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DISCRETE MULTI-HARMONIC GREEN FUNCTIONS

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Abstract. The harmonic Green function g_a of an infinite network defined as the unique Dirichlet potential which satisfies $\Delta g_a = -\delta_a$. The biharmonic Green function $g_a^{(2)}(x)$ is defined by the convolution of g_x and g_a in [6]. It is known that $\Delta^2 g_a^{(2)} = \delta_a$ if $g_a^{(2)}$ is finite and that $g_a^{(2)}$ is a Dirichlet potential if g_a has a finite Green energy. In this paper, we define the k-harmonic Green function $g_a^{(k)}(x)$ as the convolution of $g_x^{(k-1)}$ and g_a if it converges. We study some potential theoretic properties related to $g_a^{(k)}$.

1. Introduction with Preliminaries

Let $\mathcal{N} = \langle V, E, K, r\rangle$ be an infinite network which is connected and locally finite and has no self-loop, where V is the set of nodes, E is the set of arcs, and the resistance r is a strictly positive function on E. For $x \in V$ and for $e \in E$ the node-arc incidence matrix K is defined by $K(x, e) = 1$ if x is the initial node of e; $K(x, e) = -1$ if x is the terminal node of e; $K(x, e) = 0$ otherwise. Let $L(V)$ be the set of all real valued functions on V, $L^+(V)$ the set of all non-negative real valued functions on V, and $L_0(V)$ the set of all $u \in L(V)$ with finite support. We similarly define $L(E)$, $L^+(E)$, and $L_0(E)$. For $u \in L(V)$ we define the *discrete derivative* $\nabla u \in L(E)$ and the *Laplacian* $\Delta u \in L(V)$ as

$$\nabla u(e) = -r(e)^{-1} \sum_{x\in V} K(x, e)u(x),$$

$$\Delta u(x) = \sum_{e\in E} K(x, e)\nabla u(e).$$

For convenience we give specific forms. For $e \in E$ let $x^+ \in V$ be the initial node of e and $x^- \in V$ the terminal node of e. Then

$$\nabla u(e) = \frac{u(x^-) - u(x^+)}{r(e)}.$$

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For $x \in V$ let $\{e_1, \dots, e_d\}$ be the set of arcs adjacent to x and let y_j be the other node of e_j for each j. Then

$$\Delta u(x) = \sum_{j=1}^{d} \frac{u(y_j) - u(x)}{r(e_j)}.$$

We denote by

$$\Delta^0 u = u, \qquad \Delta^k u = \Delta(\Delta^{k-1} u)$$

for $k \in \mathbb{N}$. For $u, v \in L(V)$ we put

$$\begin{gathered} \langle u, v\rangle_{\mathbf{D}} = \sum_{e \in E} r(e)\nabla u(e)\nabla v(e), \\ \|u\|_{\mathbf{D}} = \langle u, u\rangle_{\mathbf{D}}^{1/2} \quad \text{(Dirichlet sum)}, \\ \langle u, v\rangle_{l^2} = \sum_{x \in V} u(x)v(x), \\ \|u\|_{l^2} = \langle u, u\rangle_{l^2}^{1/2}. \end{gathered}$$

We define two classes of functions on V as

$$\begin{gathered} \mathbf{D} = \{u \in L(V) \mid \|u\|_{\mathbf{D}} < \infty\}, \\ \mathbf{H}^{(k)} = \{u \in L(V) \mid \Delta^k u = 0 \text{ on } V\}. \end{gathered}$$

Note that $\langle u, v\rangle_{\mathbf{D}}$ is a degenerate bilinear form in $\mathbf{D}$; for example, $\langle 1, u\rangle_{\mathbf{D}} = 0$ and $\|u + 1\|_{\mathbf{D}} = \|u\|_{\mathbf{D}}$ for $u \in \mathbf{D}$, where 1 stands for the constant function. It was shown in [5, Theorem 1.1] that $\mathbf{D}$ is a Hilbert space with respect to the inner product $\langle u, v\rangle_{\mathbf{D}} + u(o)v(o)$ for a fixed node $o \in V$. We easily verify that a sequence $\{u_n\}_n \subset \mathbf{D}$ converges to u in $\mathbf{D}$ if and only if $\lim_{n\to\infty} \|u_n - u\|_{\mathbf{D}} = 0$ and $\{u_n\}_n$ converges pointwise to u. Denote by $\mathbf{D}_0$ the closure of $L_0(V)$ in $\mathbf{D}$. We call a function in $\mathbf{D}$ and in $\mathbf{D}_0$ a *Dirichlet function* and a *Dirichlet potential*, respectively.

We always assume that the network is hyperbolic, i.e., for each $a \in V$ there exists the harmonic Green function g_a with pole at a. Also we assume that the network satisfies the following condition: There exists a constant $c_{\mathrm{LD}} > 0$ such that

$$\text{(LD)} \qquad \|\Delta f\|_{\mathbf{D}} \le c_{\mathrm{LD}} \|f\|_{\mathbf{D}} \qquad \text{for all } f \in L_0(V).$$

We define the k-harmonic Green function $g_a^{(k)}(x)$ in Section 2 as a convolution of $g_x^{(k-1)}$ and g_a and study some fundamental relations between these functions under the assumption that $g_a^{(k)}$ is finite. In Section 3 we see that $g_a^{(k)}$ is a Dirichlet potential if $\mathcal{N}$ satisfies conditions (LD) and (CLD) studied in [2]. Some potential theoretic results related to multi-harmonic Green functions in Section 4. We propose some sufficient conditions which assure the finiteness of $g_a^{(k)}$ in case where $\mathcal{N}$ does not satisfy (LD) or (CLD) in Section 5. As for the limit of $\{g_a^{(k)}\}_k$, we show a partial result in Section 6. The explicit form of $g_a^{(2)}$ is given in Section 7 for the infinite linear network and in Section 8 for the homogeneous tree of order three.

2. Multi-Harmonic Green Functions

We construct an *multi-harmonic Green function* as follows. For $a \in V$ let $g_a^{(0)} = \delta_a$, where δ_a is the characteristic function of the singleton $\{a\}$, and let $g_a^{(1)}$ be the harmonic Green function g_a of $\mathcal{N}$ with pole at $a \in V$, i.e., $g_a \in \mathbf{D}_0$ is a unique function with $\Delta g_a = -\delta_a$. For $k \geq 2$ we define the k-harmonic Green function $g_a^{(k)}$ of $\mathcal{N}$ with pole at a as the convolution of $g_x^{(k-1)}$ and g_a, i.e.,

$$g_a^{(k)}(x) = \langle g_x^{(k-1)}, g_a \rangle_{l^2}$$

if it converges.

Theorem 2.1. *If $g_a^{(k)}$ is finite, then $g_a^{(k)}(b) = g_b^{(k)}(a)$ for every $a, b \in V$.*

Proof. It is obvious for $k = 0$. It is well-known that $g_a(b) = g_b(a)$ holds, which shows the case $k = 1$. Let $k \geq 2$ and assume $g_a^{(j)}(b) = g_b^{(j)}(a)$ for $j \leq k-1$ and for $a, b \in V$. Then

$$\begin{aligned} g_a^{(k)}(b) &= \sum_{x \in V} g_b^{(k-1)}(x) g_a(x) = \sum_{x \in V} g_a(x) \sum_{y \in V} g_x^{(k-2)}(y) g_b(y) \\ &= \sum_{y \in V} \sum_{x \in V} g_a(x) g_y^{(k-2)}(x) g_b(y) = \sum_{y \in V} g_a^{(k-1)}(y) g_b(y) \\ &= g_b^{(k)}(a). \end{aligned}$$

□

Corollary 2.2. $g_a^{(k)}(x) = \langle g_a^{(k-1)}, g_x \rangle_{l^2}$.

Lemma 2.3. *If $g_a^{(k)}$ is finite, then $g_a^{(k)}$ is superharmonic and satisfies $\Delta g_a^{(k)} = -g_a^{(k-1)}$ for $k \geq 1$.*

Proof. Theorem 2.1 shows that

$$\begin{aligned} \Delta g_a^{(k)}(x) &= \sum_{y \in V} (\Delta g_y(x)) g_a^{(k-1)}(y) = -\sum_{y \in V} \delta_y(x) g_a^{(k-1)}(y) \\ &= -g_a^{(k-1)}(x) \leq 0 \end{aligned}$$

as required. □

Theorem 2.4. *If $g_a^{(k)}$ is finite, then $\Delta^k g_a^{(k)} = (-1)^k \delta_a$.*

Proof. Applying Lemma 2.3 repeatedly we have the result. □

Proposition 2.5. *If $g_a^{(k+l)}$ is finite, then $g_a^{(k+l)}(b) = \langle g_a^{(k)}, g_b^{(l)} \rangle_{l^2}$.*

Proof. We show it by induction on l. It is obvious for each k if $l = 0$. We assume the assertion holds for $l - 1$. Then

$$\begin{aligned} g_a^{(k+l)}(b) &= \sum_{x\in V} g_a^{(k+l-1)}(x) g_b(x) = \sum_{x\in V}\sum_{y\in V} g_a^{(k)}(y) g_x^{(l-1)}(y) g_b(x) \\ &= \sum_{y\in V} g_a^{(k)}(y) \sum_{x\in V} g_y^{(l-1)}(x) g_b(x) = \sum_{y\in V} g_a^{(k)}(y) g_b^{(l)}(y) \\ &= \langle g_a^{(k)}, g_b^{(l)}\rangle_{l^2}. \end{aligned}$$

□

Proposition 2.6. *If $g_a^{(k)}, g_b^{(l)} \in \mathbf{D}_0$, then $\langle g_a^{(k)}, g_b^{(l)}\rangle_{\mathbf{D}} = g_a^{(k+l-1)}(b)$.*

Proof. Note that $\langle u, v\rangle_{\mathbf{D}} = -\langle u, \Delta v\rangle_{l^2}$ for $u, v \in \mathbf{D}_0$ (see [4, Lemma 3]). Using Lemma 2.3 and Proposition 2.5, we obtain

$$\langle g_a^{(k)}, g_b^{(l)}\rangle_{\mathbf{D}} = -\langle g_a^{(k)}, \Delta g_b^{(l)}\rangle_{l^2} = \langle g_a^{(k)}, g_b^{(l-1)}\rangle_{l^2} = g_a^{(k+l-1)}(b).$$

□

3. Conditions for $g_a^{(k)} \in \mathbf{D}_0$

For $\mu, \nu \in L^+(V)$ we define the *Green potential* $G\mu$ and the *mutual Green energy* $G(\mu, \nu)$ by

$$\begin{aligned} G\mu(x) &= \langle g_x, \mu\rangle_{l^2}, \\ G(\mu, \nu) &= \langle G\mu, \nu\rangle_{l^2}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{M} &= \{\mu \in L^+(V) \mid G\mu < \infty \text{ on } V\}, \\ \mathcal{E} &= \{\mu \in L^+(V) \mid G(\mu, \mu) < \infty\}. \end{aligned}$$

Corollary 2.2 shows that

Lemma 3.1. $g_a^{(k)} = G g_a^{(k-1)}$.

We recall a lemma.

Lemma 3.2 ([6, Lemma 3.1])**.** *Let $\mu \in L^+(V)$. If $G\mu \in \mathbf{D}$, then $\mu \in \mathcal{E}$, $G\mu \in \mathbf{D}_0$, and $\|G\mu\|_{\mathbf{D}}^2 = G(\mu, \mu)$.*

Proposition 3.3. *$g_a^{(k)} \in \mathbf{D}$ implies $g_a^{(k)} \in \mathbf{D}_0$.*

Proof. Since $g_a^{(k-1)} \in L^+(V)$, Lemmas 3.1 and 3.2 show the assertion. □

Theorem 3.4. *$g_a^{(k)} \in \mathbf{D}_0$ if and only if $g_a^{(k-1)} \in \mathcal{E}$. In this case the formula $\|g_a^{(k)}\|_{\mathbf{D}}^2 = G(g_a^{(k-1)}, g_a^{(k-1)})$ holds.*

Proof. First assume $g_a^{(k)} \in \mathbf{D}_0$. Lemmas 3.1 and 3.2 show $g_a^{(k-1)} \in \mathcal{E}$. Next assume $g_a^{(k-1)} \in \mathcal{E}$. Then $G g_a^{(k-1)} \in \mathbf{D}_0$ by [5, Theorem 5.2]. Lemma 3.1 implies $g_a^{(k)} \in \mathbf{D}_0$. In this case Lemma 3.2 shows $\|g_a^{(k)}\|_{\mathbf{D}}^2 = G(g_a^{(k-1)}, g_a^{(k-1)})$. □

Lemma 3.5. $\mathbf{D}_0 \cap \mathbf{H}^{(k)} = \{0\}$.

Proof. Let $u \in \mathbf{D}_0 \cap \mathbf{H}^{(k)}$ and $u_j = \Delta^j u$ for $0 \le j \le k$. Then $u_j \in \mathbf{D}_0$ by [1, Lemma 3.1]. Since $u_k = 0$, we have $u_{k-1} \in \mathbf{D}_0 \cap \mathbf{H}^{(1)}$, so that $u_{k-1} = 0$ by [5, Lemma 1.3]. Repeating this argument we have $u = u_0 = 0$ □

Corollary 3.6. *If $g_a^{(k-1)} \in \mathcal{E}$, then $g_a^{(k)}$ is the unique function $u \in \mathbf{D}_0$ with $\Delta^k u = (-1)^k \delta_a$.*

Proof. Theorems 3.4 and 2.4 show that $g_a^{(k)} \in \mathbf{D}_0$ and $\Delta^k g_a^{(k)} = (-1)^k \delta_a$. Lemma 3.5 implies the uniqueness. □

We introduced in [2] the following condition: There exists a constant $c_{\mathrm{CLD}} > 0$ such that

$$\text{(CLD)} \qquad \|f\|_{\mathbf{D}} \le c_{\mathrm{CLD}} \|\Delta f\|_{\mathbf{D}} \qquad \text{for all } f \in L_0(V).$$

We need a lemma.

Lemma 3.7 ([2, Theorem 3.2])**.** *If* (LD) *and* (CLD) *are fulfilled, then* $\mathbf{D}_0 \cap L^+(V) = \mathcal{E}$.

Theorem 3.8. *If* (LD) *and* (CLD) *are fulfilled, then* $g_a^{(k)} \in \mathcal{E}$ *for* $k \in \mathbb{N}$.

Proof. Since $g_a^{(1)} \in \mathbf{D}_0$, Lemma 3.7 implies $g_a^{(1)} \in \mathcal{E}$. Theorem 3.4 shows $g_a^{(2)} \in \mathbf{D}_0$. Repeating this argument, we have our assertion. □

4. Multi-Harmonic Green Potential

We define the *k-harmonic Green potential* $G^{(k)}\mu$ of $\mu \in L^+(V)$ and the *mutual k-harmonic Green energy* $G^{(k)}(\mu, \nu)$ of $\mu, \nu \in L^+(V)$ as

$$G^{(k)}\mu(x) = \langle g_x^{(k)}, \mu \rangle_{l^2},$$
$$G^{(k)}(\mu, \nu) = \langle G^{(k)}\mu, \nu \rangle_{l^2}.$$

It is obvious to see that

$$G^{(1)}\mu = G\mu, \qquad G^{(1)}(\mu, \nu) = G(\mu, \nu).$$

We put

$$\mathcal{M}^{(k)} = \{\mu \in L^+(V) \mid G^{(k)}\mu < \infty \text{ on } V\},$$
$$\mathcal{E}^{(k)} = \{\mu \in L^+(V) \mid G^{(k)}(\mu, \mu) < \infty\}.$$

Proposition 4.1. *For $\mu, \nu \in L^+(V)$ we have*

(1) $\Delta G^{(k)}\mu = -G^{(k-1)}\mu$ *if* $\mu \in \mathcal{M}^{(k)}$;
(2) $G^{(k+l)}\mu = G^{(k)}G^{(l)}\mu$ *if* $\mu \in \mathcal{M}^{(k+l)}$;
(3) $G^{(k+l)}(\mu, \nu) = \langle G^{(k)}\mu, G^{(l)}\nu \rangle_{l^2}$.

Proof. Lemma 2.3 shows

$$\begin{aligned}\Delta G^{(k)}\mu(x) &= \Delta \sum_{y\in V} g_x^{(k)}(y)\mu(y) = \sum_{y\in V}(\Delta g_y^{(k)}(x))\mu(y) \\ &= -\sum_{y\in V} g_y^{(k-1)}(x)\mu(y) = -G^{(k-1)}\mu(x).\end{aligned}$$

Proposition 2.5 implies

$$\begin{aligned}G^{(k+l)}\mu(x) &= \sum_{y\in V} g_x^{(k+l)}(y)\mu(y) = \sum_{y\in V}\sum_{z\in V} g_x^{(k)}(z)g_y^{(l)}(z)\mu(y) \\ &= \sum_{z\in V} g_x^{(k)}(z)G^{(l)}\mu(z) = G^{(k)}G^{(l)}\mu(x).\end{aligned}$$

By Proposition 2.5 again

$$\begin{aligned}G^{(k+l)}(\mu,\nu) &= \sum_{x\in V}\sum_{y\in V} g_x^{(k+l)}(y)\mu(x)\nu(y) \\ &= \sum_{x\in V}\sum_{y\in V}\sum_{z\in V} g_x^{(k)}(z)g_y^{(l)}(z)\mu(x)\nu(y) \\ &= \sum_{z\in V}(G^{(k)}\mu(z))(G^{(l)}\nu(z)).\end{aligned}$$

□

Corollary 4.2. *The following statements hold:*

(1) $\mu \in \mathcal{M}^{(k+l)}$ *if and only if* $G^{(l)}\mu \in \mathcal{M}^{(k)}$*;*
(2) $g_a^{(k+l)}$ *is finite if and only if* $g_a^{(l)} \in \mathcal{M}^{(k)}$*.*
(3) $\mu \in \mathcal{E}^{(k+l)}$ *if and only if* $\langle G^{(k)}\mu, G^{(l)}\mu\rangle_{l^2}$ *converges.*

Proof. Proposition 4.1 (2) and (3) immediately show (1) and (3). Using Lemma 3.1 and Proposition 4.1 (2) we have $g_a^{(k+l)} = G^{(k)}g_a^{(l)}$, which shows (2). □

5. Sufficient Conditions for the Finiteness of Multi-Harmonic Green Functions

Even in the case where $\mathcal{N}$ does not satisfy (LD) or (CLD), some conditions are sufficient to assure the finiteness of multi-harmonic Green function. We say that condition (GB) is fulfilled if

$$\text{(GB)} \qquad c_{\mathrm{GB}} := \sup_{x\in V} G1(x) = \sup_{x\in V}\sum_{y\in V} g_x(y) < \infty.$$

Proposition 5.1. (GB) *implies* $g_x^{(k)}(y) \le c_{\mathrm{GB}}^k$ *for* $k \ge 1$.

Proof. Let (GB) be fulfilled. We show that $g_x^{(k)}(y) \le c_{\mathrm{GB}}^k$ for $x, y \in V$ by induction on k. The base case $k = 1$ is trivial. Assume that $g_x^{(k-1)}(y) \le c_{\mathrm{GB}}^{k-1}$ for $x, y \in V$.

Then

$$g_x^{(k)}(y) = \sum_{z\in V} g_x^{(k-1)}(z)g_y(z) \le c_{\mathrm{GB}}^{k-1}\sum_{z\in V} g_y(z) = c_{\mathrm{GB}}^{k-1}G1(y) \le c_{\mathrm{GB}}^{k}.$$

□

Let

$$\beta(x) = \sum_{y\in V} g_x(y)^2$$

and consider a condition

$$B := \sum_{x\in V}\beta(x) < \infty. \tag{SG}$$

Proposition 5.2. (SG) *implies* $g_x^{(k)}(y)^2 \le \beta(x)\beta(y)B^{k-2}$ *for* $x, y \in V$ *and for* $k \ge 2$.

Proof. We show the assertion by induction on k. If $k = 2$, then

$$\begin{aligned} g_x^{(2)}(y)^2 &= \left(\sum_{z\in V} g_x(z)g_y(z)\right)^2 \le \left(\sum_{z\in V} g_x(z)^2\right)\left(\sum_{z\in V} g_y(z)^2\right) \\ &= \beta(x)\beta(y). \end{aligned}$$

Suppose that $g_x^{(k-1)}(y)^2 \le \beta(x)\beta(y)B^{k-3}$ for $x, y \in V$. Then

$$\begin{aligned} g_x^{(k)}(y)^2 &= \left(\sum_{z\in V} g_x(z)g_y^{(k-1)}(z)\right)^2 \le \left(\sum_{z\in V} g_x(z)^2\right)\left(\sum_{z\in V} g_y^{(k-1)}(z)^2\right) \\ &\le \beta(x)\sum_{z\in V}\beta(y)\beta(z)B^{k-3} = \beta(x)\beta(y)B^{k-2}. \end{aligned}$$

□

Corollary 5.3. (SG) *implies* $g_x^{(k)}(y) \le B^{k/2}$ *for* $x, y \in V$ *and for* $k \ge 2$.

Proposition 5.4. $1 \in \mathcal{E}$ *implies* $g_x \in \mathcal{E}$ *for* $x \in V$.

Proof. Since $g_x(y) \le g_x(x)$ for all $y \in V$,

$$G(g_x, g_x) = \sum_{y\in V}\sum_{z\in V} g_y(z)g_x(z)g_x(y) \le g_x(x)^2 G(1,1).$$

□

Proposition 5.5. (GB) *implies* $g_x \in \mathcal{E}$ *for* $x \in V$.

Proof. (GB) shows $g_y(z) \le c_{\mathrm{GB}}$, and that

$$G(g_x, g_x) = \sum_{y\in V}\sum_{z\in V} g_y(z)g_x(y)g_x(z) \le c_{\mathrm{GB}}\sum_{y\in V} g_x(y)\sum_{z\in V} g_x(z) \le c_{\mathrm{GB}}^3.$$

□

Proposition 5.6. (SG) *implies* $g_x \in \mathcal{E}$ *for* $x \in V$.

Proof.

$$\begin{aligned}
G(g_x, g_x)^2 &= \left(\sum_{y\in V}\sum_{z\in V} g_y(z)g_x(y)g_x(z)\right)^2 \\
&\leq \left(\sum_{y\in V}\sum_{z\in V} g_y(z)^2\right)\left(\sum_{y\in V}\sum_{z\in V} g_x(y)^2 g_x(z)^2\right) \\
&= \left(\sum_{y\in V} \beta(y)\right)\beta(x)^2 \leq B^3.
\end{aligned}$$

□

6. The Limit of Multi-Harmonic Green Functions

We give some general results related to the limit function of $\{g_a^{(k)}\}_k$. Propositions 5.1 and 5.2 show

Theorem 6.1. *If $c_{\mathrm{GB}} < 1$ or $B < 1$, then $\lim_{k\to\infty} g_x^{(k)}(y) = 0$.*

Lemma 6.2. *Let $\{u_k\}_k$ be a sequence of non-negative superharmonic functions on V. If $\lim_{k\to\infty} u_k(x_0) = \infty$ for some $x_0 \in V$, then $\lim_{k\to\infty} u_k(x) = \infty$ for each $x \in V$.*

Proof. Let x_1 be a node adjacent to x_0. Let $\{e_0, \ldots, e_{d-1}\}$ be the set of arcs adjacent to x_1 and let y_j be the other node of e_j for each j. We assume $y_0 = x_0$. Since $\Delta u_k(x_1) = \sum_{j=0}^{d-1} r(e_j)^{-1}(u_k(y_j) - u_k(x_1)) \leq 0$, we have

$$r(e_0)^{-1}u_k(x_0) \leq \sum_{j=0}^{d-1} r(e_j)^{-1}u_k(y_j) \leq \left(\sum_{j=0}^{d-1} r(e_j)^{-1}\right) u_k(x_1),$$

which implies $\lim_{k\to\infty} u_k(x_1) = \infty$. Repeating this argument, we have the assertion. □

Proposition 6.3. *If $g_{x_0}^{(k_0)}(x_0) > 1$ for some $x_0 \in V$ and $k_0 \in \mathbb{N}$, then $\lim_{k\to\infty} g_x^{(k)}(y) = \infty$ for each $x, y \in V$.*

Proof. By Proposition 2.5 we have

$$g_x^{(k)}(x) = \sum_{y\in V} g_x^{(k-l)}(y)g_x^{(l)}(y) \geq g_x^{(k-l)}(x)g_x^{(l)}(x)$$

for $l < k$. Let $\alpha = g_{x_0}^{(k_0)}(x_0) > 1$. For each $k \in \mathbb{N}$ we take $p, q \in \mathbb{N} \cup \{0\}$ such that $k = pk_0 + q$ and $0 \leq q < k_0$. We have

$$g_{x_0}^{(k)}(x_0) \geq \alpha^p \cdot g_{x_0}^{(q)}(x_0) \geq \alpha^p \cdot \min_{0\leq q<k_0} g_{x_0}^{(q)}(x_0) \to \infty$$

as $k \to \infty$. Lemma 6.2 implies $\lim_{k\to\infty} g_x^{(k)}(x_0) = \lim_{k\to\infty} g_{x_0}^{(k)}(x) = \infty$. Using Lemma 6.2 again, we have $\lim_{k\to\infty} g_x^{(k)}(y) = \infty$. □

Let

$$\lambda(\mathcal{N}) = \inf\left\{ \frac{\|f(x)\|_{\mathbf{D}}^2}{\|f(x)\|_{l^2}^2} \mid f \in L_0(V) \right\}.$$

and recall the following:

Lemma 6.4 ([3, Theorem 3.3])**.** *The largest number of $\lambda \geq 0$ for which the equation $\Delta u + \lambda u = 0$ has a positive solution is equal to $\lambda(\mathcal{N})$.*

Theorem 6.5. *Assume $\lambda(\mathcal{N}) < 1$. If $u(x) := \lim_{k\to\infty} g_{x_0}^{(k)}(x)$ is finite for each $x \in V$, then $u = 0$.*

Proof. Lemma 2.3 shows $\Delta u = -u$, which means that the equation $\Delta u + \lambda u = 0$ has a non-negative solution for $\lambda = 1$. Lemma 6.4 implies $u = 0$. □

Lemma 6.6. *If $g_a \leq g_a^{(2)}$ on V, then $g_a^{(k)} \leq g_a^{(k+1)}$ on V. Especially, $u(x) = \lim_{k\to\infty} g_a^{(k)}(x)$ exists for each $x \in V$ and $0 < u(x) \leq \infty$.*

Proof. We show the assertion by induction on k. The base case $k = 1$ is the assumption. Assume that $g_a^{(k-1)} \leq g_a^{(k)}$ on V. Then

$$g_a^{(k)}(x) = \langle g_a^{(k-1)}, g_x\rangle_{l^2} \leq \langle g_a^{(k)}, g_x\rangle_{l^2} = g_a^{(k+1)}(x).$$

Let $u(x) = \lim_{k\to\infty} g_a^{(k)}(x)$. Then $u(x) \geq g_a(x) > 0$. □

7. The Case of the Linear Network

Let $\mathcal{N} = \langle V, E, K, r\rangle$ be the linear network; $V = \{x_n \mid n \geq 0\}$ and $E = \{e_n \mid n \geq 1\}$. Let $K(x_{n-1}, e_n) = 1$ and $K(x_n, e_n) = -1$ for each $n \geq 1$, and let $K(x, e) = 0$ for any other pairs. Suppose $\sum_{j=1}^{\infty} r(e_j) < \infty$ and let $\rho_n := \sum_{j=n+1}^{\infty} r(e_j)$. Then

$$g_{x_m}(x_n) = \begin{cases} \rho_m & \text{if } 0 \leq n < m; \\ \rho_n & \text{if } n \geq m, \end{cases}$$

$$g_{x_m}^{(2)}(x_n) = \begin{cases} n\rho_m\rho_n + \rho_m \sum_{j=n}^{m-1} \rho_j + \sum_{j=m}^{\infty} \rho_j^2 & \text{if } 0 \leq n < m; \\ m\rho_m^2 + \sum_{j=m}^{\infty} \rho_j^2 & \text{if } n = m; \\ m\rho_m\rho_n + \rho_n \sum_{j=m}^{n-1} \rho_j + \sum_{j=n}^{\infty} \rho_j^2 & \text{if } n > m, \end{cases}$$

$$g_{x_0}^{(3)}(x_0) = \sum_{j=0}^{\infty} \rho_j^2 \rho_0 + \sum_{n=1}^{\infty} \left(\rho_n \sum_{j=0}^{n-1} \rho_j + \sum_{j=n}^{\infty} \rho_j^2 \right) \rho_n.$$

7.1. **In the Case $r(e_n) = n^{-\alpha}$.** We show $g_a^{(2)} \in L(V) \setminus \mathbf{D}_0$.

Proposition 7.1. *If $r(e_n) = n^{-\alpha}$ for $n \geq 1$ with $3/2 < \alpha < 5/3$, then $g_{x_m}^{(2)}(x_n) < \infty$ for each m and n and $g_{x_0}^{(2)} \notin \mathbf{D}_0$.*

Proof. Observe that

$$\frac{(n+1)^{1-\alpha}}{\alpha-1} \le \rho_n \le \frac{n^{1-\alpha}}{\alpha-1}, \tag{1}$$

$$\sum_{j=0}^{n-1} \rho_j \ge \frac{1}{(\alpha-1)(2-\alpha)}((n+1)^{2-\alpha}-1), \tag{2}$$

$$\rho_n^2 \sum_{j=0}^{n-1} \rho_j \ge \frac{1}{(\alpha-1)^3(2-\alpha)}((n+1)^{4-3\alpha}-(n+1)^{2-2\alpha}) \tag{3}$$

for $n \ge 1$. Since $2(1-\alpha) < -1$, (1) shows that $\sum_{j=0}^{\infty} \rho_j^2 < \infty$, which assures $g_{x_m}^{(2)}(x_n) < \infty$ for each m and n. Since $4-3\alpha > -1$, (3) implies that $g_{x_0}^{(3)}(x_0) = \infty$. Proposition 2.6 shows $g_{x_0}^{(2)} \notin \mathbf{D}_0$. □

7.2. **In the Case** $r(e_n) = t^n$**.**

Lemma 7.2. *If* $r(e_n) = t^n$ *with* $0 < t < 1$*, then*

$$c_{\mathrm{GB}} = \frac{t}{(1-t)^2} \quad \text{and} \quad B = \frac{t^2(t^2+1)}{(1-t)^4(1+t)^2}.$$

Proof. Let $c(x) = \sum_{y \in V} g_x(y)$. Then $c_{\mathrm{GB}} = \sup_{x \in V} c(x)$. We know that $\rho_n = t^{n+1}/(1-t)$ and

$$\begin{aligned} c(x_m) &= \sum_{n=0}^{\infty} g_{x_m}(x_n) = m\rho_m + \sum_{n=m}^{\infty} \rho_n = \frac{mt^{m+1}}{1-t} + \sum_{n=m}^{\infty} \frac{t^{n+1}}{1-t} \\ &= \frac{mt^{m+1}}{1-t} + \frac{t^{m+1}}{(1-t)^2}. \end{aligned}$$

We easily have that $\{c(x_m)\}_m$ decreases and that

$$c_{\mathrm{GB}} = c(x_0) = \frac{t}{(1-t)^2}.$$

Next we have

$$\begin{aligned} \beta(x_m) &= m\rho_m^2 + \sum_{n=m}^{\infty} \rho_n^2 = \frac{mt^{2(m+1)}}{(1-t)^2} + \sum_{n=m}^{\infty} \frac{t^{2(n+1)}}{(1-t)^2} \\ &= \frac{mt^{2(m+1)}}{(1-t)^2} + \frac{t^{2(m+1)}}{(1-t)^2(1-t^2)} \end{aligned}$$

and

$$B = \sum_{m=0}^{\infty} \left(\frac{mt^{2(m+1)}}{(1-t)^2} + \frac{t^{2(m+1)}}{(1-t)^2(1-t^2)} \right) = \frac{t^2(t^2+1)}{(1-t)^4(1+t)^2}.$$

□

There exists a unique solution $t = t_1 \approx 0.428$ to $B = 1$ with $0 < t < 1$. Theorem 6.1 shows that

Proposition 7.3. *If* $r(e_n) = t^n$ *with* $0 < t < t_1$*, then* $\lim_{k \to \infty} g_{x_m}^{(k)}(x_n) = 0$.

Remark 7.4. There exists a unique solution $t = t_1' \approx 0.382$ to $c_{\mathrm{GB}} = 1$ with $0 < t < 1$. Theorem 6.1 shows that $\lim_{k\to\infty} g^{(k)}_{x_m}(x_n) = 0$ if $0 < t < t_1'$; this is a weaker result than Proposition 7.3.

Proposition 7.5. *Assume that $\{g^{(k)}_{x_0}(x_n)\}_k$ converges to a finite value $u(x_n)$ for each n. If $r(e_n) = t^n$ with $t_2 \approx 0.4331 < t < 1$, then $u(x_n) = 0$.*

Proof. Since $\Delta g^{(k)}_{x_0} = -g^{(k-1)}_{x_0}$, we have

$$\Delta u = -u \quad \text{on } V.$$

For simplicity we let $u_n = u(x_n)$. By the definition we have

$$\frac{u_1 - u_0}{t} = -u_0,$$
$$\frac{u_{n+1} - u_n}{t^{n+1}} + \frac{u_{n-1} - u_n}{t^n} = -u_n \quad \text{for } n \ge 1,$$

so that

$$u_1 = (1-t)u_0,$$
$$u_{n+1} = (1 + t - t^{n+1})u_n - tu_{n-1} \quad \text{for } n \ge 1.$$

Let $f_n(t)$ be the polynomials defined by

$$f_0(t) = 1, \qquad f_1(t) = 1 - t,$$
$$f_{n+1}(t) = (1 + t - t^{n+1})f_n(t) - tf_{n-1}(t) \quad \text{for } n \ge 1.$$

Then $u_n = f_n(t)u_0$. Since $u(x) \ge 0$, we must have $f_n(t) \ge 0$ unless $u_0 = 0$. On the other hand Mathematica teaches us that $f_2(t) < 0$ if $0.6 < t < 1$ and $f_{12}(t) < 0$ if $0.4331 < t < 0.7$. These imply that $u_0 = 0$ if $t_2 < t < 1$, and that $u_n = 0$ for all n. $\square$

Proposition 7.6. *Let $t_3 \approx 0.445$ be a unique solution to $t^3 - t^2 - 2t + 1 = 0$ with $0 < t < 1$. If $t_3 < t < 1$, then $\lim_{k\to\infty} g^{(k)}_{x_0}(x_n) = \infty$ for each n.*

Proof. First observe that

$$g_{x_0}(x_n) = \rho_n = \frac{t^{n+1}}{1-t},$$
$$g^{(2)}_{x_0}(x_n) = \rho_n \sum_{j=0}^{n-1} \rho_j + \sum_{j=n}^{\infty} \rho_j^2 = \frac{t^{n+2}(1 + t - t^{n+1})}{(1-t)^3(1+t)}.$$

Let $\varphi_n(t) = -t^{n+2} - t^3 + 2t^2 + 2t - 1$. Then

$$g^{(2)}_{x_0}(x_n) - g_{x_0}(x_n) = \frac{t^{n+1}\varphi_n(t)}{(1-t)^3(1+t)}$$

and $\{\varphi_n(t)\}_n$ increases for each $0 < t < 1$. The fact $\varphi_0(t) > 0$ for $t_3 < t < 1$ implies $\varphi_n(t) > 0$ for each n, and $g_{x_0} \le g^{(2)}_{x_0}$ on V. Lemma 6.6 shows that $u(x) = \lim_{k\to\infty} g^{(k)}_{x_0}(x)$ exists and $0 < u(x) \le \infty$. Proposition 7.5 implies $u(x) = \infty$. $\square$

Remark 7.7. In summary

If	then
$0 < t < 0.428$	$\lim_{k\to\infty} g_a^{(k)} = 0$;
$0.428 \le t \le 0.4331$	no information;
$0.4331 < t \le 0.445$	$\lim_{k\to\infty} g_a^{(k)} = 0$ provided that it is finite;
$0.445 < t < 1$	$\lim_{k\to\infty} g_a^{(k)} = \infty$.

8. The Case of the Homogeneous Tree of Order Three

In this section let $\mathcal{N}$ be the homogeneous tree of order three with $r(e) = 1$ for all $e \in E$. In this case [2, Example 4.4] shows that (LD) and (CLD) are fulfilled. Theorem 3.8 and Lemma 3.7 imply $g_a^{(k)} \in \mathbf{D}_0$.

Denote by $d(a, b)$ the geodesic metric between nodes a and b and by $C_n(a) = \{x \in V \,|\, d(a, x) = n\}$ the discrete circle around $a \in V$ with radius $n \in \mathbb{N}$. We set $C_0(a) = \{a\}$.

Lemma 8.1. *The harmonic Green function g_a is given by*

$$g_a(x) = \frac{2^{1-m}}{3} \quad \textit{for } x \in C_m(a).$$

Proof. By the symmetricity of $\mathcal{N}$ we may set $g_a(x) = t_m$ for $x \in C_m(a)$ with $m \ge 0$. The equations $\Delta g_a(a) = -1$ and $\Delta g_a(x) = 0$ for $x \in C_m(a)$ with $m \ge 1$ imply

$$3(t_1 - t_0) = -1, \qquad 2t_{m+1} + t_{m-1} - 3t_m = 0 \quad \text{for } m \ge 1.$$

These lead to

$$t_m = t_0 - \frac{2}{3} + \frac{2^{1-m}}{3}.$$

The condition $g_a \in \mathbf{D}_0$ gives $\lim_{m\to\infty} t_m = 0$. We have $t_0 = 2/3$ and $t_m = 2^{1-m}/3$. □

Remark 8.2. We see easily $\sum_{x\in V} g_a(x) = \infty$, which means $c_{\mathrm{GB}} = \infty$.

Proposition 8.3. *The 2-harmonic Green function $g_a^{(2)}$ is given by*

$$g_a^{(2)}(a) = 10/9,$$
$$g_a^{(2)}(x) = \frac{2^{1-m}(3m+5)}{9} \quad \textit{for } x \in C_m(a) \textit{ with } m \ge 1.$$

Proof. We denote by $|A|$ the cardinality of a set A. We have $|C_m(a)| = 3 \cdot 2^{m-1}$ and

$$g_a^{(2)}(a) = \sum_{x\in V} g_a(x)^2 = t_0^2 + \sum_{m=1}^{\infty} |C_m(a)| t_m^2 = \frac{4}{9} + \sum_{m=1}^{\infty} \frac{2^{1-m}}{3} = \frac{10}{9}.$$

Fix $a \in V$ and put $\mu(C_n(x)) = \sum_{y\in C_n(x)} g_a(y)$. Let $x \in C_m(a)$ with $m \ge 1$. We claim

$$\mu(C_n(x)) = \begin{cases} 2^{1-m}/3 & \text{if } n = 0; \\ 2^{n-m} & \text{if } 1 \le n < m; \\ 1 & \text{if } n \ge m. \end{cases}$$

Indeed, first we see that $\mu(C_0(x)) = g_a(x) = t_m = 2^{1-m}/3$. For $1 \le n < m$ we have

$$\mu(C_n(x)) = t_{m-n} + \sum_{j=1}^{n-1} 2^{j-1} t_{m-n+2j} + 2^n t_{m+n} = 2^{n-m}.$$

For $n \ge m$ we have

$$\mu(C_n(x)) = 2^{n-m} t_{n-m} + \sum_{j=1}^{m-1} 2^{j-1+n-m} t_{n-m+2j} + 2^n t_{m+n} = 1.$$

The symmetricity of the network implies $g_x(y) = t_n$ for $y \in C_n(x)$. We have

$$\begin{aligned} g_a^{(2)}(x) &= \sum_{n=0}^{\infty} \sum_{y \in C_n(x)} g_x(y) g_a(y) = \sum_{n=0}^{\infty} t_n \mu(C_n(x)) \\ &= \frac{2}{3} \frac{2^{1-m}}{3} + \sum_{n=1}^{m-1} \frac{2^{1-n}}{3} 2^{n-m} + \sum_{n=m}^{\infty} \frac{2^{1-n}}{3} \\ &= \frac{2^{1-m}(3m+5)}{9}. \end{aligned}$$

□

Remark 8.4. It is easy to see that $g_a^{(2)}(x)$ is decreasing with respect to m. Especially $g_a^{(2)}(a) \ge g_a^{(2)}(x)$.

Proposition 8.5. *$u(x) = \lim_{k\to\infty} g_a^{(k)}(x)$ exists for each $x \in V$ and $0 < u(x) \le \infty$.*

Proof. We see that $g_a(a) = 2/3 < 10/9 = g_a^{(2)}(a)$ and that

$$g_a(x) = \frac{2^{1-m}}{3} < \frac{2^{1-m}(3m+5)}{9} = g_a^{(2)}(x)$$

for $x \in C_m(a)$ with $m \ge 1$. Lemma 6.6 shows the assertion. □

Recall that

$$\lambda(\mathcal{N}) = \inf \left\{ \frac{\|f(x)\|_{\mathbf{D}}^2}{\|f(x)\|_{l^2}^2} \mid f \in L_0(V) \right\}.$$

Proposition 8.6. $\lambda(\mathcal{N}) = 3 - 2\sqrt{2}$.

Proof. Let $\lambda^* = 3 - 2\sqrt{2}$. We consider the recurrence equations

$$3(t_1^* - t_0^*) = -\lambda^* t_0^*, \qquad 2t_{m+1}^* + t_{m-1}^* - 3t_m^* = -\lambda^* t_m^* \quad \text{for } m \ge 1,$$

which have a solution

$$t_m^* = (1 + m/3) 2^{-m/2} t_0^* \quad \text{for } m \ge 0.$$

Define u^* as the function $u^*(x) = t_m^*$ for $x \in C_m(a)$. Then u^* is positive and satisfies $\Delta u^* = -\lambda^* u^*$. Lemma 6.4 shows $\lambda^* \le \lambda(\mathcal{N})$.

On the other hand we consider a sequence $\{u^{(n)}\}_n$ defined by $u^{(n)}(x) = 2^{-m/2}$ if $0 \le m \le n$ and $u^{(n)}(x) = 0$ if $m > n$ for $x \in C_m(a)$. Then $u^{(n)} \in L_0(V)$ and

$$\|u^{(n)}\|_{l^2}^2 = 1 + \sum_{m=1}^{n} 3 \cdot 2^{m-1}(2^{-m/2})^2 = 1 + \frac{3n}{2},$$

$$\begin{aligned}\|u^{(n)}\|_{\mathbf{D}}^2 &= \sum_{m=0}^{n-1} 3 \cdot 2^m (2^{-m/2} - 2^{-(m+1)/2})^2 + 3 \cdot 2^n (2^{-n/2} - 0)^2 \\ &= 3n(1 - 2^{-1/2})^2 + 3.\end{aligned}$$

We have

$$\begin{aligned}\lambda(\mathcal{N}) &\le \frac{\|u^{(n)}\|_{\mathbf{D}}^2}{\|u^{(n)}\|_{l^2}^2} = (3n(1 - 2^{-1/2})^2 + 3) \cdot \frac{2}{3n+2} \\ &\to 2(1 - 2^{-1/2})^2 = \lambda^*\end{aligned}$$

as $n \to \infty$. Therefore $\lambda(\mathcal{N}) = \lambda^*$. □

Proposition 8.7. $\lim_{k\to\infty} g_a^{(k)}(x) = \infty$.

Proof. Propositions 8.3 and 6.3 show the assertion. □

Remark 8.8. The proposition above can be obtained by Propositions 8.5 and 8.6 and Theorem 6.5.

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