{14}$$

where m^2 represents the typical scale of soft SUSY breaking masses, and Λ denotes the scale at which initial values of the squark masses are given. We have $|y_c^2 - y_u^2 + (Y_d Y_d^\dagger)_{c_L c_L} - (Y_d Y_d^\dagger)_{u_L u_L}| < 10^{-3}$ for $\tan\beta = 50$ and at any renormalization scale. Hence, we get $|\Delta m^2_{\tilde{c}_L} - \Delta m^2_{\tilde{u}_L}| < 1.3 \times 10^{-3} \, m^2$ even when Λ is the Planck scale and m is 1 TeV. The tiny mass splitting assumed in Eq. (13) does not affect the results presented in the rest of the paper.

The contribution of the $C_{5L}^{ijkl}(Q_k Q_l)(Q_i L_j)$ term to the $p \to K^+ \bar{\nu}_\mu$ and the $p \to K^0 \mu^+$ decays is given by [39]

$$\Gamma(p \to K^+ \bar{\nu}_\mu) \; = \; \mathcal{C} \left| \beta_H(\mu_{\rm had}) \frac{1}{f_\pi} \left\{ \left(1 + \frac{D}{3} + F\right) C_{LL}^{s\mu\, du}(\mu_{\rm had}) + \frac{2D}{3} C_{LL}^{d\mu\, su}(\mu_{\rm had}) \right\} \right|^2, \tag{15}$$

$$\Gamma(p \to K^0 \mu^+) \; = \; \mathcal{C} \left| \beta_H(\mu_{\rm had}) \frac{1}{f_\pi} \, (1 - D + F) \, \overline{C}_{LL}^{u\mu\, us}(\mu_{\rm had}) \right|^2, \tag{16}$$

where $\mathcal{C} = (m_N/64\pi)\left[1 - (m_K^2/m_N^2)\right]^2$, β_H denotes a hadronic matrix element, D, F are parameters of the baryon chiral Lagrangian, and $C_{LL}, \overline{C}_{LL}$ are Wilson coefficients of the effective Lagrangian $-\mathcal{L}_6 \supset C_{LL}^{ijkl}(\psi_{u_L^k}\psi_{d_L^l})(\psi_{d_L^i}\psi_{\nu_L^j}) + \overline{C}_{LL}^{ijkl}(\psi_{d_L^k}\psi_{u_L^l})(\psi_{u_L^i}\psi_{e_L^j})$ (ψ denotes a SM Weyl fermion and spinor index is summed in each bracket). We have neglected the mass splittings among nucleons and hyperons. The Wilson coefficients $C_{LL}, \overline{C}_{LL}$ satisfy

$$C_{LL}^{s\mu\, du}(\mu_{\rm had}) = A_{LL}(\mu_{\rm had}, \mu_{\rm SUSY}) \frac{M_{\widetilde{W}}}{m^2_{\tilde{q}}} \mathcal{F} \, g_2^2 \left(C_{5L}^{s\mu\, ud} - C_{5L}^{u\mu\, sd} \right) |_{\mu = \mu_{\rm SUSY}}, \tag{17}$$

$$C_{LL}^{d\mu\, su}(\mu_{\rm had}) = A_{LL}(\mu_{\rm had}, \mu_{\rm SUSY}) \frac{M_{\widetilde{W}}}{m^2_{\tilde{q}}} \mathcal{F} \, g_2^2 \left(C_{5L}^{d\mu\, us} - C_{5L}^{u\mu\, ds} \right) |_{\mu = \mu_{\rm SUSY}}, \tag{18}$$

$$\overline{C}_{LL}^{u\mu\, us}(\mu_{\rm had}) = A_{LL}(\mu_{\rm had}, \mu_{\rm SUSY}) \frac{M_{\widetilde{W}}}{m^2_{\tilde{q}}} \mathcal{F} \, g_2^2 \left(-C_{5L}^{u\mu\, us} + C_{5L}^{s\mu\, uu} \right) |_{\mu = \mu_{\rm SUSY}}, \tag{19}$$

where $\mathcal{F}$ is a common loop function factor $\mathcal{F} = (1/x - y)[(x/1 - x)\log x - (y/1 - y)\log y]/16\pi^2 + (1/x - 1)[(x/1 - x)\log x + 1]/16\pi^2$ with $x = |M_{\widetilde{W}}|^2/m^2_{\tilde{q}}$ and $y = m^2_{\tilde{\ell}}/m^2_{\tilde{q}}$, and $m_{\tilde{q}}$ denotes the 1st- and 2nd-generation left-handed squark masses (which are assumed to be degenerate) and $m_{\tilde{\ell}}$ denotes the mass of the left-handed smuon and muon sneutrino.[4] $A_{LL}(\mu_{\rm had}, \mu_{\rm SUSY})$ accounts for renormalization group (RG) corrections in the evolution from soft SUSY breaking scale $\mu_{\rm SUSY}$ to

[4] When writing $C_{5L}^{u\mu\, us}$, we mean that Q_i is in the flavor basis where the up-type quark Yukawa coupling Y_u is diagonal and that the up-type quark component of Q_i is exactly u quark (then the down-type quark component

a hadronic scale where the value of β_H is reported.[5] C_{5L} are related to the colored Higgs Yukawa couplings as

$$C_{5L}^{s\mu\, ud}(\mu_{\rm SUSY}) - C_{5L}^{u\mu\, sd}(\mu_{\rm SUSY}) = A_L(\mu_{\rm SUSY}, \mu_{H_C}) \sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \frac{3}{2} \left\{ (Y_L^A)_{ud} (\overline{Y}_L^B)_{s\mu} - (Y_L^A)_{ds} (\overline{Y}_L^B)_{u\mu} \right\} |_{\mu=\mu_{H_C}}, \tag{20}$$

$$C_{5L}^{d\mu\, us}(\mu_{\rm SUSY}) - C_{5L}^{u\mu\, ds}(\mu_{\rm SUSY}) = A_L(\mu_{\rm SUSY}, \mu_{H_C}) \sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \frac{3}{2} \left\{ (Y_L^A)_{us} (\overline{Y}_L^B)_{d\mu} - (Y_L^A)_{ds} (\overline{Y}_L^B)_{u\mu} \right\} |_{\mu=\mu_{H_C}}, \tag{21}$$

$$C_{5L}^{u\mu\, us}(\mu_{\rm SUSY}) - C_{5L}^{s\mu\, uu}(\mu_{\rm SUSY}) = A_L(\mu_{\rm SUSY}, \mu_{H_C}) \sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \frac{3}{2} \left\{ (Y_L^A)_{us} (\overline{Y}_L^B)_{u\mu} - (Y_L^A)_{uu} (\overline{Y}_L^B)_{s\mu} \right\} |_{\mu=\mu_{H_C}}, \tag{22}$$

where $A_L(\mu_{\rm SUSY}, \mu_{H_C})$ accounts for RG corrections in the evolution from colored Higgs mass scale $\mu_{H_C} \sim M_{H_C}$ to soft SUSY breaking scale $\mu_{\rm SUSY}$.[6]

We relate the flavor-dependent part of Eqs. (20)–(22) to Y_{10}, Y_{126}. Since $Y_L^A, \overline{Y}_L^A$ are proportional to either Y_{10} or Y_{126}, we can write without loss of generality

$$\begin{aligned}
&\sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \left\{ (Y_L^A)_{ud} (\overline{Y}_L^B)_{s\mu} - (Y_L^A)_{ds} (\overline{Y}_L^B)_{u\mu} \right\} \\
&= \frac{1}{M_{H_C}} \Big[a \left\{ (Y_{10})_{u_L d_L} (Y_{10})_{s_L \mu_L} - (Y_{10})_{d_L s_L} (Y_{10})_{u_L \mu_L} \right\} + b \left\{ (Y_{10})_{u_L d_L} (Y_{126})_{s_L \mu_L} - (Y_{10})_{d_L s_L} (Y_{126})_{u_L \mu_L} \right\} \\
&+ c \left\{ (Y_{126})_{u_L d_L} (Y_{10})_{s_L \mu_L} - (Y_{126})_{d_L s_L} (Y_{10})_{u_L \mu_L} \right\} + d \left\{ (Y_{126})_{u_L d_L} (Y_{126})_{s_L \mu_L} - (Y_{126})_{d_L s_L} (Y_{126})_{u_L \mu_L} \right\} \Big],
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\sum_{A,B} (\mathcal{M}_{H_C}^{-1})_{AB} \left\{ (Y_L^A)_{us} (\overline{Y}_L^B)_{d\mu} - (Y_L^A)_{ds} (\overline{Y}_L^B)_{u\mu} \right\} \\
&= \frac{1}{M_{H_C}} \Big[a \left\{ (Y_{10})_{u_L s_L} (Y_{10})_{d_L \mu_L} - (Y_{10})_{d_L s_L} (Y_{10})_{u_L \mu_L} \right\} + b \left\{ (Y_{10})_{u_L s_L} (Y_{126})_{d_L \mu_L} - (Y_{10})_{d_L s_L} (Y_{126})_{u_L \mu_L} \right\} \\
&+ c \left\{ (Y_{126})_{u_L s_L} (Y_{10})_{d_L \mu_L} - (Y_{126})_{d_L s_L} (Y_{10})_{u_L \mu_L} \right\} + d \left\{ (Y_{126})_{u_L s_L} (Y_{126})_{d_L \mu_L} - (Y_{126})_{d_L s_L} (Y_{126})_{u_L \mu_L} \right\} \Big],
\end{aligned} \tag{24}$$

of Q_i is a mixture of d, s, b). Likewise, Q_k is in the flavor basis where the down-type quark Yukawa coupling Y_d is diagonal and its down-type quark component is exactly s quark, and Q_l is in the flavor basis where the up-type quark Yukawa coupling is diagonal and its up-type quark component is exactly u quark. The same rule applies to other Wilson coefficients.

[5] RG corrections involving SM Yukawa couplings are negligible for $C_{LL}^{s\mu\, du}, C_{LL}^{d\mu\, su}, C_{LL}^{u\mu\, us}$, and hence their RG corrections are approximately flavor-universal.

[6] Again, RG corrections involving MSSM Yukawa couplings are negligible for $C_{5L}^{d\mu\, us}$, $C_{5L}^{u\mu\, ds}$, $C_{5L}^{u\mu\, su}$, $C_{5L}^{s\mu\, uu}$ and hence their RG corrections are approximately flavor-universal.

$$\sum_{A,B}(\mathcal{M}_{HC}^{-1})_{AB}\left\{(Y_L^A)_{us}(\overline{Y}_L^B)_{u\mu}-(Y_L^A)_{uu}(\overline{Y}_L^B)_{s\mu}\right\}$$
$$=\frac{1}{M_{H_C}}\left[a\left\{(Y_{10})_{u_Ls_L}(Y_{10})_{u_L\mu_L}-(Y_{10})_{u_Lu_L}(Y_{10})_{s_L\mu_L}\right\}+b\left\{(Y_{10})_{u_Ls_L}(Y_{126})_{u_L\mu_L}-(Y_{10})_{u_Lu_L}(Y_{126})_{s_L\mu_L}\right\}\right.$$
$$\left.+c\left\{(Y_{126})_{u_Ls_L}(Y_{10})_{u_L\mu_L}-(Y_{126})_{u_Lu_L}(Y_{10})_{s_L\mu_L}\right\}+d\left\{(Y_{126})_{u_Ls_L}(Y_{126})_{u_L\mu_L}-(Y_{126})_{u_Lu_L}(Y_{126})_{s_L\mu_L}\right\}\right], \tag{25}$$

where M_{H_C} is a typical value of the eigenvalues of $\mathcal{M}_{H_C}$, and a, b, c, d are numbers common for Eqs. (23)–(25). Here $(Y_{10})_{u_Ls_L}$ denotes the $(1,2)$-component of Y_{10} of the term $(Y_{10})_{ij}\,\Psi_i H \Psi_j$ in the flavor basis where the left-handed up-type quark component of Ψ_i has the diagonalized up-type quark Yukawa coupling, and the left-handed down-type quark component of Ψ_j has the diagonalized down-type quark Yukawa coupling. $(Y_{10})_{d_L\mu_L}$, $(Y_{126})_{u_Ls_L}$ and others are defined analogously.

In principle, numbers a, b, c, d are determined from the colored Higgs mass matrix [33]–[38]. However, as we do not specify fields responsible for breaking the $SU(5)$ subgroup of $SO(10)$, we treat a, b, c, d as independent $O(1)$ parameters.

We observe that each term in Eq. (25) is given by $(Y_{10})_{u_Lj}/(Y_{10})_{d_Lj}$ or $(Y_{126})_{u_Lj}/(Y_{126})_{d_Lj}$ times some term in Eqs. (23),(24), as advertised in the Introduction. For example, the term $(Y_{10})_{u_Ls_L}(Y_{10})_{u_L\mu_L}$ in Eq. (25) equals $(Y_{10})_{u_Ls_L}/(Y_{10})_{d_Ls_L}$ times the term $(Y_{10})_{d_Ls_L}(Y_{10})_{u_L\mu_L}$ in Eq. (23), and also equals $(Y_{10})_{u_L\mu_L}/(Y_{10})_{d_L\mu_L}$ times the term $(Y_{10})_{u_Ls_L}(Y_{10})_{d_L\mu_L}$ in Eq. (24).

3. Estimates on $\Gamma(p\to K^0\mu^+)/\Gamma(p\to K^+\bar{\nu}_\mu)$

We estimate $\Gamma(p\to K^0\mu^+)/\Gamma(p\to K^+\bar{\nu}_\mu)$ in the minimal $SU(5)$ GUT and in the minimal renormalizable SUSY $SO(10)$ GUT without numerically determining Y_{10}, Y_{126}. In the minimal $SU(5)$ GUT, we assume, as usual, that the splitting between the down-type quark Yukawa coupling Y_d and the charged lepton Yukawa coupling Y_e is realized by non-renormalizable terms.

3.1. *Estimate in the minimal $SU(5)$ GUT*

In the minimal $SU(5)$ GUT, we have only one pair of colored Higgs fields, and Y_L and $\overline{Y}_L$ are proportional to the Yukawa couplings for **5** and $\overline{\mathbf{5}}$ Higgs fields, respectively. Hence, Eqs. (23)–(25) are altered to

$$\sum_{A,B}(\mathcal{M}_{HC}^{-1})_{AB}\left\{(Y_L^A)_{ud}(\overline{Y}_L^B)_{s\mu}-(Y_L^A)_{ds}(\overline{Y}_L^B)_{u\mu}\right\}=\frac{1}{M_{H_C}}\left\{(Y_5)_{u_Ld_L}(Y_{\bar{5}})_{s_L\mu_L}-(Y_5)_{d_Ls_L}(Y_{\bar{5}})_{u_L\mu_L}\right\}, \tag{26}$$

$$\sum_{A,B}(\mathcal{M}_{HC}^{-1})_{AB}\left\{(Y_L^A)_{ud}(\overline{Y}_L^B)_{s\mu}-(Y_L^A)_{ds}(\overline{Y}_L^B)_{u\mu}\right\}=\frac{1}{M_{H_C}}\left\{(Y_5)_{u_Ls_L}(Y_{\bar{5}})_{d_L\mu_L}-(Y_5)_{d_Ls_L}(Y_{\bar{5}})_{u_L\mu_L}\right\}, \tag{27}$$

$$\sum_{A,B}(\mathcal{M}_{HC}^{-1})_{AB}\left\{(Y_L^A)_{us}(\overline{Y}_L^B)_{u\mu}-(Y_L^A)_{uu}(\overline{Y}_L^B)_{s\mu}\right\}=\frac{1}{M_{H_C}}\left\{(Y_5)_{u_Ls_L}(Y_{\bar{5}})_{u_L\mu_L}-(Y_5)_{u_Lu_L}(Y_{\bar{5}})_{s_L\mu_L}\right\}, \tag{28}$$

where Y_5 and $Y_{\bar{5}}$ denote the Yukawa couplings for **5** and $\overline{\mathbf{5}}$ Higgs fields, respectively, and M_{H_C} denotes the mass for the colored Higgs fields.

The key fact is that since Y_5 is identical to the up-type quark Yukawa coupling matrix, the components of Y_5 with flavor index u_L are given by the up-quark Yukawa coupling times a mixing angle. Hence, they are estimated to be

$$(Y_5)_{u_L u_L},\ (Y_5)_{u_L d_L} \sim y_u(\mu = \mu_{H_C}) \tag{29}$$

$$(Y_5)_{u_L s_L} \sim y_u(\mu = \mu_{H_C}) \cdot \lambda, \tag{30}$$

where $\mu_{H_C} \sim M_{H_C}$, and λ denotes the Cabibbo angle $\lambda \simeq |V_{us}| \simeq |V_{cd}| \simeq 0.22$. On the other hand, $(Y_5)_{d_L s_L}$ is estimated to be the second generation Yukawa coupling times a mixing angle as

$$(Y_5)_{d_L s_L} \ \sim \ y_c(\mu_{H_C}) \cdot \lambda \tag{31}$$

Although the unification of down-type quark Yukawa coupling and charged lepton Yukawa coupling is unsuccessful at the renormalizable level (but the unification can always be achieved with non-renormalizable terms), we can estimate components of $Y_{\bar{5}}$ as

$$(Y_{\bar{5}})_{s_L \mu_L} \sim y_s(\mu_{H_C}) \ \text{ or } \ y_\mu(\mu_{H_C}), \tag{32}$$

$$(Y_{\bar{5}})_{u_L \mu_L} \ \sim \ (Y_{\bar{5}})_{d_L \mu_L} \sim y_s(\mu_{H_C}) \cdot \lambda \ \text{ or } \ y_\mu(\mu_{H_C}) \cdot \lambda. \tag{33}$$

From formulas (15)–(22) and estimates (26)–(33), we estimate the partial widths as

$$\Gamma(p \to K^+ \bar{\nu}_\mu) \ = \ \mathcal{C} \left| (1 + \frac{D}{3} + F) c_1\, \lambda^2\, y_c\, y_\mu + \frac{2D}{3} c_2\, \lambda^2\, y_c\, y_\mu \right|^2 \tag{34}$$

$$\Gamma(p \to K^0 \mu^+) \ = \ \mathcal{C} \ \left| (1 - D + F) c_3\, y_u\, y_\mu \right|^2 \tag{35}$$

or

$$\Gamma(p \to K^+ \bar{\nu}_\mu) \ = \ \mathcal{C} \left| (1 + \frac{D}{3} + F) c_1\, \lambda^2\, y_c\, y_s + \frac{2D}{3} c_2\, \lambda^2\, y_c\, y_s \right|^2 \tag{36}$$

$$\Gamma(p \to K^0 \mu^+) \ = \ \mathcal{C} \ |(1 - D + F) c_3\, y_u\, y_s|^2 \tag{37}$$

where $\mathcal{C}$ is a common constant, c_1, c_2, c_3 are $O(1)$ numbers, and y_u, y_c, y_μ, y_s are the up, charm, muon and strange quark Yukawa couplings at scale $\mu = \mu_{H_C}$.[7] We have discarded subleading terms. The partial width ratio is then estimated as

$$\left(\frac{1 - D + F}{1 + D + F}\right)^2 \left(\frac{y_u}{\lambda^2 y_c}\right)^2 \lesssim \frac{\Gamma(p \to K^0 \mu^+)}{\Gamma(p \to K^+ \bar{\nu}_\mu)} \lesssim \left(\frac{1 - D + F}{1 - D/3 + F}\right)^2 \left(\frac{y_u}{\lambda^2 y_c}\right)^2, \tag{38}$$

where the variation is due to unknown relative phase between c_1 and c_2. Numerically, the above estimate becomes

$$\left(\frac{1 - D + F}{1 + D + F}\right)^2 \cdot 0.002 \lesssim \frac{\Gamma(p \to K^0 \mu^+)}{\Gamma(p \to K^+ \bar{\nu}_\mu)} \lesssim \left(\frac{1 - D + F}{1 - D/3 + F}\right)^2 \cdot 0.002. \tag{39}$$

We find that the $p \to K^0 \mu^+$ partial width is quite suppressed compared to the $p \to K^+ \bar{\nu}_\mu$ partial width because of the factor 0.002 coming from the ratio of y_u and $\lambda^2 y_c$; namely, the large hierarchy between the up and charm quark Yukawa couplings suppresses the partial width ratio. Also, baryon chiral Lagrangian parameters give $(1 - D + F)^2/(1 + D + F)^2 = 0.085$ and $(1 - D + F)^2/(1 - D/3 + F)^2 = 0.3$, and they provide further suppression.

[7] We neglect the small difference between hyperon masses and the nucleon mass.

3.2. *Estimate in the minimal renormalizable SUSY SO(10) GUT*

In the minimal renormalizable SUSY $SO(10)$ GUT, we can rewrite the right-hand side of Eqs. (23)–(25) using the relation $Y_u = Y_{10} + r_2\, Y_{126}$, as

$$\sum_{A,B}(\mathcal{M}_{H_C}^{-1})_{AB}\left\{(Y_L^A)_{ud}(\overline{Y}_L^B)_{s\mu} - (Y_L^A)_{ds}(\overline{Y}_L^B)_{u\mu}\right\}$$
$$= \frac{1}{M_{H_C}}\left[a\left\{(Y_u)_{u_Ld_L}(Y_u)_{s_L\mu_L} - (Y_u)_{d_Ls_L}(Y_u)_{u_L\mu_L}\right\} + b'\left\{(Y_u)_{u_Ld_L}(Y_{126})_{s_L\mu_L} - (Y_u)_{d_Ls_L}(Y_{126})_{u_L\mu_L}\right\}\right.$$
$$\left.+c'\left\{(Y_{126})_{u_Ld_L}(Y_u)_{s_L\mu_L} - (Y_{126})_{d_Ls_L}(Y_u)_{u_L\mu_L}\right\} + d'\left\{(Y_{126})_{u_Ld_L}(Y_{126})_{s_L\mu_L} - (Y_{126})_{d_Ls_L}(Y_{126})_{u_L\mu_L}\right\}\right], \tag{40}$$

$$\sum_{A,B}(\mathcal{M}_{H_C}^{-1})_{AB}\left\{(Y_L^A)_{us}(\overline{Y}_L^B)_{d\mu} - (Y_L^A)_{ds}(\overline{Y}_L^B)_{u\mu}\right\}$$
$$= \frac{1}{M_{H_C}}\left[a\left\{(Y_u)_{u_Ls_L}(Y_u)_{d_L\mu_L} - (Y_u)_{d_Ls_L}(Y_u)_{u_L\mu_L}\right\} + b'\left\{(Y_u)_{u_Ls_L}(Y_{126})_{d_L\mu_L} - (Y_u)_{d_Ls_L}(Y_{126})_{u_L\mu_L}\right\}\right.$$
$$\left.+c'\left\{(Y_{126})_{u_Ls_L}(Y_u)_{d_L\mu_L} - (Y_{126})_{d_Ls_L}(Y_u)_{u_L\mu_L}\right\} + d'\left\{(Y_{126})_{u_Ls_L}(Y_{126})_{d_L\mu_L} - (Y_{126})_{d_Ls_L}(Y_{126})_{u_L\mu_L}\right\}\right], \tag{41}$$

$$\sum_{A,B}(\mathcal{M}_{H_C}^{-1})_{AB}\left\{(Y_L^A)_{us}(\overline{Y}_L^B)_{u\mu} - (Y_L^A)_{uu}(\overline{Y}_L^B)_{s\mu}\right\}$$
$$= \frac{1}{M_{H_C}}\left[a\left\{(Y_u)_{u_Ls_L}(Y_u)_{u_L\mu_L} - (Y_u)_{u_Lu_L}(Y_u)_{s_L\mu_L}\right\} + b'\left\{(Y_u)_{u_Ls_L}(Y_{126})_{u_L\mu_L} - (Y_u)_{u_Lu_L}(Y_{126})_{s_L\mu_L}\right\}\right.$$
$$\left.+c'\left\{(Y_{126})_{u_Ls_L}(Y_u)_{u_L\mu_L} - (Y_{126})_{u_Lu_L}(Y_u)_{s_L\mu_L}\right\} + d'\left\{(Y_{126})_{u_Ls_L}(Y_{126})_{u_L\mu_L} - (Y_{126})_{u_Lu_L}(Y_{126})_{s_L\mu_L}\right\}\right], \tag{42}$$

where

$$b' = b - r_2\, a, \quad c' = c - r_2\, a, \quad d' = d - r_2(b+c) + r_2^2\, a. \tag{43}$$

We still have $b', c', d' = O(1)$, since we have $|r_2| = O(1)$ to fit the charged lepton Yukawa coupling. The right-hand sides of Eqs. (40)–(42) contain terms analogous to Eqs. (26)–(28) [note that Y_u in Eqs. (40)–(42) corresponds to Y_5 in Eqs. (26)–(28)], plus non-analogous terms in the form $(Y_{126})_{ij}(Y_{126})_{kl}$. Each component is estimated as follows. $(Y_u)_{s_L\mu_L}$ is estimated to be the charm quark Yukawa coupling and $(Y_u)_{d_Ls_L}$ is estimated to be the charm quark Yukawa coupling times the Cabibbo angle,

$$(Y_u)_{s_L\mu_L} \sim y_c(\mu_{H_C}), \tag{44}$$
$$(Y_u)_{d_Ls_L} \sim y_c(\mu_{H_C}) \cdot \lambda. \tag{45}$$

The components of Y_u with flavor index u_L are always given by the up Yukawa coupling y_u times a mixing angle, and hence we get

$$(Y_u)_{u_Lu_L},\ (Y_u)_{u_Ld_L} \sim y_u(\mu_{H_C}), \tag{46}$$
$$(Y_u)_{u_Ls_L},\ (Y_u)_{u_L\mu_L} \sim y_u(\mu_{H_C}) \cdot \lambda. \tag{47}$$

In contrast, the components of Y_{126} do not follow the rule and are estimated as

$$(Y_{126})_{u_L u_L},\ (Y_{126})_{u_L d_L} \sim \frac{1}{r_1} y_d(\mu_{H_C}), \tag{48}$$

$$(Y_{126})_{u_L s_L},\ (Y_{126})_{d_L s_L},\ (Y_{126})_{u_L \mu_L} \sim \frac{1}{r_1} y_s(\mu_{H_C}) \cdot \lambda, \tag{49}$$

$$(Y_{126})_{s_L \mu_L} \sim \frac{1}{r_1} y_s(\mu_{H_C}). \tag{50}$$

We have estimated $(Y_{126})_{s_L\mu_L}$ to be $y_s(\mu_{H_C})/r_1$, because we empirically have $y_\mu/y_s|_{\mu=10^{16}\ \text{GeV}} \simeq 4$ and this factor 4 is mostly explained by the factor 3 in Eq. (5). We have estimated $(Y_{126})_{u_L u_L}$ to be $y_d(\mu_{H_C})/r_1$, not $y_u(\mu_{H_C})$, based on the following argument: Recall that components of Y_{10} and Y_{126} reproduce the up and down Yukawa couplings as

$$(Y_{10})_{u_R u_L} + r_2 (Y_{126})_{u_R u_L} = y_u(\mu_{H_C}), \tag{51}$$

$$r_1 \left((Y_{10})_{d_R d_L} + (Y_{126})_{d_R d_L} \right) = y_d(\mu_{H_C}). \tag{52}$$

Since the unification of the top and bottom Yukawa couplings requires $\tan\beta/r_1 \simeq m_t/m_b \simeq 50$, we get

$$\frac{(Y_{10})_{u_R u_L} + r_2 (Y_{126})_{u_R u_L}}{(Y_{10})_{d_R d_L} + (Y_{126})_{d_R d_L}} = r_1 \frac{y_u}{y_d} = \frac{r_1}{\tan\beta} \frac{m_u}{m_d} \simeq \frac{m_b}{m_t} \frac{m_u}{m_d} \simeq 0.01. \tag{53}$$

$(Y_{10})_{u_R u_L}/(Y_{10})_{d_R d_L}$ and $(Y_{126})_{u_R u_L}/(Y_{126})_{d_R d_L}$ are estimated to be $1 - \lambda^2 \simeq 1$. Then, the only way to realize Eq. (53) is to take

$$(Y_{10})_{d_R d_L} \simeq -r_2 (Y_{126})_{d_R d_L} \simeq \frac{1}{r_1} \frac{r_2}{r_2 - 1} y_d(\mu_{H_C}) \tag{54}$$

and impose a fine-tuning between $(Y_{10})_{u_R u_L}$ and $r_2(Y_{126})_{u_R u_L}$ to realize the small value 0.01 in Eq. (53). Here we cannot assume $r_2 \simeq 0$ because we need $|r_2| = O(1)$ to reproduce the charged lepton Yukawa coupling, as will be confirmed numerically in Fig. 1. From Eq. (54), we find

$$(Y_{10})_{u_R u_L} \simeq -r_2 (Y_{126})_{u_R u_L} \simeq \frac{1}{r_1} \frac{r_2}{r_2 - 1} y_d(\mu_{H_C}). \tag{55}$$

Using $|r_2| = O(1)$, we estimate $(Y_{10})_{u_L u_L}$, $(Y_{126})_{u_L u_L}$ as

$$(Y_{10})_{u_L u_L},\ (Y_{126})_{u_L u_L} \sim \frac{1}{r_1} y_d(\mu_{H_C}). \tag{56}$$

From formulas (15)–(22) and estimates (44)–(50), we estimate the partial widths as

$$\begin{aligned}\Gamma(p \to K^+ \bar{\nu}_\mu) = \mathcal{C} \Big| &(1 + \frac{D}{3} + F)(a\beta_1\, y_u y_c + b'\beta_2\, y_c y_s \lambda^2/r_1 + c'\beta_3\, y_c y_s \lambda^2/r_1 + d'\beta_4\, y_s^2 \lambda^2/r_1^2) \\ &+ \frac{2D}{3}(a\gamma_1\, y_u y_c \lambda^2 + b'\gamma_2\, y_c y_s \lambda^2/r_1 + c'\gamma_3\, y_c y_s \lambda^2/r_1 + d'\gamma_4\, y_s^2 \lambda^2/r_1^2) \Big|^2,\end{aligned} \tag{57}$$

$$\Gamma(p \to K^0 \mu^+) = \mathcal{C} \left| (1 - D + F)(a\delta_1\, y_u y_c + b'\delta_2\, y_u y_s/r_1 + c'\delta_3\, y_c y_s \lambda^2/r_1 + d'\delta_4\, y_s^2 \lambda^2/r_1^2) \right|^2, \tag{58}$$

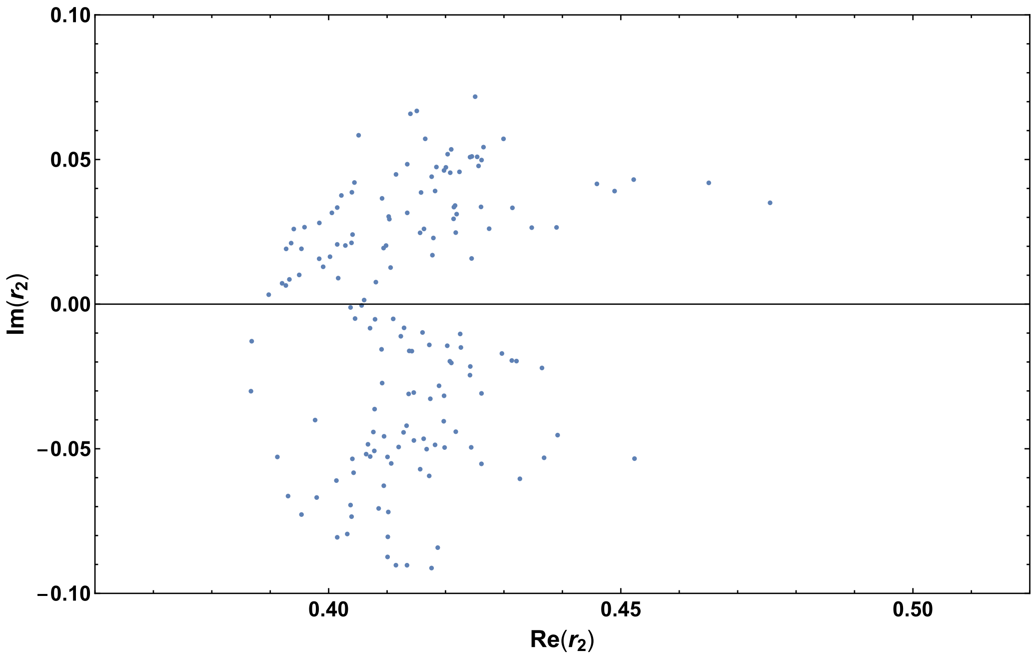

Fig. 1. Distribution of r_2 [defined in Eq. (3)] in the fitting results satisfying the constraints of Table 2.

where $\mathcal{C}$ is a common constant, y_u, y_s, y_c are the up, strange and charm quark Yukawa couplings at scale $\mu = \mu_{H_C}$, and $\beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3, \delta_4$ are $O(1)$ numbers. We have used the empirical relation $m_s\lambda^2 \simeq m_d$ and let $y_s\lambda^2$ represent both $y_s\lambda^2$ and y_d.

In Eqs. (57) and (58), $y_s\lambda^2/r_1^2$ and $y_c y_s\lambda^2/r_1$ are much larger than the other terms containing y_u. Hence, in generic cases where $d' = O(1)$ and/or $c' = O(1)$, the partial width ratio is estimated as

$$\left(\frac{1-D+F}{1+D+F}\right)^2 \lesssim \frac{\Gamma(p\to K^0\mu^+)}{\Gamma(p\to K^+\bar{\nu}_\mu)} \lesssim \left(\frac{1-D+F}{1-D/3+F}\right)^2 \tag{59}$$

[in minimal renormalizable $SO(10)$ GUT with $d' = O(1)$ and/or $c' = O(1)$],

where the variation is due to unknown relative phases among $\beta_2, \beta_3, \beta_4, \gamma_2, \gamma_3, \gamma_4$. We find that the suppression factor of 0.002 in Eq. (39) is absent in Eq. (59). This means that in the minimal renormalizable SUSY $SO(10)$ GUT with $d' = O(1)$ and/or $c' = O(1)$, $\Gamma(p\to K^0\mu^+)/\Gamma(p\to K^+\bar{\nu}_\mu)$ is highly enhanced compared to the minimal $SU(5)$ GUT.

In the non-generic case where c' and d' are both fine-tuned to 0, the partial width ratio is quite suppressed as

$$\left(\frac{1-D+F}{1+D+F}\right)^2\left(\frac{y_u}{\lambda^2 y_c}\right)^2 \lesssim \frac{\Gamma(p\to K^0\mu^+)}{\Gamma(p\to K^+\bar{\nu}_\mu)} \lesssim \left(\frac{1-D+F}{1-D/3+F}\right)^2\left(\frac{y_u}{\lambda^2 y_c}\right)^2 \tag{60}$$

[in minimal renormalizable $SO(10)$ GUT with $c' = d' = 0$],

which is the same as in the minimal $SU(5)$ GUT. This is reasonable because when $c' = d' = 0$ the contribution of $(\mathbf{3}, \mathbf{1}, -1/3)$ fields to dimension-5 proton decay is dictated by the up-type quark Yukawa matrix, just as in the minimal $SU(5)$ GUT.

In the next section, we numerically confirm the estimates Eqs. (59) and (60) through a fitting of the quark and lepton masses and flavor mixings in terms of Y_{10}, Y_{126}.

4. Numerical analysis

4.1. Overview

Our first task is to fit the MSSM Yukawa matrices with $Y_{10}, Y_{126}, r_1, r_2$ through Eqs. (3)–(5), and fit the neutrino mass matrix with Y_{10}, Y_{126}, r_2. When calculating the Type-1 seesaw contribution to the Weinberg operator $L_i H_u L_j H_u$, we have to integrate out each right-handed neutrino N_i^c at its respective mass scale. This requires information on the eigenvalues of Y_{126}, but that is obtained only after the fitting is complete. Hence, it is technically difficult to integrate out each right-handed neutrino separately. In this paper, therefore, we make an approximation that the three right-handed neutrinos are integrated out at one scale. Accordingly, the neutrino mass matrix M_ν is related to Y_{126} and Y_D in Eq. (6) as

$$(M_\nu)_{ij} \propto R_{ik} \left\{ r_L (Y_{126})_{kl} + (Y_D)_{km} (Y_{126}^{-1})_{mn} (Y_D)_{ln} \right\} R_{jl},$$

where r_L is a complex number that parametrizes the ratio of the Type-1 and Type-2 seesaw contributions, and R_{ij} denotes the flavor-dependent RG correction to the coefficient of the Weinberg operator $L_i H_u L_j H_u$ when it evolves from a $SO(10)$ breaking scale to electroweak scale. Since the flavor-dependent RG correction R_{ij} is at most 3% (see Table 1) while the errors of the neutrino data we employ are much larger (see Table 2), we expect that the approximation of integrating out right-handed neutrinos at one scale does not affect the results.

We repeat the above fitting analysis many times and obtain as many fitting results. We compute $\Gamma(p \to K^+ \bar{\nu}_\mu)$ and $\Gamma(p \to K^0 \mu^+)$ from each fitting result of $Y_{10}, Y_{126}, r_1, r_2, r_L$ using Eqs. (15)–(22) and Eqs. (40)–(42), with coefficients a, b', c', d' treated as independent $O(1)$ parameters. The fitting results are plotted with respect to the ratio $\Gamma(p \to K^0 \mu^+)/\,\Gamma(p \to K^+ \bar{\nu}_\mu)$. From the plot, we read out the range of the ratio $\Gamma(p \to K^0 \mu^+)/\,\Gamma(p \to K^+ \bar{\nu}_\mu)$ predicted by the minimal renormalizable $SO(10)$ GUT.

We assume a benchmark SUSY particle mass spectrum to evaluate the MSSM Yukawa couplings at a $SO(10)$ breaking scale as well as R_{ij}, and to compute the individual partial widths $\Gamma(p \to K^+ \bar{\nu}_\mu)$ and $\Gamma(p \to K^0 \mu^+)$. However, we emphasize that the purpose of this paper is to predict the ratio $\Gamma(p \to K^0 \mu^+)/\,\Gamma(p \to K^+ \bar{\nu}_\mu)$, which is not very dependent on the SUSY particle mass spectrum due to the cancellations of the RG corrections and the factors coming from Wino-dressing.

4.2. Procedures

First, we numerically calculate the MSSM Yukawa matrices Y_u, Y_d, Y_e at scale $\mu = 2 \times 10^{16}$ GeV in a $\overline{\text{DR}}$ scheme, and the flavor-dependent RG correction to the coefficient of the Weinberg operator R_{ij}. Specifically, we calculate R_{ij} for the evolution from $\mu = 2 \times 10^{16}$ GeV to $\mu = M_Z$. We assume a high-scale split SUSY particle mass spectrum below for concreteness;

$$m_{\tilde{q}} = m_{\tilde{\ell}} = m_{H^0} = m_{H^\pm} = m_A = 2000 \text{ TeV}, \;\; M_{\tilde{g}} = M_{\widetilde{W}} = \mu_H = 100 \text{ TeV}, \;\; \tan\beta \;=\; 50. \tag{61}$$

For the calculation of the quark Yukawa couplings, we adopt the following input values for quark masses and Cabibbo–Kobayashi–Maskawa (CKM) matrix parameters: The isospin-averaged quark mass and strange quark mass in $\overline{\text{MS}}$ scheme are obtained from lattice calculations in Refs. [40–45] as $1/2(m_u + m_d)(2 \text{ GeV}) = 3.373(80)$ MeV and $m_s(2 \text{ GeV}) = 92.0(2.1)$ MeV. The up and down quark mass ratio is obtained from an estimate in Ref. [46] as $m_u/m_d = 0.46(3)$. The $\overline{\text{MS}}$ charm and bottom quark masses are obtained from quantum chromodynamics (QCD) sum rule calculations

Table 1. The singular values of MSSM Yukawa couplings Y_u, Y_d, Y_e, and the mixing angles and charge conjugation parity (CP) phase of CKM matrix, at $\mu = 2 \times 10^{16}$ GeV in the $\overline{\rm DR}$ scheme. Also shown is the flavor-dependent RG correction R_{ij} for the Weinberg operator [defined in Eq. (61)] in the evolution from $\mu = 2 \times 10^{16}$ GeV to $\mu = M_Z$, in the flavor basis where Y_e is diagonal (R_{ij} is also diagonal in this basis). For each singular value of the quark Yukawa matrices, we present the 1σ error that has propagated from experimental error of the corresponding input quark mass, and for the CKM parameters we present 1σ errors that have propagated from experimental errors of the input Wolfenstein parameters.

	Value at $\mu = 2 \times 10^{16}$ GeV in $\overline{\rm DR}$ scheme
y_u	$2.74(14)\times 10^{-6}$
y_c	0.001407(14)
y_t	0.4620(84)
y_d	0.0002998(94)
y_s	0.00597(14)
y_b	0.3376(19)
y_e	0.00012486
y_μ	0.026364
y_τ	0.50319
$\cos\theta_{13}^{\rm ckm}\sin\theta_{12}^{\rm ckm}$	0.22475(25)
$\cos\theta_{13}^{\rm ckm}\sin\theta_{23}^{\rm ckm}$	0.0421(11)
$\sin\theta_{13}^{\rm ckm}$	0.00372(22)
$\delta_{\rm km}$ (rad)	1.147(33)
R_{ee}	1.00
$R_{\mu\mu}$	1.00
$R_{\tau\tau}$	0.974

in Ref. [47] as $m_c(3~\text{GeV}) = 0.986 - 9(\alpha_s^{(5)}(M_Z) - 0.1189)/0.002 \pm 0.010$ GeV and $m_b(m_b) = 4.163 + 7(\alpha_s^{(5)}(M_Z) - 0.1189)/0.002 \pm 0.014$ GeV. The top quark pole mass is obtained from $t\bar{t}$+jet events measured by ATLAS [48] as $M_t = 171.1 \pm 1.2$ GeV. The CKM mixing angles and CP phase are calculated from the Wolfenstein parameters in the latest CKM fitter result [49].[8] For the QCD and quantum electrodynamics gauge couplings, we use $\alpha_s^{(5)}(M_Z) = 0.1181$ and $\alpha^{(5)}(M_Z) = 1/127.95$. For the lepton and W, Z, Higgs pole masses, we use the values from the Particle Data Group [50].

The results are given in terms of the singular values of Y_u, Y_d, Y_e and the CKM mixing angles and CP phase at $\mu = 2 \times 10^{16}$ GeV, as well as R_{ij} in the flavor basis where Y_e is diagonal (R_{ij} is also diagonal in this basis), tabulated in Table 1. For each singular value of Y_u, Y_d, we present the 1σ error that has propagated from experimental error of the corresponding input quark mass. For the CKM mixing angles and CP phase, we present 1σ errors that have propagated from experimental errors of the input Wolfenstein parameters.

To facilitate the fitting analysis, we rearrange Eqs. (3)–(5) as follows. We fix the flavor basis such that the left-handed up-type quark components in both Ψ_i and Ψ_j have the diagonalized up-type quark Yukawa matrix with real positive components. Y_d, which is still symmetric, is then written as

$$Y_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\,a_2} & 0 \\ 0 & 0 & e^{i\,a_3} \end{pmatrix} V_{\rm CKM}^* \begin{pmatrix} y_d\,e^{2i\,b_1} & 0 & 0 \\ 0 & y_s\,e^{2i\,b_2} & 0 \\ 0 & 0 & y_b\,e^{2i\,b_3} \end{pmatrix} V_{\rm CKM}^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\,a_2} & 0 \\ 0 & 0 & e^{i\,a_3} \end{pmatrix}, \quad (62)$$

[8] Updated results and plots are available at `http://ckmfitter.in2p3.fr`.

where a_2, a_3, b_1, b_2, b_3 are unknown phases.[9] In the same flavor basis, Y_e is written from Eqs. (3)–(5) as

$$\frac{1}{r_1} Y_e = \frac{4}{1-r_2} \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix} - \frac{3+r_2}{1-r_2} \frac{1}{r_1} Y_d, \tag{63}$$

with Y_d given in Eq. (62). We can also write

$$Y_{126} \propto \frac{1}{r_1} Y_d - \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \tag{64}$$

$$Y_D = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix} - \frac{4r_2}{1-r_2} \left(\frac{1}{r_1} Y_d - \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix} \right). \tag{65}$$

Finally, we perform the singular value decomposition of Y_e as

$$Y_e = U_{eL} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix} U_{eR}^\dagger, \tag{66}$$

and calculate the active neutrino mass matrix (up to overall constant) as

$$(M_\nu)_{\ell\ell'} \ \propto \ R_{\ell\ell} \, [\, U_{eL}^T (r_L \, Y_{126} + Y_D Y_{126}^{-1} Y_D^T) U_{eL} \,]_{\ell\ell'} \, R_{\ell'\ell'}, \qquad \ell, \ell' = e, \mu, \tau, \tag{67}$$

where ℓ, ℓ' denote flavor indices for the left-handed charged leptons. From Eq. (67), we derive the three neutrino mixing angles $\theta_{12}^{\rm pmns}, \theta_{13}^{\rm pmns}, \theta_{23}^{\rm pmns}$ and the ratio of the neutrino masses $m_1 : m_2 : m_3$.

Now we perform the fitting with $Y_{10}, Y_{126}, r_1, r_2, r_L$. It proceeds as follows. We fix y_u, y_c, y_t and the CKM matrix by the values in Table 1, while we vary y_d/r_1, y_s/r_1, y_b/r_1, unknown phases a_2, a_3, b_1, b_2, b_3 in Eq. (62) and complex number r_2. Here we eliminate r_1 by requiring that the central value of the electron Yukawa coupling y_e be reproduced. In this way, we try to reproduce the correct values of y_d, y_s, y_μ, y_τ, $\theta_{12}^{\rm pmns}, \theta_{13}^{\rm pmns}, \theta_{23}^{\rm pmns}$ and neutrino mass difference ratio $\Delta m_{21}^2/\Delta m_{32}^2$. Specifically, we require y_d, y_s to fit within their respective 3σ ranges, while we do not constrain y_b because y_b can receive sizable SUSY particle and GUT-scale threshold corrections. Since the experimental errors of y_μ, y_τ are tiny, we only require that their reproduced values fit within $\pm 0.1\%$ ranges of their central values. We require $\sin^2\theta_{12}^{\rm pmns}$, $\sin^2\theta_{13}^{\rm pmns}$, $\sin^2\theta_{23}^{\rm pmns}$, $\Delta m_{21}^2/\Delta m_{32}^2$ to fit within their respective 3σ ranges reported by NuFIT 4.1 [51,52]. However, we do not constrain the Dirac CP phase $\delta_{\rm pmns}$, since its measurement is still at a primitive stage. We only consider the normal mass hierarchy case, because we cannot obtain a good fitting with the inverted mass hierarchy. We have confirmed that our fitting analysis always gives small values for m_1 that are not in tension with cosmological observations or searches for neutrinoless double-beta decay, and hence no constraint is imposed on α_2, α_3, m_1. The constraints are summarized in Table 2.

We collect sets of values of $Y_{10}, Y_{126}, r_1, r_2, r_L$ that satisfy the constraints of Table 2. From these values, we reconstruct the MSSM Yukawa couplings Y_u, Y_d, Y_e, perform flavor basis changes and

[9] Note that Y_d in Eq. (2) is the complex conjugate of Y_d in SM defined as $-\mathcal{L} = \bar{q}_L Y_d d_R \, i\sigma_2 H^*$.

Table 2. Allowed ranges of quantities in the analysis.

Quanitity	Allowed range
y_u	2.74×10^{-6} (fixed)
y_c	0.001407 (fixed)
y_t	0.4620 (fixed)
y_d	$0.0002998 \pm 0.0000094 \cdot 3$
y_s	$0.00597 \pm 0.00014 \cdot 3$
y_b	Unconstrained
y_e	0.00012486 (used to fix r_1)
y_μ	$0.026364 \pm 0.1\%$
y_τ	$0.50319 \pm 0.1\%$
$\cos\theta_{13}^{\rm ckm}\sin\theta_{12}^{\rm ckm}$	0.22475 (fixed)
$\cos\theta_{13}^{\rm ckm}\sin\theta_{23}^{\rm ckm}$	0.0421 (fixed)
$\sin\theta_{13}^{\rm ckm}$	0.00372 (fixed)
$\delta_{\rm km}$ (rad)	1.147 (fixed)
$\sin^2\theta_{12}^{\rm pmns}$	$[0.275,\ 0.350]$
$\sin^2\theta_{13}^{\rm pmns}$	$[0.02044,\ 0.02435]$
$\sin^2\theta_{23}^{\rm pmns}$	$[0.433,\ 0.609]$
$\Delta m_{21}^2/\Delta m_{32}^2$	$[0.0267,\ 0.0339]$
$\delta_{\rm pmns},\ \alpha_2,\ \alpha_3,\ m_1$	Unconstrained
$a_2,\ a_3,\ b_1,\ b_2,\ b_3$	Unconstrained
r_1	Eliminated in favor of y_e
r_2	Unconstrained

calculate the following components:

$$(Y_u)_{u_L d_L},\quad (Y_u)_{s_L \mu_L},\quad (Y_u)_{d_L s_L},\quad (Y_u)_{u_L \mu_L},$$
$$(Y_{126})_{u_L d_L},\quad (Y_{126})_{s_L \mu_L},\quad (Y_{126})_{d_L s_L},\quad (Y_{126})_{u_L \mu_L}.$$

From the values above, we calculate $\Gamma(p \to K^+\bar{\nu}_\mu)$ and $\Gamma(p \to K^0\mu^+)$ through Eqs. (15)–(22) and Eqs. (40)–(42), by considering various $O(1)$ values for coefficients a, b', c', d' in Eqs. (40)–(42). Here we take $M_{H_C} = 2\times10^{16}$ GeV and assume the SUSY particle mass spectrum of Eq. (61). We employ the following data and formulas. For the hadronic matrix element β_H, we adopt the value in Ref. [53], which reads $\beta_H = 0.0144$ GeV^3 at $\mu = 2$ GeV in the $\overline{\rm MS}$ scheme. The baryon chiral Lagrangian parameters are given by $D = 0.804$, $F = 0.463$, and we include the mass splittings among nucleon and hyperon masses found in Particle Data Group [50]. When computing RG corrections to the dimension-5 operators and the dimension-6 operators after Wino-dressing, we choose $\mu_{\rm SUSY} =$ 2000 TeV and $\mu_{H_C} = 2\times10^{16}$ GeV, and use one-loop formulas in Ref. [54].

4.3. Results

We have obtained 158 sets of values of $Y_{10}, Y_{126}, r_1, r_2, r_L$ that satisfy the constraints of Table 2.

Before presenting the main results, we show in Fig. 1 the distribution of r_2 in the fitting results, to confirm the relation $|r_2| = O(1)$ used in Sect. 3.

Now we plot the sets of values of $Y_{10}, Y_{126}, r_1, r_2, r_L$ satisfying Table 2, on the plane of $p \to K^+\bar{\nu}_\mu$ partial lifetime versus the ratio of the partial widths of $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$. From the plots, we read out the range of the partial width ratio predicted by the model.

We first study the contribution of individual terms in Eqs. (40)–(42) by taking $(a, b', c', d') =$ $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$. The plots are in Fig. 2. We caution that although some

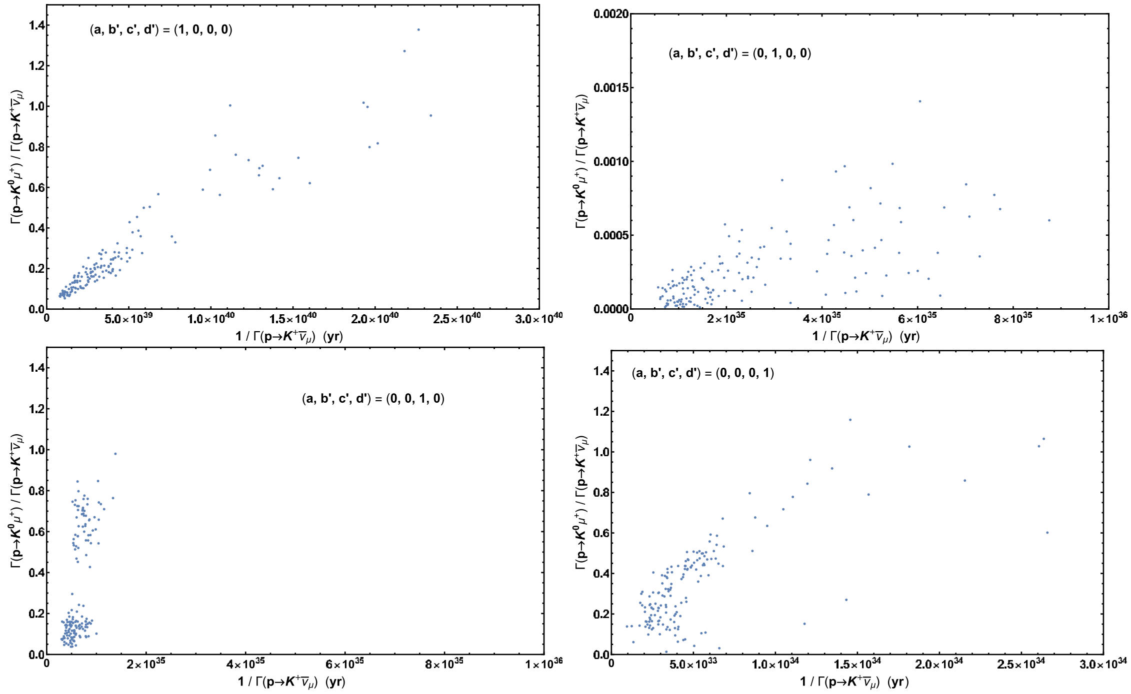

Fig. 2. $p \to K^+\bar{\nu}_\mu$ partial lifetime versus the ratio of the partial widths of $p \to K^0\mu^+$ and $p \to K^+\bar{\nu}_\mu$. Each dot corresponds to a set of values of $Y_{10}, Y_{126}, r_1, r_2, r_L$ that satisfy the constraints of Table 2. We take $(a, b', c', d') = (1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ in Eqs. (40)–(42). Note that the vertical scale of the panel of $(a, b', c', d') = (0, 1, 0, 0)$ is different because the partial width ratio is quite suppressed in this case. Also, the horizontal scale is different for the four panels, due to the large hierarchy of $p \to K^+\bar{\nu}_\mu$ partial lifetime in the four cases. Although some points are apparently excluded by the current 90% CL experimental bound $1/\Gamma(p \to K^+\nu) > 5.9 \times 10^{33}$ yr [55], these points are revived if (a, b', c', d') are reduced due to the mixing of $(\mathbf{3}, \mathbf{1}, -1/3)$, $(\bar{\mathbf{3}}, \mathbf{1}, 1/3)$ components of fields other than $H, \overline{\Delta}$, or if SUSY particles are slightly heavier than the spectrum of Eq. (61).

points are apparently excluded by the current 90% confidence limit (CL) experimental bound $1/\Gamma(p \to K^+\nu) > 5.9 \times 10^{33}$ yr [55], these points are revived if (a, b', c', d') are reduced due to the mixing of $(\mathbf{3}, \mathbf{1}, -1/3)$, $(\bar{\mathbf{3}}, \mathbf{1}, 1/3)$ components of fields other than $H, \overline{\Delta}$, or if SUSY particles are slightly heavier than the spectrum of Eq. (61) by factor $O(1)$.

We find that the predictions for $\Gamma(p \to K^+\bar{\nu}_\mu)$ in the cases with $(a, b', c', d') = (1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ exhibit the following hierarchy:

$$[\text{Case with } (1,0,0,0)] \ll [\text{Case with } (0,1,0,0)] \lesssim [\text{Case with } (0,0,1,0)] \ll [\text{Case with } (0,0,0,1)].$$

On the other hand, the predictions for the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ follow the following pattern:

$$[\text{Case with } (0,1,0,0)] \ll [\text{Case with } (1,0,0,0)] \sim [\text{Case with } (0,0,1,0)] \sim [\text{Case with } (0,0,0,1)].$$

From the above hierarchy patterns, we infer $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ for general values of (a, b', c', d') as follows.

- When $d' = O(1)$, the partial width $\Gamma(p \to K^+\bar{\nu}_\mu)$ is dominated by the contribution from the term with coefficient d'. Since the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ with $(a, b', c', d') = (0, 0, 0, 1)$ is comparable to or larger than in the other cases, we expect that $\Gamma(p \to K^0\mu^+)$ is also dominated by the contribution from the term with d'. Therefore, we

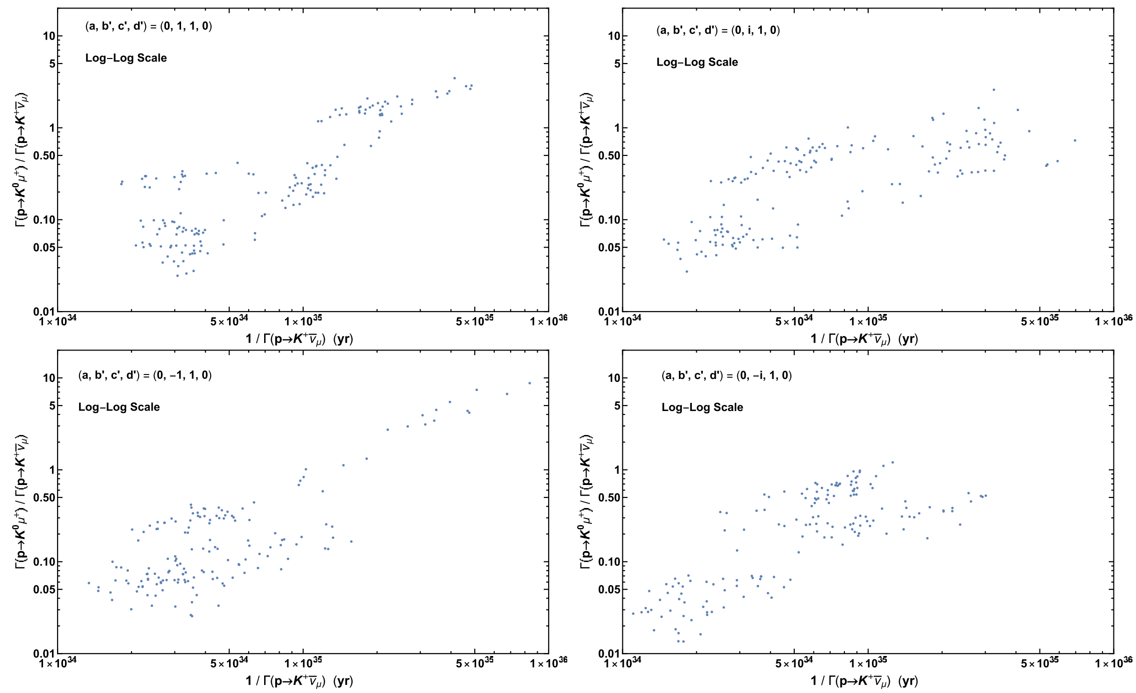

Fig. 3. Same as Fig. 2 except that we take $(a, b', c', d') = (0, 1, 1, 0)$, $(0, i, 1, 0)$, $(0, -1, 1, 0)$, $(0, -i, 1, 0)$ in Eqs. (40)–(42).

conclude that *when* $d' = O(1)$, *irrespectively of the values of* a, b', c', *the prediction on the partial width ratio is given by the lower right-hand panel of Fig. 2, where the partial width ratio mostly varies in the range* 0.05–0.6. This result is consistent with our estimate (59).

- When $d' = 0$, the partial width $\Gamma(p \to K^+\bar{\nu}_\mu)$ receives comparable contributions from the terms with c' and b'. On the other hand, since the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ with $(a, b', c', d') = (0, 1, 0, 0)$ is much smaller than that with $(a, b', c', d') = (0, 0, 1, 0)$, $\Gamma(p \to K^0\mu^+)$ receives contribution solely from the term with c'. Hence, when $c' = O(1)$ and $b' = O(1)$, the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is suppressed if the contributions of the terms with c' and b' to $\Gamma(p \to K^+\bar{\nu}_\mu)$ interfere constructively, and the partial width ratio is enhanced if they interfere destructively. To examine these possibilities, we present plots for cases with $(a, b', c', d') = (0, 1, 1, 0)$, $(0, i, 1, 0)$, $(0, -1, 1, 0)$, $(0, -i, 1, 0)$ in Fig. 3.
  We observe that *when* $d' = 0$, $c' = O(1)$ *and* $b' = O(1)$, *the prediction on the partial width ratio varies considerably with the relative phase of* b' *and* c' *and with different fitting results.* Still, we can assert that the ratio is above 0.01. The absence of strong suppression factor $0.3 \cdot 0.002$ is consistent with our estimate (59).
- When $d' = b' = 0$, both $\Gamma(p \to K^+\bar{\nu}_\mu)$ and $\Gamma(p \to K^0\mu^+)$ are dominated by the contribution from the term with c'. We thus conclude that *when* $d' = b' = 0$, *irrespectively of the value of* a, *the prediction on the partial width ratio is given by the lower left-hand panel of Fig. 2, where it varies in the ranges* 0.03–0.2 *and* 0.4–0.8.
- When $d' = c' = 0$, the partial width $\Gamma(p \to K^+\bar{\nu}_\mu)$ is dominated by the contribution from the term with b'. On the other hand, since the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is much larger with $(a, b', c', d') = (1, 0, 0, 0)$ than with $(a, b', c', d') = (0, 1, 0, 0)$, $\Gamma(p \to K^0\mu^+)$ might receive a larger contribution from the term with a than from the term with b'. However, we

have inspected cases with $(a, b', c', d') = (1, 1, 0, 0)$, $(i, 1, 0, 0)$, $(-1, 1, 0, 0)$ and $(-i, 1, 0, 0)$ and found that the distribution in these cases is almost identical to that with $(a, b', c', d') = (0, 1, 0, 0)$. We thus conclude that *when $d' = c' = 0$, irrespectively of the value of a, the prediction on the partial width ratio is given by the upper right-hand panel of Fig. 2, where it is mostly suppressed below* 0.0005. This result agrees with our estimate (60).

◦ Only in the very special case with $d' = c' = b' = 0$ do we obtain the distribution of the upper left-hand panel of Fig. 2, where the ratio is above 0.05.

To summarize, if $d' = O(1)$, the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is mostly in the range 0.05–0.6. If $d' = 0$, $c' = O(1)$ and $b' = O(1)$, the partial width ratio varies in a wide range, still it is above 0.01. If $d' = b' = 0$ and $c' = O(1)$, it is in the ranges 0.03–0.2 and 0.4–0.8. If $d' = c' = b' = 0$, it is above 0.05. Only when $d' = c' = 0$ and $b' = O(1)$ is the partial width ratio mostly highly suppressed below 0.0005.

Because there is no particular reason to believe $d' = 0$, our most important result is the lower right-hand panel of Fig. 2, which covers the case with $d' = O(1)$. Accordingly, our main prediction is

$$0.6 \gtrsim \frac{\Gamma(p \to K^0\mu^+)}{\Gamma(p \to K^+\bar{\nu}_\mu)} \gtrsim 0.05. \tag{68}$$

Considering the current 90% CL bound $1/\Gamma(p \to K^+\nu) > 5.9 \times 10^{33}$ yr [55], we can at best observe the $p \to K^0\mu^+$ decay at a rate $1/\Gamma(p \to K^0\mu^+) = 1 \times 10^{34}$ yr.

5. Summary

The ratio of the partial widths of some dimension-5 proton decay modes can be predicted without knowledge of SUSY particle masses, and thus serves as a probe for various SUSY GUT models even when SUSY particles are not discovered. We have focused on the partial width ratio $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ in the minimal renormalizable SUSY $SO(10)$ GUT. In the model, the Wilson coefficients of dimension-5 operators responsible for the $p \to K^0\mu^+$ and the $p \to K^+\bar{\nu}_\mu$ decays are on the same order, and $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu)$ is largely determined by the ratio of baryon chiral Lagrangian parameters and is estimated to be $O(0.1)$. This is in striking contrast to the minimal $SU(5)$ GUT, where this partial width ratio is further suppressed by a factor of $y_u^2/(\lambda^2 y_c)^2 \simeq 0.002$. To confirm that $\Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu) = O(0.1)$ in the minimal renormalizable SUSY $SO(10)$ GUT, we have numerically determined Y_{10}, Y_{126} through a fitting of the quark and charged lepton Yukawa couplings and neutrino mass matrix, and calculated the partial width ratio based on the fitting results. Our most important finding is that the partial width ratio generally varies in the range $0.6 \gtrsim \Gamma(p \to K^0\mu^+)/\Gamma(p \to K^+\bar{\nu}_\mu) \gtrsim 0.05$ in the most generic case where $d' = O(1)$ in Eqs. (40)–(42).

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