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Abstract In the research of optimization problems, optimality conditions play an important role. By using some derivatives, various types of necessary and/or sufficient optimality conditions have been introduced by many researchers. Especially, in convex programming, necessary and sufficient optimality conditions in terms of the subdifferential have been studied extensively. Recently, necessary and sufficient optimality conditions for quasiconvex programming have been investigated by the authors. However, there are not so many results concerned with Karush-Kuhn-Tucker type optimality conditions for non-differentiable quasiconvex programming.

In this paper, we study a Karush-Kuhn-Tucker type optimality condition for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. We show some closedness properties for Greenberg-Pierskalla subdifferential. Under the Slater constraint qualification, we show a necessary and sufficient optimality condition for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. Additionally, we introduce a necessary and sufficient constraint qualification of the optimality condition. As a corollary, we show a necessary and sufficient optimality condition for convex programming in terms of the subdifferential.

Keywords Optimality condition · quasiconvex programming · subdifferential · constraint qualification

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1 Introduction

In this paper, we consider the following optimization problem (P):

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } g_i(x) \leq 0, \forall i \in I,\end{cases}$$

where I is an index set, f and g_i are extended real-valued functions on $\mathbb{R}^n$, and $A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$ is a constraint set. In the research of the above problem, optimality conditions play an important role. By using some derivatives, various types of necessary and/or sufficient optimality conditions have been introduced by many researchers, see [1–14]. Especially, in convex programming, the following necessary and sufficient optimality condition in terms of the subdifferential has been studied extensively: under some constraint qualifications, $x_0 \in A$ is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0),$$

where $\mathbb{R}^{(I)}_+ = \{\lambda \in \mathbb{R}^I : \forall i \in I, \lambda_i \geq 0, \{i \in I : \lambda_i \neq 0\} \text{ is finite}\}$, and $I(x_0) = \{i \in I : g_i(x_0) = 0\}$. The above optimality condition is called Karush-Kuhn-Tucker (KKT, in short) type optimality condition. The best known constraint qualification is the Slater constraint qualification, and the basic constraint qualification is a necessary and sufficient constraint qualification for the above optimality condition, see [4, 8].

In quasiconvex programming, various types of subdifferentials and optimality conditions have been investigated. Especially, in [2], Danillidis, Hadjisavvas, and Martínez-Legaz introduced quasiconvex subdifferential and show some properties of quasiconvex subdifferential. Additionally, in [5], Linh and Penot investigated necessary and/or sufficient optimality conditions in terms of lower subdifferential and Plastria subdifferential. Recently, necessary and sufficient optimality conditions for quasiconvex programming have been investigated by the authors, see [10, 13, 14]. In [13], the authors show the following necessary and sufficient optimality condition for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential: $x_0 \in A$ is a global minimizer of f over A if and only if

$$0 \in \partial^{GP} f(x_0) + N_A(x_0),$$

where $\partial^{GP} f(x_0)$ is the Greenberg-Pierskalla subdifferential of f at x_0, and $N_A(x_0)$ is the normal cone of A at x_0. In [14], the authors show similar optimality conditions for non-essentially quasiconvex programming in terms of Martínez-Legaz subdifferential. Additionally, in [10], the author studies optimality conditions for quasiconvex programming in terms of generators of quasiconvex functions. However, the following KKT type optimality condition has not been investigated yet: $x_0 \in A$ is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Since there are not so many results concerned with KKT type optimality conditions for non-differentiable quasiconvex programming, it is expected to investigate the above condition and its constraint qualifications.

In this paper, we study a Karush-Kuhn-Tucker type optimality condition for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. We show some closedness properties for Greenberg-Pierskalla subdifferential. Under the Slater constraint qualification, we show a necessary and sufficient optimality condition for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential. Additionally, we introduce a necessary and sufficient constraint qualification of the optimality condition. As a corollary, we show a necessary and sufficient optimality condition for convex programming in terms of the subdifferential.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we show some important lemmas. In Section 4, we study a KKT type optimality condition for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential under two types of constraint qualifications. In Section 5, we compare our results with previous ones. Especially, we show a necessary and sufficient optimality condition for convex programming in terms of the subdifferential as a corollary of our results.

2 Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given a nonempty set S, we denote the interior, the convex hull, and the conical hull, generated by S, by $\mathrm{int}S$, $\mathrm{co}S$, and $\mathrm{cone}S$, respectively. A cone K is said to be pointed if $K \cap (-K) = \{0\}$. The normal cone of S at $x \in S$ is denoted by $N_S(x) := \{v \in \mathbb{R}^n : \forall y \in S, \langle v, y - x\rangle \leq 0\}$. Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$. f is said to be convex if for each x, $y \in \mathbb{R}^n$, and $\alpha \in (0, 1)$, $f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$. The subdifferential of f at x is defined as $\partial f(x) := \{v \in \mathbb{R}^n : \forall y \in \mathbb{R}^n, f(y) \geq f(x) + \langle v, y - x\rangle\}$. Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\mathrm{lev}(f, \diamond, \beta) := \{x \in \mathbb{R}^n : f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. f is said to be quasiconvex if $\mathrm{lev}(f, \leq, \beta)$ is a convex set for all $\beta \in \mathbb{R}$. Additionally, f is said to be essentially quasiconvex if it is quasiconvex and each local minimizer $x \in \mathbb{R}^n$ of f over $\mathbb{R}^n$ is a global minimizer of f over $\mathbb{R}^n$. Clearly, all convex functions are essentially quasiconvex. There are many characterizations of essentially quasiconvexity. For example, a real-valued continuous quasiconvex function is essentially quasiconvex if and only if it is semistrictly quasiconvex [1], a pseudoconvex differentiable function is essentially quasiconvex [3,15,16], and a real-valued quasiconvex function is essentially quasiconvex if and only if it is neatly quasiconvex [17].

In quasiconvex analysis, various types of subdifferentials have been introduced, see [5,7,11,12,18–26]. In [18], Greenberg and Pierskalla introduce the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) := \{v \in \mathbb{R}^n : \langle v, x\rangle \geq \langle v, x_0\rangle \text{ implies } f(x) \geq f(x_0)\}.$$

We need the following relation between the subdifferential and the Greenberg-Pierskalla subdifferential in [25].

Theorem 1 *[25] Let f be a real-valued convex function on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ is not a global minimizer of f over $\mathbb{R}^n$, then*

$$\mathbb{R}_{++}\partial f(x) = \partial^{GP} f(x),$$

where $\mathbb{R}_{++} := \{t \in \mathbb{R} : t > 0\}$.

We introduce the following necessary and sufficient optimality condition for essentially quasiconvex programming.

Theorem 2 *[13] Let f be an extended real-valued upper semicontinuous (usc) essentially quasiconvex function on $\mathbb{R}^n$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$.*

Then, the following statements are equivalent:

(i) $f(x_0) = \min_{x \in A} f(x)$,

(ii) $0 \in \partial^{GP} f(x_0) + N_A(x_0)$.

In Corollary 9.1.3 in [8], the following result for the closedness of the sum of cones is given.

Theorem 3 *[8] Let K_1, K_2 be non-empty closed convex cones in $\mathbb{R}^n$. Assume that if $z_i \in K_i$ for $i = 1, 2$ and $z_1 + z_2 = 0$, then z_i belongs the linearity space of K_i, the largest subspace contained in the recession cone of K_i, for $i = 1, 2$. Then, $K_1 + K_2$ is closed.*

In Corollary 1.2.3 in [27], the following result is given.

Theorem 4 *[27] Let K be a non-empty closed convex pointed cone in $\mathbb{R}^n$. Then, there exists $v \in \mathbb{R}^n$ such that $\langle v, x \rangle > 0$ for each $x \in K \setminus \{0\}$.*

3 Important lemmas

In this section, we show some important lemmas for our main results. Especially, we study the closedness of Greenberg-Pierskalla subdifferential precisely.

Lemma 1 *Let g be an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $v \in \mathbb{R}^n \setminus \{0\}$. Assume that there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < g(x)$. Then, $v \in \partial^{GP} g(x)$ if and only if $v \in N_{\mathrm{lev}(g,\leq,g(x))}(x)$.*

Proof Let $v \in \partial^{GP} g(x)$ and $y \in \mathrm{lev}(g, \leq, g(x))$. We show that $\langle v, y - x \rangle \leq 0$. By the definition of Greenberg-Pierskalla subdifferential, if $g(y) < g(x)$, then $\langle v, y \rangle < \langle v, x \rangle$. We assume that $g(y) = g(x)$. Since $\mathrm{lev}(g, <, g(x))$ is nonempty, y is not a global minimizer of g over $\mathbb{R}^n$. By the essentially quasiconvexity of g, y is not a local minimizer of f over $\mathbb{R}^n$. Hence, for each $k \in \mathbb{N}$, there exists $y_k \in \mathbb{R}^n$ such that $g(y_k) < g(y)$ and $\|y - y_k\| < \frac{1}{k}$. Then, we can check that $\langle v, y_k \rangle < \langle v, x \rangle$ and y_k converges to y. This shows that $\langle v, y \rangle \leq \langle v, x \rangle$, that is, $\langle v, y - x \rangle \leq 0$.

Conversely, let $v \in N_{\mathrm{lev}(g,\leq,g(x))}(x)$ and $y \in \mathrm{lev}(v, \geq, \langle v, x \rangle)$. We show that $g(y) \geq g(x)$. If $\langle v, y \rangle > \langle v, x \rangle$, then $g(y) > g(x)$. We assume that $\langle v, y \rangle = \langle v, x \rangle$

and $g(y) < g(x)$. Since g is usc, there exists $r > 0$ such that for each $z \in \mathbb{R}^n$ satisfying $\|y-z\| < r$, $g(z) < g(x)$. This shows that $g(y+\frac{r}{2\|v\|}v) < g(x)$. However, this contradicts to $v \in N_{\mathrm{lev}(g,\leq,g(x))}(x)$ since

$$\left\langle v, y + \frac{r}{2\|v\|}v \right\rangle > \langle v, y \rangle = \langle v, x \rangle .$$

Hence, $g(y) \geq g(x)$. This completes the proof. □

Lemma 2 *Let g be an extended real-valued usc quasiconvex function on $\mathbb{R}^n$, and $x \in \mathbb{R}^n$. Then, $\partial^{GP} g(x)$ is nonempty.*

Proof Let $x \in \mathbb{R}^n$. If $\mathrm{lev}(g,<,g(x))$ is empty, then we can check easily that $\partial^{GP} g(x) = \mathbb{R}^n$. We assume that $\mathrm{lev}(g,<,g(x))$ is nonempty. By the assumption, $\mathrm{lev}(g,<,g(x))$ is nonempty open convex. By the separation theorem, there exist $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that for each $y \in \mathrm{lev}(g,<,g(x))$,

$$\langle v, x \rangle \leq \alpha < \langle v, y \rangle .$$

This shows that if $g(y) < g(x)$ then $\langle v, x \rangle < \langle v, y \rangle$, that is, $\inf\{g(y) : \langle v, x \rangle \geq \langle v, y \rangle\} \geq g(x)$. Hence, $v \in \partial^{GP} g(x)$. This completes the proof. □

Lemma 3 *Let g be an extended real-valued function on $\mathbb{R}^n$, and $x \in \mathbb{R}^n$. Then, $\partial^{GP} g(x) \cup \{0\}$ is a convex cone.*

Proof Let v, $w \in \partial^{GP} g(x) \cup \{0\}$, and $\lambda \geq 0$. If $v \in \partial^{GP} g(x)$ and $\lambda > 0$,

$$\inf\{g(y) : \langle \lambda v, y \rangle \geq \langle \lambda v, x \rangle\} = \inf\{g(y) : \langle v, y \rangle \geq \langle v, x \rangle\} \geq g(x).$$

Hence, $\lambda v \in \partial^{GP} g(x)$. If $v = 0$ or $\lambda = 0$, then $\lambda v = 0 \in \partial^{GP} g(x) \cup \{0\}$. This shows that $\partial^{GP} g(x) \cup \{0\}$ is a cone. Next, we show that $\partial^{GP} g(x) \cup \{0\}$ is convex. If $v = 0$ or $w = 0$, then it is clear that $v + w \in \partial^{GP} g(x) \cup \{0\}$. If $v \neq 0$ and $w \neq 0$, then for each $y \in \mathbb{R}^n$ satisfying $\langle v + w, y \rangle \geq \langle v + w, x \rangle$, $\langle v, y \rangle \geq \langle v, x \rangle$ or $\langle w, y \rangle \geq \langle w, x \rangle$. Hence,

$$\begin{aligned} &\inf\{g(y) : \langle v + w, y \rangle \geq \langle v + w, x \rangle\} \\ \geq\ &\min\{\inf\{g(y) : \langle v, y \rangle \geq \langle v, x \rangle\}, \inf\{g(y) : \langle w, y \rangle \geq \langle w, x \rangle\}\} \\ \geq\ &g(x). \end{aligned}$$

This shows that $\partial^{GP} g(x) \cup \{0\}$ is convex. This completes the proof. □

Next, we study the closedness of Greenberg-Pierskalla subdifferential of an usc function.

Lemma 4 *Let g be an extended real-valued usc function on $\mathbb{R}^n$, and $x \in \mathbb{R}^n$. Then, $\partial^{GP} g(x) \cup \{0\}$ is closed.*

Proof Let $\{v_k\} \subset \partial^{GP} g(x) \cup \{0\}$ be a sequence such that $\{v_k\}$ converges to $v_0 \in \mathbb{R}^n$. We assume that $v_0 \notin (\partial^{GP} g(x) \cup \{0\})$. This implies that $v_0 \neq 0$ and $\inf\{g(y) : \langle v_0, y \rangle \geq \langle v_0, x \rangle\} < g(x)$. Then, there exists $\bar{y} \in \mathbb{R}^n$ such that $g(\bar{y}) < g(x)$ and $\langle v_0, \bar{y} \rangle \geq \langle v_0, x \rangle$. By the upper semicontinuity of g, there exists $r > 0$ such that

$g(\bar{y}+rv_0) < g(x)$. Additionally, there exists $\bar{k} \in \mathbb{N}$ such that $\langle v_{\bar{k}}, \bar{y} + rv_0\rangle > \langle v_{\bar{k}}, x\rangle$ since $\{v_k\}$ converges to v_0 and $\langle v_0, \bar{y} + rv_0\rangle > \langle v_0, \bar{y}\rangle \geq \langle v_0, x\rangle$. This shows that

$$\inf\{g(y) : \langle v_{\bar{k}}, y\rangle \geq \langle v_{\bar{k}}, x\rangle\} \leq g(\bar{y} + rv_0) < g(x).$$

This contradicts to $v_{\bar{k}} \in \partial^{GP} g(x)$. Hence, $v_0 \in \partial^{GP} g(x) \cup \{0\}$, that is, $\partial^{GP} g(x) \cup \{0\}$ is closed. □

Lemma 5 *Let S be a subset of $\mathbb{R}^n$, $x \in S$, and assume that S has nonempty interior. Then $N_S(x)$ is a pointed cone.*

Proof Let $v \in N_S(x) \setminus \{0\}$ and $\bar{x} \in \mathrm{int}S$. Since $v \neq 0$ and $\bar{x} \in \mathrm{int}S$, we can check that $\langle v, \bar{x}\rangle < \langle v, x\rangle$. This shows that $-v \notin N_S(x)$. □

The following lemma plays an important role in our main results.

Lemma 6 *Let I be an index set, g_i an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$ for each $i \in I$, $x \in A = \{y \in \mathbb{R}^n : \forall i \in I, g_i(y) \leq 0\}$, and $I(x) = \{i \in I : g_i(x) = 0\}$. Assume that I is finite and there exists $\bar{x} \in \mathbb{R}^n$ such that $g_i(\bar{x}) < 0$ for each $i \in I$. Then, $\mathrm{co}\bigcup_{i\in I(x)}(\partial^{GP} g(x) \cup \{0\})$ is closed.*

Proof By the assumption, we can check that A has nonempty interior. Let $i \in I(x)$, then by Lemma 3 and Lemma 4, $\partial^{GP} g_i(x) \cup \{0\}$ is a closed convex cone. By Lemma 1,

$$N_A(x) \supset N_{\mathrm{lev}(g_i,\leq,0)}(x) = \partial^{GP} g_i(x) \cup \{0\}.$$

Additionally, by Lemma 5, $N_A(x)$ is pointed.

Let $i_1, i_2 \in I(x)$. Then, $\partial^{GP} g_{i_1}(x) \cup \{0\}$ and $\partial^{GP} g_{i_2}(x) \cup \{0\}$ are closed convex cones. Let $v_1 \in \partial^{GP} g_{i_1}(x) \cup \{0\}$ and $v_2 \in \partial^{GP} g_{i_2}(x) \cup \{0\}$ satisfying $v_1 + v_2 = 0$. Then, by the above inclusion, $v_1, v_2 \in N_A(x)$. Since $N_A(x)$ is pointed, $v_1 = v_2 = 0$. This shows that v_1 and v_2 belong the linearity spaces of $\partial^{GP} g_{i_1}(x) \cup \{0\}$ and $\partial^{GP} g_{i_2}(x) \cup \{0\}$, respectively. Hence, by Theorem 3, $\partial^{GP} g_{i_1}(x) \cup \{0\} + \partial^{GP} g_{i_2}(x) \cup \{0\}$ is closed. Since I is finite,

$$\mathrm{co} \bigcup_{i\in I(x)} (\partial^{GP} g(x) \cup \{0\}) = \sum_{i\in I(x)} (\partial^{GP} g(x) \cup \{0\})$$

is closed. This completes the proof. □

4 Main results

In this section, we show our main results. At first, we show a necessary and sufficient optimality condition for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential under the Slater constraint qualification.

Theorem 5 *Let I be an index set, g_i an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$ for each $i \in I$, $x_0 \in A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$, $I(x_0) = \{i \in I : g_i(x_0) = 0\}$, and f an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$. Assume that I is finite and there exists $\bar{x} \in \mathbb{R}^n$ such that $g_i(\bar{x}) < 0$ for each $i \in I$. Then, x_0 is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that*

$$0 \in \partial^{GP} f(x_0) + \sum_{i\in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Proof By Lemma 1,

$$N_A(x_0) \supset \mathrm{co} \bigcup_{i \in I(x_0)} N_{\mathrm{lev}(g_i, \leq, 0)}(x_0) = \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}).$$

Next, we show

$$N_A(x_0) \subset \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}).$$

By Lemma 6, $\mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$ is a closed convex cone. Additionally, by Lemma 5, $N_A(x_0)$ is pointed, that is, $\mathrm{co} \bigcup_{i \in I(x_0)} \partial^{GP} g_i(x_0) \cup \{0\}$ is also pointed. Let $v \in N_A(x_0)$, and assume that $v \notin \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$. By the separation theorem, there exists $\hat{x} \in \mathbb{R}^n \setminus \{0\}$ such that for each $w \in \bigcup_{i \in I(x_0)} \partial^{GP} g_i(x_0)$, $\langle v, \hat{x} \rangle > 0 \geq \langle w, \hat{x} \rangle$ since $\mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$ is a closed convex cone. Additionally, by Theorem 4, there exists $\tilde{x} \in \mathbb{R}^n \setminus \{0\}$ such that for each $w \in \bigcup_{i \in I(x_0)} \partial^{GP} g_i(x_0) \setminus \{0\}$, $0 > \langle w, \tilde{x} \rangle$. For sufficiently small $r > 0$, we put $x^* = \hat{x} + r\tilde{x}$, then, for each $w \in \bigcup_{i \in I(x_0)} \partial^{GP} g_i(x_0) \setminus \{0\}$,

$$\left\langle v, x^* \right\rangle > 0 > \left\langle w, x^* \right\rangle.$$

We show that for each $i \in I(x_0)$, there exists $r_i > 0$ such that for each $r \in (0, r_i]$, $g_i(x_0 + rx^*) \leq 0$. Actually, if $\mathrm{lev}(g_i, <, 0) \cap (x_0 + \mathbb{R}_+\{x^*\})$ is empty, then, by the separation theorem for $\mathrm{lev}(g_i, <, 0)$ and $x_0 + \mathbb{R}_+\{x^*\}$, there exist $\bar{w} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that for each $y \in \mathrm{lev}(g_i, <, 0)$ and $t \geq 0$,

$$\langle \bar{w}, y \rangle < \beta \leq \left\langle \bar{w}, x_0 + tx^* \right\rangle.$$

This shows that $\bar{w} \in \partial^{GP} g_i(x_0)$ and $\langle \bar{w}, x^* \rangle \geq 0$. This is a contradiction. Hence there exists $r_i > 0$ such that $g_i(x_0 + r_i x^*) < 0$. By the quasiconvexity of g_i, for each $r \in (0, r_i]$, $g_i(x_0 + rx^*) \leq 0 = g_i(x_0)$. Additionally, by the upper semicontinuity of g_i, for each $i \notin I(x_0)$, there exists $r_i > 0$ such that for each $r \in (0, r_i]$, $g_i(x_0 + rx^*) < 0$. Since I is finite, $\bar{r} = \min\{r_i : i \in I\} > 0$ and $x_0 + \bar{r}x^* \in A$. However, by the above separation inequality, $\langle v, x_0 + \bar{r}x^* \rangle > \langle v, x_0 \rangle$. This contradicts to $v \in N_A(x_0)$. Hence, $v \in \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$.

By Theorem 2, x_0 is a global minimizer of f over A if and only if

$$0 \in \partial^{GP} f(x_0) + N_A(x_0) = \partial^{GP} f(x_0) + \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}).$$

Hence, if x_0 is a global minimizer of f over A, then there exist $v \in \partial^{GP} f(x_0)$, $w_i \in \partial^{GP} g_i(x_0) \cup \{0\}$ and $\alpha_i \geq 0$ for each $i \in I(x_0)$ such that $v + \sum_{i \in I(x_0)} \alpha_i w_i = 0$ and $\sum_{i \in I(x_0)} \alpha_i = 1$. Put $\lambda \in \mathbb{R}^I_+$ as follows: for each $i \in I$,

$$\lambda_i = \begin{cases} 0, & i \notin I(x_0), \\ 0, & i \in I(x_0) \text{ and } w_i = 0, \\ \alpha_i, & i \in I(x_0) \text{ and } w_i \neq 0. \end{cases}$$

Then,

$$0 = v + \sum_{i \in I(x_0)} \lambda_i w_i \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Assume that there exists $\lambda \in \mathbb{R}^I_+$ such that $0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0)$. If $\sum_{i \in I(x_0)} \lambda_i = 0$, then,

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0) = \partial^{GP} f(x_0) + \{0\} \subset \partial^{GP} f(x_0) + N_A(x_0).$$

Hence, by Theorem 2, x_0 is a global minimizer of f over A. If $\sum_{i \in I(x_0)} \lambda_i > 0$, then

$$\begin{aligned} 0 &\in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0) \\ &= \partial^{GP} f(x_0) + \left(\sum_{i \in I(x_0)} \lambda_i \right) \sum_{i \in I(x_0)} \frac{\lambda_i}{\sum_{i \in I(x_0)} \lambda_i} \partial^{GP} g_i(x_0) \\ &\subset \partial^{GP} f(x_0) + \operatorname{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}) \end{aligned}$$

since $\operatorname{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$ is a cone. This completes the proof. □

Next, we show necessary and sufficient constraint qualifications for the KKT type optimality condition in terms of Greenberg-Pierskalla subdifferential.

Theorem 6 *Let I be an index set, g_i an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$ for each $i \in I$, $x_0 \in A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$, $I(x_0) = \{i \in I : g_i(x_0) = 0\}$. Assume that there exists $\hat{x} \in A \setminus \{x_0\}$.*

Then, the following statements are equivalent:

(i)

$$N_A(x_0) = \operatorname{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}),$$

(ii) for each extended real-valued usc essentially quasiconvex function f on $\mathbb{R}^n$, x_0 is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Proof Assume that (i) holds and let f be an extended real-valued usc essentially quasiconvex function on $\mathbb{R}^n$. By Theorem 2, x_0 is a global minimizer of f over A if and only if

$$0 \in \partial^{GP} f(x_0) + N_A(x_0).$$

Since $N_A(x_0) = \operatorname{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$, the above condition holds if and only if there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0),$$

This shows that (ii) holds.

Assume that the statement (ii) holds. At first, we show that $\mathrm{lev}(g_i, <, 0)$ is nonempty for each $i \in I$. Actually, if there exists $i_0 \in I$ such that $\mathrm{lev}(g_{i_0}, <, 0)$ is empty, then, $g_{i_0}(x_0) = 0$, that is, $i_0 \in I(x_0)$ since $x_0 \in A$. Let $f(x) = \langle x_0 - \hat{x}, x \rangle$ for each $x \in \mathbb{R}^n$. Then, $f(x_0) > f(\hat{x})$, $\partial^{GP} f(x_0) = \{l(x_0 - \hat{x}) : l > 0\}$, $\partial^{GP} g_{i_0}(x_0) = \mathbb{R}^n$ and

$$0 \in \mathbb{R}^n = \partial^{GP} f(x_0) + \partial^{GP} g_{i_0}(x_0).$$

This is a contradiction since x_0 is not a global minimizer of f over A. Hence, $\mathrm{lev}(g_i, <, 0)$ is nonempty for each $i \in I$. By Lemma 1,

$$N_A(x_0) \supset \mathrm{co} \bigcup_{i \in I(x_0)} N_{\mathrm{lev}(g_i, \leq, 0)}(x_0) = \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}).$$

Let $v \in N_A(x_0)$, then x_0 is a global minimizer of $-v$ over A. Let $f(x) = \langle -v, x \rangle$ for each $x \in \mathbb{R}^n$. By the statement (ii), there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Since $\partial^{GP} f(x_0) = \{-lv : l > 0\}$, there exists $l > 0$ such that $-lv \in \partial^{GP} f(x_0)$. Hence,

$$v \in \sum_{i \in I(x_0)} \frac{\lambda_i}{l} \partial^{GP} g_i(x_0) \subset \sum_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\}) = \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$$

since $\partial^{GP} g_i(x_0) \cup \{0\}$ is a convex cone for each $i \in I(x_0)$. This completes the proof. □

5 Discussion

In this section, we discuss about our results. We compare our results with previous ones. Additionally, we show two examples. At first, we show a necessary and sufficient optimality condition for convex programming in terms of the subdifferential as a corollary of our result.

Corollary 1 *Let I be an index set, g_i a real-valued convex function on $\mathbb{R}^n$ for each $i \in I$, $x_0 \in A = \{x \in \mathbb{R}^n : \forall i \in I, g_i(x) \leq 0\}$, $I(x_0) = \{i \in I : g_i(x_0) = 0\}$, and f a real-valued convex function on $\mathbb{R}^n$. Assume that I is finite and there exists $\bar{x} \in \mathbb{R}^n$ such that $g_i(\bar{x}) < 0$ for each $i \in I$. Then, x_0 is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that*

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0). \tag{1}$$

Proof It is well known that a real-valued convex function on $\mathbb{R}^n$ is continuous essentially quasiconvex. Hence, by Theorem 5, x_0 is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

Since $g_i(\bar{x}) < 0$ for each $i \in I$, x_0 is not a global minimizer of g_i over $\mathbb{R}^n$ for each $i \in I(x_0)$. Hence, by Theorem 1, for each $i \in I(x_0)$,

$$\mathbb{R}_{++}\partial g_i(x_0) = \partial^{GP} g_i(x_0).$$

This shows that x_0 is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0).$$

Assume that x_0 is a global minimizer of f over A. If x_0 is not a global minimizer of f over $\mathbb{R}^n$, then $\mathbb{R}_{++}\partial f(x_0) = \partial^{GP} f(x_0)$. Hence, we can show that Equation (1) holds. On the other hand, if x_0 is a global minimizer of f over $\mathbb{R}^n$, then $0 \in \partial f(x_0)$. In this case, put $\lambda = 0$, then Equation (1) holds. Conversely, we can check easily that if Equation (1) holds then x_0 is a global minimizer of f over A. This completes the proof. □

Remark 1 Lemma 1 is equivalent to the following set containment characterization: the following statements are equivalent;

(i) $\{x \in \mathbb{R}^n : g(x) \leq g(x_0)\} \subset \{x \in \mathbb{R}^n : \langle v, x\rangle \leq \langle v, x_0\rangle\}$,
(ii) $v \in \partial^{GP} g(x_0)$.

By using such a set containment characterization, necessary and sufficient constraint qualifications for duality results have been investigated, see [28–36].

Remark 2 In Theorem 6, we assume that there exists $\hat{x} \in A \setminus \{x_0\}$. If the assumption does not hold, then x_0 is always a global minimizer of f over A. Hence, we need the assumption in this theorem.

Remark 3 In Theorem 6, we show the following necessary and sufficient constraint qualification:

(i) $N_A(x_0) = \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$.

If $\mathrm{lev}(g_i, <, 0)$ is nonempty for each $i \in I$, then the above constraint qualification (i) is equivalent to the following conditions:

(ii) $N_A(x_0) \subset \mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$,
(iii) $\mathrm{co} \bigcup_{i \in I(x_0)} (\partial^{GP} g_i(x_0) \cup \{0\})$ is closed.

In Lemma 6, we show that the Slater constraint qualification implies the above condition (iii).

Remark 4 In [4], the basic constraint qualification is introduced as follows:

$$N_A(x_0) = \mathrm{cone}\ \mathrm{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0).$$

The basic constraint qualification is known as a necessary and sufficient constraint qualification for KKT type optimality condition via convex programming. Our constraint qualification in Theorem 6 is very similar to the basic constraint qualification. Actually, if g_i is real-valued convex and $\mathrm{lev}(g_i, <, 0)$ is nonempty for each $i \in I$, our constraint qualification is equivalent to the basic constraint qualification.

However, we can not show that the basic constraint qualification is a necessary and sufficient constraint qualification for KKT type optimality condition via convex programming. When we compare $\partial g_i(x_0)$ and $\partial^{GP} g_i(x_0)$, we need the assumption, $\text{lev}(g_i, <, 0)$ is nonempty. Hence, we can not show this important result in [4] as a corollary of our result. On the other hand, in Corollary 1, we show KKT type optimality condition via convex programming under the Slater constraint qualification as a corollary of our result.

Finally, we show the following two examples.

Example 1 Let $I = \{1, 2\}$, f, g_1 and g_2 be the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = |x - 2|^5, \quad g_1(x) = (x - 1)^3, \quad g_2(x) = -(x + 1)^3.$$

Then, we can easily check that f and g_i are real-valued continuous essentially quasiconvex and $g_i(0) < 0$ for each $i \in I$. Additionally, $A = \{x \in \mathbb{R} : \forall i \in I, g_i(x) \leq 0\} = [-1, 1]$. By Theorem 5, $x_0 \in A$ is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}^I_+$ such that

$$0 \in \partial^{GP} f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial^{GP} g_i(x_0).$$

We can calculate that for each $x \in A$,

$$\partial^{GP} f(x) = (-\infty, 0), \quad \partial^{GP} g_1(x) = (0, \infty), \quad \partial^{GP} g_2(x) = (-\infty, 0)$$

Let $x_0 = 1 \in A$. Then $I(x_0) = \{1\}$ and for $\bar{\lambda} = (1, 0)$,

$$0 \in (-\infty, 0) + (0, \infty) = \partial^{GP} f(x_0) + \partial^{GP} g_1(x_0).$$

By the above optimality condition, $x_0 = 1$ is a global minimizer of f over A.

Example 2 Let $I = \mathbb{Z} \setminus \{0\}$, and g_i be the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$g_i(x_1, x_2) = i x_2^{2|i|+1}.$$

Then, we can check that g_i are real-valued continuous essentially quasiconvex and $A = \{(x_1, x_2) \in \mathbb{R}^2 : \forall i \in I, g_i(x_1, x_2) \leq 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Additionally, there does not exist $x \in \mathbb{R}^2$ such that $g_i(x) < 0$ for each $i \in I$.

On the other hand, we can show that for each $\bar{x} \in A$,

$$N_A(\bar{x}) = \text{co} \bigcup_{i \in I(\bar{x})} (\partial^{GP} g_i(\bar{x}) \cup \{0\}).$$

Actually, $N_A(\bar{x}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$, $I(\bar{x}) = I$, and

$$\partial^{GP} g_i(\bar{x}) = \begin{cases} \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 > 0\} & \text{if } i > 0, \\ \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 < 0\} & \text{if } i < 0. \end{cases}$$

This shows that the constraint qualification in Theorem 6 is satisfied.

Hence, by Theorem 6, for each extended real-valued usc essentially quasiconvex function f on $\mathbb{R}^n$, $\bar{x} \in A$ is a global minimizer of f over A if and only if there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that

$$0 \in \partial^{GP} f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^{GP} g_i(\bar{x}).$$

We can calculate that

$$\sum_{i \in I(\bar{x})} \lambda_i \partial^{GP} g_i(\bar{x}) = \left(\sum_{i \in I_+} \lambda_i \{(0, x_2) : x_2 > 0\} \right) + \left(\sum_{i \in I_-} \lambda_i \{(0, x_2) : x_2 < 0\} \right),$$

where $I_+ = \{i \in I : i > 0\}$ and $I_- = \{i \in I : i > 0\}$. Hence, if there exists $(v_1, v_2) \in \partial^{GP} f(\bar{x})$ such that $v_1 = 0$, then $\bar{x}$ is a global minimizer of f over A. Additionally, if $v_1 \neq 0$ for each $(v_1, v_2) \in \partial^{GP} f(\bar{x})$, then $\bar{x}$ is not a global minimizer of f over A.

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