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S^1-Actions on Vector Bundles over Spheres

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In the previous papers [6], [7] and [8], we studied some group actions on sphere bundles over spheres and proved some non existence theorems. In this paper, we shall study S^1-vector bundle structures on vector bundles over spheres. We shall fix a representation of S^1 on the fiber over the north pole.

Section 1 provides some preliminaries and we shall prove some classification theorems for S^1-vector bundles with a designated action on the fibre over the north pole (Theorem 1 and Theorem 2). As a corollary, we shall obtain a non existence theorem.

In section 2, we shall construct two kinds of lifting actions. One of them is a lifting of a linear action on a sphere and the other is a lifting of a quasi linear action.

In the last section, we shall show that some action on a sphere bundle over a sphere can not be derived from an S^1-vector bundle with a specified action on the fibre over the north pole.

§1. Preliminaries

We denote the $(n+1)$-dimensional complex number space by C^{n+1} and the real number field by R respectively. We use symbols ρ_{S^1} and θ_R for the standard representation $S^1 \to U(1)$ and the trivial real representation of S^1. Consider the action on $S^{2n+2} \subset C^{n+1} \oplus R$ given by a representation $a\rho_{S^1} \oplus (2n-2a+3)\theta_R$, $a \leqq n$. Then the upper and lower hemispheres e_+^{2n+2} and e_-^{2n+2}, contract equivariantly to the north and south poles O_+, O_- respectively. Let $C^k \to B \to S^{2n+2}$ be a complex S^1-vector bundle. By Proposition 1.3 in [9], the portions $B|e_\pm^{2n+2}$ yield equivariant isomorphisms, $\alpha_\pm$: $B|e_\pm^{2n+2} \to e_\pm^{2n+2} \times C_\pm^k$, where $C_\pm^k$ are S^1-modules such that

$$g(v_1, \dots, v_k) = (g^{\varepsilon_1(\pm)} v_1, \dots, g^{\varepsilon_k(\pm)} v_k),$$

for $g \in S^1$, $(v_1, \dots, v_k) \in C_\pm^k$ and $\varepsilon_1(\pm), \dots, \varepsilon_k(\pm)$ are some integers. Define an S^1-action on $U(k)$ by $(g, A) \to D_-(g) A D_+(g)$ for $g \in S^1$, $A \in U(k)$, where $D_\pm(g) = (g^{\varepsilon_1(\pm)}) \times \cdots \times (g^{\varepsilon_k(\pm)})$. Consider the commutative diagram

$$\begin{array}{ccccccc} & \overset{\alpha_+}{\nearrow} & S^{2n+1} \times C_+^k & \xrightarrow{g} & S^{2n+1} \times C_+^k & \overset{\alpha_+}{\searrow} & \\ B|^{2n+1} & & & & & & B|S^{2n+1} \\ & \underset{\alpha_-}{\searrow} & S^{2n+1} \times C_-^k & \xrightarrow{g} & S^{2n+1} \times C_-^k & \underset{\alpha_-}{\nearrow} & \end{array} \qquad \cdots(1)$$

Define a map χ: $S^{2n+1} \to U(k)$ by $\alpha_- \circ \alpha_+^{-1}(x, v) = (x, \chi(x)(v))$ for $(x, v) \in S^{2n+1} \times C_+^k$, then we have $\chi(gx) = D_-(g)\chi(x)D_+(g)^{-1}$. Therefore the map χ is S^1-equivariant. Let

$C^k \to B' \to S^{2n+2}$ be another S^1-vector bundle. We have another diagram (1′) similar to (1). If an S^1-isomorphism $h: B \to B'$ is given, then we obtain a cubic diagram connecting (1) with (1′). Define $h_\pm: e_\pm^{2n+2} \to \mathrm{Iso}\ (C_\pm^k, C_\pm'^k)$ by $\alpha'_\pm \circ h \circ (\alpha_\pm)^{-1}(x, v) = (x, h_\pm(x)(v))$ for $x \in e_\pm^{2n+2}$, $v \in C_\pm^k$, where $C_\pm'^k$ are S^1-modules which appear in equivariant trivializations $\alpha'_\pm: B' \mid S^{2n+1} \xrightarrow{\approx} S^{2n+1} \times C_\pm'^k$. But we can choose a basis in $C_\pm'^k$ such that $C_\pm'^k = C_\pm^k$. By the cubic diagram, we obtain

$$h_-(x)\chi(x) = \chi'(x)h_+(x), \qquad x \in S^{2n+1} \qquad \cdots(2)$$

The maps $h_\pm: e_\pm^{2n+2} \to U(k)$ are equivariantly homotopic to constant maps $h_\pm(O_\pm) = A_\pm$. By (2), χ' is equivariantly homotopic to $A_-^{-1}\chi A_+$. We can normalize maps $h_\pm$ by $(A_\pm)^{-1}h_\pm$ to obtain $\chi' \simeq \chi$, S^1-homotopic.

Conversely, suppose that there is an equivariant homotopy $H_t: S^{2n+1} \to U(k)$ such that $H_0 = \chi$ and $H_1 = \chi'$. Set $h_- = H_0 \circ H_1^{-1}$ and $h_+ = I_k$, which is the unit matrix in $U(k)$. Then h_- is equivariantly homotopic to the constant map, and we obtain extended maps $h_\pm: e_\pm^{2n+2} \to U(k)$. Clearly, $h_-\chi' = H_0 = \chi h_+$ on S^{2n+1}. Let B' be an S^1-vector bundle with characteristic map χ'. Then we obtain an S^1-isomorphism $h: B \to B'$ such that $h \mid e_\pm^{2n+2} = (\alpha'_\pm)^{-1} \circ h_\pm \circ (\alpha_\pm)$. Set $(\varepsilon_1(+), \dots, \varepsilon_k(+)) = \varepsilon$ and denote S^1-equivalence classes of S^1-C^k-bundles over S^{2n+2} with the specified action on the fibre over O_+ by $Vect_{S^1}^{\varepsilon}(S^{2n+2})$. Then we have

THEOREM 1. *The set $Vect_{S^1}^{\varepsilon}(S^{2n+2})$ corresponds bijectively to the equivariant homotopy set $[\![S^{2n+1}, U(k)]\!]$, where the S^1-action on $U(k)$ is given by $(g, A) \to D_+(g)AD_+(g)^{-1}$.*

PROOF. Since $a \leqq n$, O_+ and O_- can be combined by a curve in the fixed point set, there exists $A \in U(k)$ such that $D_-(g) = A^{-1}D_+(g)A$. From the relation $\chi(gx)D_+(g) = D_-(g)\chi(x)$, we have $\chi(gx) = A^{-1}D_+(g)A\chi(x)D_+(g)^{-1}$, and $(A\chi(gx))D_+(g) = D_+(g)\cdot(A\chi(x))$. The S^1-bundle $e_+^{2n+2} \times C_+^k \cup_{A\chi} e_-^{2n+2} \times C_+^k$ is equivariantly isomorphic to $e_+^{2n+2} \times C_+^k \cup_\chi e_-^{2n+2} \times C_-^k$. Thus we obtain the theorem.

For S^1-maps $\gamma, \gamma': S^{2n+1} \to U(k)$, we define a product by $(\gamma \circ \gamma')(x) = \gamma(x) \cdot \gamma'(x)$, $x \in S^{2n+1}$. Then the set $[\![S^{2n+1}, U(k)]\!]$ admits a group structure.

Now consider the case $a = 1$. Then S^{2n+2} is an S^1-manifold, where the action is given by the representation $\rho_{S^1} \oplus (2n+1)\theta_R$. The fixed point set is the $2n$-sphere S^{2n} and the orbit space is a $(2n+1)$-disc D^{2n+1}. We have an equivariant decomposition $S^{2n+2} = D^2 \times S^{2n} \cup S^1 \times D^{2n+1}$, where $D^2 \times S^{2n}$ is an equivariant tublar neighborhood of the fixed point set. Let $p: S^1 \times S^{2n} \to S^{2n}$ be the projection onto the second factor and $\pi: S^{2n+2} \to D^{2n+1}$ be the orbit map. S^{2n+2} is a special S^1-manifold ([3]). The image $\pi(S^1 \times S^{2n})$ is a $2n$-sphere S_ε^{2n} in D^{2n+1} with the same center as the one of D^{2n+1}. Consider the commutative diagram

$$\begin{array}{ccc} S^1\times S^{2n} & \xrightarrow{p} & S^{2n} \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ S^{2n}_{\varepsilon} & \xrightarrow{p'} & S^{2n}, \end{array} \quad \text{the identity map}$$

where p' is the map induced from p. For an S^1-vector bundle $C^k\to B\to S^{2n+2}$, its data (cf. §3 in [5]) is

$$B\,|\,S^{2n}\xrightarrow{r^*p'^*} r^*p'^*(B\,|\,S^{2n})\xrightarrow{\alpha} S^{2n}_{\varepsilon}\times C^k\subset D^{2n+1}_{\varepsilon}\times C^k,$$

where r^* denotes the forgetful map and α is an isomorphism of vector bundles. We have an S^1-isomorphism $B\,|\,S^{2n}\cong e^{2n}_+\times C^k_+\cup_f e^{2n}_-\times C^k$. Since $S(C^{n+1}\oplus R)\supset S(C\oplus R)\ni O_{\pm}$, it follows from Proposition 1.3 in [9], that there exists $A\in U(k)$ such that $D_-=A^{-1}D_+A$. From the relation $D_-f(x)D_+^{-1}=f(x)$, we have $D_+(Af(x))D_+^{-1}=Af(x)$ for $x\in S^{2n-1}$. Suppose that $\varepsilon_1=\cdots=\varepsilon_{m_1}$, $\varepsilon_{m_1+1}=\cdots=\varepsilon_{m_2},\dots$, $\varepsilon_{m_{s-1}+1}=\cdots=\varepsilon_{m_s}=\varepsilon_k$. By the proof of Theorem 1, we can assume that $f\colon S^{2n-1}\to U(m_1)\times\cdots\times U(m_s)$. Let $i_*\colon \pi_{2n-1}(U(m_1)\times\cdots\times U(m_s))\to\pi_{2n-1}(U(k))$ be the homomorphism induced from the inclusion map $i\colon U(m_1)\times\cdots\times U(m_s)\to U(k)$. By the trivialization α, $[f]\in\text{kernel } i^*$. Let $\alpha_1\colon r^*p'^*(B\,|\,S^{2n})\to S^{2n}_{\varepsilon}\times C^k\subset D^{2n+1}_{\varepsilon}\times C^k$ be another trivialization. Then $\alpha_1\circ\alpha^{-1}$ determines an element of $\pi_{2n}(U(k))$. Conversely, for a fixed α, each element of $\pi_{2n}(U(k))$ gives a homotopy class of trivializations. By the theorem 2 in [5], we have

THEOREM 2. *We have a bijection*

$$S\colon \mathit{Vect}^{\varepsilon}_{S^1}(S^{2n+2})\longrightarrow\{\mathit{kernel}\ i^*\times\pi_{2n}(U(k))\}/(\sim),$$

where $\sim$ denotes equivalences of data (Definition 1 in [5]).

Let $\Psi_{\#}\colon \mathit{Vect}^{\varepsilon}_{S^1}(S^{2n+2})\to \mathit{Vect}^k(S^{2n+2})$ be the forgetful homomorphism. Then we have

COROLLARY. If $k<2n$, then the image $\Psi_{\#}$ is a finite subgroup of $\mathit{Vect}^k(S^{2n+2})$.

PROOF. $\pi_{2n}(U(k))$ is a finite group, and if $m_i\geqq n$, then $m_j<n$ for $j\neq i$. Therefore the kernel i_* is a finite subgroup of $\pi_{2n-1}(U(m_1)\times\cdots\times U(m_s))$. Thus we have the corollary.

If $n+1\leqq k<2n$, then the image $\Psi_{\#}$ is trivial.

§ 2. Construction of lifting group actions

Consider an S^1-action on S^{2n+2} given by the representation $\rho_{S^1}\oplus(2n+1)\theta_R$. Let $C^k\to B\to S^{2n+2}$ be an S^1-vector bundle. Define $\Psi_{S^1}\colon S^1\times(B\,|\,S^{2n})\to S^1\times(r^*p'^*(B\,|\,S^{2n}))$ by $\Psi_{S^1}(x, v)=(x, x^{-1}v)$ for $(x, v)\in S^1\times(B\,|\,S^{2n})$. Then Ψ_{S^1} is an equivariant isomorphism. Consider the composite isomorphism

$$S^1\times(B\,|\,S^{2n})\xrightarrow{\Psi_{S^1}} S^1\times(r^*p'^*(B\,|\,S^{2n}))\xrightarrow{1_{S^1}\times\alpha} S^1\times S^{2n}\times C^k.$$

Set $\Psi_{S^1}(x, v)=(x, \pi(v), f(x^{-1}v))$. We can define $\phi\colon S^{2n}\to \mathrm{Hom}(S^1, U(k))$ by $f(x^{-1}v)=\phi_{\pi(v)}(x)f(v)$. Now suppose that $k\geqq 2n$. Then we have a unique homotopy class $[\alpha]$ of a trivialization. Since $B\cong D^2\times(B\,|\,S^{2n})\cup_{\Phi} S^1\times D^{2n+1}\times C^k$, where Φ denotes $(1_{S^1}\times\alpha)\circ\Psi_{S^1}$, the bundle B is determined by the homotopy class $[\phi]$ up to S^1-isomorphism.

Let Γ be a compact connected Lie group. For $\alpha\in\mathrm{Hom}(S^1, \Gamma)$, $\gamma\in\Gamma$, we define $\alpha^\gamma\in\mathrm{Hom}(S^1, \Gamma)$ by $\alpha^\gamma(x)=\gamma\alpha(x)\gamma^{-1}$, $x\in S^1$, and denote the set $\{\alpha^\gamma;\gamma\in\Gamma\}$ by α^Γ. Then α^Γ is included in the connected component $\mathrm{Hom}(S^1, \Gamma)_\alpha$ of α. The next proposition is due to H. Toda.

PROPOSITION 1. $\mathrm{Hom}(S^1, \Gamma)_\alpha=\alpha^\Gamma$.

PROOF. Let $\{\alpha_t\in\mathrm{Hom}(S^1, \Gamma)\}$ be a homotopy of $\alpha=\alpha_0$. The image $\alpha(S^1)$ is contained in a maximal torus T of Γ and there is $s\in T$ such that $\alpha_1(S^1)\subset sTs^{-1}$. The circle group S^1 is topologically generated by a generator g. Since Γ is connected, there is a curve $c(\tau)$ in Γ such that $c(0)=\alpha_1(g)$ and $c(1)=s^{-1}\alpha_1(g)s\in T$. The curve $\alpha_t(g)\circ c(\tau)$ connects $\alpha_0(g)$ with $s^{-1}\alpha_1(g)s$. We have $\pi_1(\Gamma/T)=0$. Therefore the curve $\alpha_t(g)\circ c(\tau)$ is deformable into a curve in T. Thus we have $\alpha_0(g)=s^{-1}\alpha_1(g)s$ and $\alpha_1=s\cdot\alpha\cdot s^{-1}$, which proves the proposition.

Now let Γ be $U(k)$. Then $\mathrm{Hom}(S^1, U(k))_\alpha=\alpha^{U(k)}$. We consider the case $k=2n$ and $\alpha(x)=D_n(x)\times D_n(x)^{-1}$, where $D_n(x)$ denotes the n-dimensional diagonal matrix with x as diagonal entries. Let C_α be the set $\{\gamma\in U(2n);\gamma\cdot\alpha(x)=\alpha(x)\cdot\gamma \text{ for any } x\in S^1\}$. Then $C_\alpha=U(n)\times U(n)$. By the correspondence $\gamma C_\alpha\to(\gamma C_\alpha)\alpha(\gamma C_\alpha)^{-1}$, the space $\alpha^{U(2n)}$ can be identified with the grassmannian $U(2n)/C_\alpha=G_{2n,n}$. The composite map $G_{2n,n}\to\alpha^{U(2n)}=\mathrm{Hom}(S^1, U(2n))_\alpha\subset\Omega_\alpha U(2n)$ is just the Bott map $\bar{f}$ in §8 of [2], where Ω_α denotes the component of α in the loop space $\Omega U(2n)$. By the consideration above, a map $\phi\colon S^{2n}\to\mathrm{Hom}(S^1, U(2n))_\alpha$ corresponds to a map $\phi'\colon S^{2n}\to G_{2n,n}$. By §8 in [2], we have $\bar{f}_*\pi_{2n}(G_{2n,n})=2\pi_{2n}(\Omega_\alpha(U(2n)))\cong 2\pi_{2n+1}(U(2n))$. Thus we have proved

THEOREM 3. *If a C^{2n}-bundle B over S^{2n+2} has the homotopy class of characteristic maps in $2\pi_{2n+1}(U(2n))$, then the S^1-action $\rho_{S^1}\oplus(2n+1)\theta_R$ on S^{2n+2} can be lifted to an action on B.*

Next we construct another bundle lifting of a quasi linear action on the sphere S^{2n+2}, where we mean by a quasi linear action a smooth action which is topologically equivalent to a linear action. We refer the construction of difference bundles due to Atiyah-Bott-Shapiro (§9 in [1]). Let $\gamma_n\colon S^{2n-1}\to U(n)$ be a representative for a generator of $\pi_{2n-1}(U(n))$. Define $\bar{\gamma}_n\colon D^{2n}\to M_n(C)$, the complex $n\times n$ matrices, by $\bar{\gamma}_n(sx)=s\gamma_n(x)$ for $(s, x)\in[0, 1]\times S^{2n-1}$. Then we have a complex of vector bundles

$$\tilde{\gamma}_n\colon D^{2n}\times C_1^n\longrightarrow D^{2n}\times C_0^n,$$

which is given by $\tilde{\gamma}_n(x, v)=(x, \bar{\gamma}_n(x)(v))$. In the case $n=1$, $\tilde{\gamma}_1$ is given by $\tilde{\gamma}_1(x, v_1)=$

(x, xv_1) and its adjoint $\tilde{\gamma}_1^*(x, v_0)=(x, \bar{x}v_0)$. We give an S^1-action on the complex $\tilde{\gamma}_1$ by

$$D^2\times C_1 \ni (x, v_1) \longrightarrow (gx, v_1)\in D^2\times C_1,$$

$$D^2\times C_0 \ni (x, v_0) \longrightarrow (gx, gv_0)\in D^2\times C_0,$$

where $g\in S^1$. Then $\tilde{\gamma}_1$ is S^1-equivariant. We use the same notations as the ones in §9 of [1]. Thus $\rho_1^*(\phi_0^*)^{-1}d(\tilde{\gamma}_n)$ gives a generator of $K(S^{2n})$ and d is an Euler characteristic (p. 22 ibid.). Now $\tilde{\gamma}_1$ gives a complex S^1-line bundle and represents a generator of $K(S^2)$. We consider the product of complexes $\tilde{\gamma}_1$ and $\tilde{\gamma}_n$:

$$\tilde{\gamma}_{n+1}\colon (D^2\times D^{2n})\times(C_0\otimes C_1^n\oplus C_1\otimes C_0^n) \longrightarrow (D^2\times D^{2n})\times(C_0\otimes C_0^n\oplus C_1\otimes C_1^n),$$

where

$$\bar{\gamma}_{n+1}(x, y)=\begin{pmatrix} I_1\otimes\bar{\gamma}_n(y) & \bar{\gamma}_1(x)\otimes I_n \\ \bar{\gamma}_1^*\otimes I_n & -I_1\otimes\bar{\gamma}_n^*(y)\end{pmatrix}=\begin{pmatrix} \bar{\gamma}_n(y) & x\cdot I_n \\ \bar{x}\cdot I_n & -\bar{\gamma}_n^*(y)\end{pmatrix}$$

and I_k denotes the unit matrix of k-dimension. For a boundary point $(x, y)\in\partial(D^2\times D^{2n})$, $(1/\sqrt{|x|^2+\|y\|^2})\bar{\gamma}_{n+1}(x, y)\in U(2n)$, where $\|\ \|$ denotes the norm, and we denotes this by $\bar{\bar{\gamma}}_{n+1}(x, y)$. For $g\in S^1$, we have

$$\bar{\bar{\gamma}}_{n+1}(gx, y)\begin{pmatrix} gv_0 & w_1 \\ v_1 & w_0\end{pmatrix}=(1/\sqrt{|x|^2+\|y\|^2})\begin{pmatrix} \bar{\gamma}_n(y) & gx\otimes I_n \\ \bar{g}\bar{x}\cdot I_n & -\bar{\gamma}_n^*(y)\end{pmatrix}\begin{pmatrix} gv_0\otimes w_1 \\ v_1\otimes w_0\end{pmatrix}.$$

Therefore $\bar{\bar{\gamma}}_{n+1}(gx, y)=(D_n(g)\times I_n)\bar{\bar{\gamma}}_{n+1}(x, y)(D_n(g)\times I_n)^{-1}$. We denote the n-fold product of the unit interval $[0, 1]$ by I^n and its boundary by ∂I^n. Define a map Φ_n: $(I^n, \partial I^n)\to(D^n, S^{n-1})$ by $\Phi_n(p)=(\max\{|t_i|, i=1,\dots, n\}/\|p\|)\cdot p$, where $p=(t_1,\dots, t_n)\neq(0,\dots, 0)$ and $\Phi_n(0,\dots, 0)=(0,\dots, 0)$. Then Φ_n is a homeomorphism. Consider the composite map $h=\Phi_{2n+2}\circ(\Phi_2^{-1}\times\Phi_{2n}^{-1})\colon D^2\times D^{2n}\to I^2\times I^{2n}\to D^{2n+2}$. Then the restriction to the boundary $h\colon \partial(D^2\times D^{2n})\to\partial D^{2n+2}$ is a homeomorphism. By Theorem M due to Smale ([10], p. 394), there exists a diffeomorphism $\tilde{h}\colon D^2\times D^{2n}\to D^{2n+2}$ with $\tilde{h}(\partial(D^2\times D^{2n}))=S^{2n+1}$. We define an S^1-action on D^{2n+2} by $gx=\tilde{h}\circ(g\tilde{h}^{-1}(x))$ for $g\in S^1$, $x\in D^{2n+2}$. Hence the map $\hat{\gamma}_{n+1}=\bar{\bar{\gamma}}_{n+1}\tilde{h}^{-1}\colon S^{2n+1}\to\partial(D^2\times D^{2n})\to U(2n)$ satisfies the relation

$$\begin{aligned}\hat{\gamma}_{n+1}(gx)&=\bar{\bar{\gamma}}_{n+1}h^{-1}(gx)=\bar{\bar{\gamma}}_{n+1}\tilde{h}^{-1}\tilde{h}g\tilde{h}^{-1}(x)\\ &=\bar{\bar{\gamma}}_{n+1}(g\tilde{h}^{-1}(x))=(D_n(g)\times I_n)\bar{\bar{\gamma}}_{n+1}(\tilde{h}^{-1}(x))(D_n(g)\times I_n)^{-1}\\ &=(D_n(g)\times I_n)\hat{\gamma}(x)(D_n(g)\times I_n)^{-1}.\end{aligned}$$

Here we give an S^1-action on $U(2n)$ by $(g, A)\to(D_n(g)\times I_n)A(D_n(g)\times I_n)^{-1}$. Then $\hat{\gamma}_{n+1}$ is S^1-equivariant. Thus we obtain an S^1-C^{2n}-bundle over S^{2n+2}. By the Bott periodicity, $\hat{\gamma}_{n+1}$ gives a generator of $K(S^{2n+2})$.

§3. Non lifting actions

The sphere S^{2n+2} is an $SO(2)$-manifold by the repersentation $\rho_{SO(2)}\oplus(2n+1)\theta_R$. Suppose that $2n+2\equiv 0 \bmod 8$. We want to construct a lifting action on R^{2n+2}-bundle over S^{2n+2}, where the action is compatible with the action on the structure group $SO(2n+2)$ given by $(g, A)\to(D_2(g)\times I_{2n})A(D_2(g)\times I_{2n})^{-1}$. Let $R^{2n+2}\to E\to S^{2n+2}$ be an $SO(2)$-bundle with the lifting action. Denote the portion of E on the fixed point set by B. Then the data (§3 in [5]) is given by

$$B \longrightarrow r^*p'^*B \xrightarrow{\alpha} S_\varepsilon^{2n}\times R^{2n+2}\subset D_\varepsilon^{2n+1}\times R^{2n+2}.$$

Let $\alpha_1, \alpha_2\colon r^*p'^*B\to S_\varepsilon^{2n}\times R^{2n+2}$ be two trivializations. Then the composite $\alpha_2\circ\alpha_1^{-1}$ determines an element of $\pi_{2n}(SO(2n+2))$, which is trivial by the assumption. We have $[\![S_{2n-1}^{2n-1}, SO(2n+2)]\!]=[S^{2n-1}, SO(2)\times SO(2n)]$, which is a cyclic group generated by the class $\{\tau_{2n}\}$ of the tangent bundle of S^{2n}. Then $B=B(k\tau_{2n})\oplus R^2(g)$, where $B(k\tau_{2n})$ denotes the bundle with the class $\{k\tau_{2n}\}$ of chracteristic maps and $R^2(g)$ denotes the product bundle with the standard S^1-action on fibres. Hence the classification theorem (§3 in [5]) shows that image (Ψ: $[\![S_{2n-1}^{2n+1}, SO(2n+2)]\!]\to\pi_{2n+1}(SO(2n+2))$) is a subgroup $\{k\tau_{2n+2}\}$ generated by the class of the tangent bundle of S^{2n+2}. Denote by σ the one of generators of $\pi_{2n+1}(SO(2n+2))$ which gives rise to a generator of the stable group $\pi_{2n+1}(SO)$. Let $E(\tau_{2n+2}+2m\sigma)$ be the sphere bundle over S^{2n+2} with the class $\{\tau_{2n+2}+2m\sigma\}$ of characteristic maps. By §5 in [4], $E(\tau_{2n+2}+2m\sigma)$ is diffeomorphic to the connected sum $E(\tau_{2n+2})\#m^2\Sigma$, where Σ is a homotopy sphere. When m is divided by the order of the group θ_{4n+3} of homotopy $4n+3$ spheres, $E(\tau_{2n+2}+2m\sigma)$ is diffeomorphic to the tangent sphere bundle $E(\tau_{2n+2})$, which admits a lifting action. By our consideration above, the derived action on $E(\tau_{2n+2}+2m\sigma)$ is not a lifting action.

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