Mem. Fac. Sci., Shimane Univ.,
17, pp. 1–8 Dec. 20, 1983

Note on Special Involutions in a Generalized Inverse Semigroup

Miyuki YAMADA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 3, 1983)

A unary operation $*: S \to S$ on a semigroup S is called a special involution if it satisfies (1) $(x^*)^*=x$, (2) $(xy)^*=y^*x^*$ and (3) $xx^*x=x$ for all $x, y \in S$. It has been shown by [5] that every special involution in a regular semigroup S is determined by the p-system in S. In this paper, we shall determine all the p-systems in a generalized inverse semigroup S, and accordingly all the special involutions in S. Further, we shall investigate the cardinality of the set of p-systems in S.

§1. Preliminary

A regular semigroup S equipped with a unary operation $*: S \to S$ is called *a special $*$-regular semigroup* if it satisfies (1) $(x^*)^*=x$, (2) $(xy)^*=y^*x^*$ and (3) $xx^*x=x$ for all $x, y \in S$. The unary operation $*$ is called *a special involution* in S. If a regular semigroup S admits at least one special involution, then S is called *specially involutive*. We sometimes denote by $(S, \#)$ a special $*$-regular semigroup S equipped with a special involution $\#$. In the previous paper [5], the concept of a p-system in a regular semigroup S has been introduced. A subset P of the set $E(S)$ of idempotents of S is called *a p-system* in S if P satisfies the following (1)–(3):

(C.1) (1) For any $x \in S$, there exists a unique $x^\# \in V(x)$ (the set of inverses of x) such that $xx^\#, x^\#x \in P$.

(2) For the operation $\#$ defined above, $x^\# Px \subset P$ for any $x \in S$.

(3) $P^2 \subset E(S)$.

It has been shown by [5] that in this case $(S, \#)$ is a special $*$-regular semigroup. Further, it has been proved that a regular semigroup admits a special involution if and only if it has at least one p-system. The operation $\#$ above is called *the special involution determined by P*, and denoted by $*_P$. Conversely, if $(S, *)$ is a special $*$-regular semigroup then the set of projections of $(S, *)$, that is, the set $P=\{e \in E(S): e^*=e\}$ is a p-system in S (see [5]). We denote this P by P_*. Now, it is easy to see that the set of projections of a special $*$-regular semigroup $(S, *_P)$, where P is a p-system in S, is P, and the special involution in S determined by P_*, where $*$ is a special involution in S, is $*$. Let $\mathscr{P}(S)$ be the set of all p-systems in a regular semigroup S, and $\mathscr{I}(S)$ the set of all possible special involutions in S. It is obvious that $\mathscr{P}(S)=\square$ if and only if

$\mathscr{I}(S)=\square$. Let $f: \mathscr{P}(S)\to\mathscr{I}(S)$ and $g: \mathscr{I}(S)\to\mathscr{P}(S)$ be the mappings defined by $Pf=*_P$ and $*g=P_*$ respectively. Then, since $Pfg=P$ and $*gf=*$, fg and gf are the identity mappings on $\mathscr{P}(S)$ and $\mathscr{I}(S)$ respectively. Hence, $|\mathscr{P}(S)|=|\mathscr{I}(S)|$ ($|\ |$ means cardinality). In [5], it has been shown that for a generalized inverse semigroup S, P is a p-system in S if and only if P is a p-system in the normal band $E(S)$. Therefore, in this case $|\mathscr{P}(S)|=|\mathscr{P}(E(S))|=|\mathscr{I}(E(S))|=|\mathscr{I}(S)|$. In the following sections, we shall investigate the cardinality of $\mathscr{P}(S)$ of a normal band S (hence, a generalized inverse semigroup S).

Remark. Let S be a generalized inverse semigroup, and $*$, $\sharp$ special involutions in S. It has been proved by [2] and [5] that $(S, *)$ and $(S, \sharp)$ are $*$-isomorphic, that is, there exists an isomorphism $f: S\to S$ such that $x^*f=(xf)^\sharp$ for all $x\in S$.

§2. Normal bands with special involution

Let S be a normal band. By Scheiblich [2] and the author [5], S admits a special involution if and only if S is isomorphic to the spined product $L\otimes L^d$ of a left normal band L and its dual semigroup L^d.

Note. That is, $L^d=L$ as set, and for any $x, y\in L^d$, $x\circ y=yx$ (the product of x, y in L), where $\circ$ means the multiplication in L^d.

Let Y be a semilattice, and A, B bands which are semilattice Y of rectangular bands $\{A_\alpha: \alpha\in Y\}$ and a semilattice Y of rectangular bands $\{B_\alpha: \alpha\in Y\}$ respectively (in this case, we say that A, B have the structure decompositions $A\sim\Sigma\{A_\alpha: \alpha\in Y\}$ and $B\sim\Sigma\{B_\alpha: \alpha\in Y\}$ respectively). Then, $C=\Sigma\{A_\alpha\times B_\alpha: \alpha\in Y\}$, where $\times$ means "direct product" and Σ means "disjoint sum", becomes a subsemigroup of $A\times B$. This C is called the spined product of A and B, and denoted by $A\otimes B$.

From the above, if a normal band S admits a special involution then we can assume that $S=L\otimes L^d$, where L is a left normal band and L^d its dual semigroup. If L is a semilattice Y of left zero semigroups $\{L_\alpha: \alpha\in Y\}$, that is, if L has the structure decomposition $L\sim\Sigma\{L_\alpha: \alpha\in Y\}$, then L^d is a right normal band and has the structure decomposition $L^d\sim\Sigma\{L_\alpha^d: \alpha\in Y\}$, where L_α^d is the dual semigroup of L_α, and $L\otimes L^d$ is a semilattice Y of square bands (see [5]) $\{L_\alpha\times L_\alpha^d: \alpha\in Y\}$, that is, $L\otimes L^d$ has the structure decomposition $L\otimes L^d\sim\Sigma\{L_\alpha\times L_\alpha^d: \alpha\in Y\}$. Hereafter, the notion "a band $B\equiv\Sigma\{B_\gamma: \gamma\in Y_1\}$" means that B is a band which is a semilattice Y_1 of rectangular bands $\{B_\gamma: \gamma\in Y_1\}$. Of course, each L_α and L_α^d above are a left zero semigroup and a right zero semigroup respectively. Now, let $L\otimes L^d\equiv\Sigma\{L_\alpha\times L_\alpha^d: \alpha\in Y\}$ be a normal band which admits a special involution, where L is a left normal band. Let $\cdot$ and $\circ$ be the multiplications in L and L^d respectively. Then, of course $x\cdot y=y\circ x$ for all $x, y\in L$ (hence, for $x, y\in L^d$). Now, it is easy to see that $P=\{(i, i)\in L_\alpha\times L_\alpha^d: \alpha\in Y\}$ is a p-system in $L\otimes L^d$. In general, if F is a p-system in $L\otimes L^d$ then F is the set of projections in $(L\otimes L^d, *_F)$. Hence, each L-class [each R-class] contains a unique element of F.

Therefore, there exists a bijection τ: $L \to L$ such that $L_\alpha \tau = L_\alpha$ for all $\alpha \in Y$ and $F = \{(i, i\tau): i \in L_\alpha, \alpha \in Y\}$ (see also [5]). It is easily seen that $*_F$ is defined by τ as follows: $(i, j)^{*F} = (j\tau^{-1}, i\tau)$. Since F satisfies (C.1), (2), $(i, j)(u, u\tau)(j\tau^{-1}, i\tau) \in F$ (where $(u, u\tau) \in L_\beta \times L_\beta^d \cap F$). Hence, $(i \cdot u)\tau = i\tau \cdot u\tau$, that is, τ is an automorphism. Thus, we have the following:

THEOREM 1. *Let $L \equiv \Sigma\{L_\alpha: \alpha \in Y\}$ be a left normal band, and $S = L \otimes L^d$ the spined product of L and its dual semigroup L^d. Let τ: $L \to L$ be an automorphism of L such that $L_\alpha \tau = L_\alpha$ for all $\alpha \in Y$. Then, S is a normal band which admits a special involution, and $F = \{(i, i\tau): i \in L_\alpha, \alpha \in Y\}$ is a p-system in S. Further, every p-system in S is obtained in this way.*

PROOF. The latter half was seen above. The first half: Let τ be an automorphism of L such that $L_\alpha \tau = L_\alpha$ for all $\alpha \in Y$. We need only to show that $F = \{(i, i\tau): i \in L_\alpha, \alpha \in Y\}$ satisfies (C.1), (1), (2). For all elements $(i, j) \in L_\alpha \times L_\alpha^d (\alpha \in Y)$, let $(i, j)^\# = (j\tau^{-1}, i\tau)$. Then, $(i, j)(j\tau^{-1}, i\tau) = (i, i\tau) \in F$ and $(j\tau^{-1}, i\tau)(i, j) = (j\tau^{-1}, j) \in F$. Hence, $(i, j)(i, j)^\#$, $(i, j)^\#(i, j) \in F$ and $(i, j)^\# \in V((i, j))$. If there exists $(u, v) \in V((i, j))$ such that $(u, v)(i, j)$, $(i, j)(u, v) \in F$, then $j = u\tau$ and $v = i\tau$. Hence $(u, v) = (j\tau^{-1}, i\tau)$. Thus, (C.1), (1) is satisfied. (C.1), (2): For any $(i, j) \in L_\alpha \times L_\alpha^d (\alpha \in Y)$ and for any $(u, u\tau) \in L_\beta \times L_\beta^d (\beta \in Y)$, $(i, j)(u, u\tau)(j\tau^{-1}, i\tau) = (i \cdot u \cdot j\tau^{-1}, j \circ u\tau \circ i\tau) = (i \cdot u, i\tau \cdot u\tau) = (i \cdot u, (i \cdot u)\tau) \in F$.

From the result above, the problem of determining all p-systems in $S = L \otimes L^d$ is reduced to that of determining all automorphisms f: $L \to L$ satisfying

(C.2) $L_\alpha f = L_\alpha$ for every $\alpha \in Y$.

An automorphism f of L satisfying (C.2) is called *a Y-restricted automorphism* or more simply *a restricted automorphism* on $L \equiv \Sigma\{L_\alpha: \alpha \in Y\}$. The set of all restricted automorphisms on $L \equiv \Sigma\{L_\alpha: \alpha \in Y\}$ is clearly a group with respect to the usual resultant composition. We denote it by G_L, and call it *the group of restricted automorphisms on $L \equiv \Sigma\{L_\alpha: \alpha \in Y\}$.*

§3. The restricted automorphisms G_L

Let $L \equiv \Sigma\{L_\alpha: \alpha \in Y\}$ be a left normal band. Then, it is well known that L is a strong semilattice Y of $\{L_\alpha: \alpha \in Y\}$. Hence, there exists a family of homomorphisms $(\phi_\beta^\alpha: \alpha \geq \beta, \alpha, \beta \in Y\}$, where each ϕ_β^α is a homomorphism of L_α into L_β (this is just a mapping), such that

(C.3) (1) ϕ_α^α = the identity mapping on L_α for all $\alpha \in Y$,

(2) $\phi_\beta^\alpha \phi_\gamma^\beta = \phi_\gamma^\alpha$ for $\alpha \geq \beta \geq \gamma$, $\alpha, \beta, \gamma \in Y$,

and (3) the multiplication $\cdot$ in L is given by

$$x \cdot y = (x\phi_{\xi\delta}^\xi) \cdot (y\phi_{\xi\delta}^\delta) = x\phi_{\xi\delta}^\xi \qquad \text{for} \quad x \in L_\xi,\ y \in L_\delta.$$

Hereafter, we shall call $\{\phi^{\alpha}_{\beta}: \alpha \geq \beta, \alpha, \beta \in Y\}$ *the characteristic family of homomorphisms for* $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$.

Now,

LEMMA 2. *$G_L \ni \tau$ if and only if τ is a bijection on L such that*

(1) *$L_{\alpha}\tau = L_{\alpha}$ for all $\alpha \in Y$, and*

(2) *$\tau\phi^{\alpha}_{\beta} = \phi^{\alpha}_{\beta}\tau$ for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$.*

PROOF. To prove "if" part, it is need only to show that τ is a homomorphism. Let $x \in L_{\alpha}$, $y \in L_{\beta}$. Then, $(x \cdot y)\tau = (x\phi^{\alpha}_{\alpha\beta})\tau = x\phi^{\alpha}_{\alpha\beta}\tau = x\tau\phi^{\alpha}_{\alpha\beta} = x\tau\phi^{\alpha}_{\alpha\beta} \cdot y\tau\phi^{\beta}_{\alpha\beta} = x\tau \cdot y\tau$. Hence, τ is a homomorphism. The "only if" part: For $x \in L_{\alpha}$ and $y \in L_{\beta}$, $(x \cdot y)\tau = x\tau \cdot y\tau$. Hence, $(x\phi^{\alpha}_{\alpha\beta} \cdot y\phi^{\beta}_{\alpha\beta})\tau = x\tau\phi^{\alpha}_{\alpha\beta} \cdot y\tau\phi^{\beta}_{\alpha\beta}$, and accordingly $x\phi^{\alpha}_{\alpha\beta}\tau = x\tau\phi^{\alpha}_{\alpha\beta}$. Therefore, $\phi^{\alpha}_{\alpha\beta}\tau = \tau\phi^{\alpha}_{\alpha\beta}$.

If a bijection τ on L satisfies (1) of Lemma 2, then τ is called *a restricted bijection on* $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$. If τ further satisfies (2) of Lemma 2, then τ is said to be *compatible with* $\Phi = \{\phi^{\alpha}_{\beta}: \alpha \geq \beta, \alpha, \beta \in Y\}$.

Now, we have the following:

THEOREM 3. *Let $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$ be a left normal band, and let $S = L \otimes L^d$. Let $\Phi = \{\phi^{\alpha}_{\beta}: \alpha \geq \beta, \alpha, \beta \in Y\}$ be the characteristic family of homomorphisms for $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$, and G_L the set of all restricted bijections on $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$ which are compatible with Φ. Then, G_L is a group, and $|\mathscr{P}(S)| = |G_L|$.*

Examples. 1. If L is a left zero semigroup, G_L is the group of all bijections on L, that is, G_L is the symmetric group on L. Therefore, $S = L \otimes L^d (= L \times L^d)$ is a square band (see [5]), and $\mathscr{P}(S)$ coincides with the cardinality of the symmetric group on the set L.

2. Let Y be a semilattice consisting of α, β and 0 such that $\alpha\beta = 0$ and 0 is a zero element. Let L be a left normal band whose structure decomposition is $L \sim \Sigma\{L_{\xi}: \xi \in Y\}$, where $L_{\alpha} = \{a\}$, $L_{\beta} = \{b\}$ and $L_0 = \{e, f\}$. Let $\{\phi^{\xi}_{\eta}: \xi \geq \eta, \xi, \eta \in Y\}$ be the characteristic family of homomorphisms for $L \equiv \Sigma\{L_{\xi}: \xi \in Y\}$, where $L_{\alpha}\phi^{\alpha}_0 = e$ and $L_{\beta}\phi^{\beta}_0 = f$. Then, in this case $G_L = 1$, and hence $L \otimes L^d$ has a unique p-system.

COROLLARY 4. *Let $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$ be a left normal band. Then, the following two conditions are equivalent.*

(1) *A restricted automorphism on $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$ is unique (hence, it is the identity mapping on L),*

(2) *A restricted bijection on L which is compatible with the characteristic family of homomorphisms for $L \equiv \Sigma\{L_{\alpha}: \alpha \in Y\}$ is unique (hence, it is the identity mapping on L).*

In this case, $L \otimes L^d$ has a unique p-system.

§4. The group of restricted bijections

Let $S=\{S_i: i\in I\}$ be a collection of sets S_i, $i\in I$, and $S=\Sigma\{S_i: i\in I\}$ the disjoint sum of all S_i. If a bijection $f: S\to S$ satisfies $S_i f=S_i$ for all $i\in I$, then f is called *a I-restricted bijection* or more simply *a restricted bijection on* $S=\Sigma\{S_i: i\in I\}$. The set $G(S)$ of all restricted bijections on $S=\Sigma\{S_i: i\in I\}$ forms a group with respect to the usual resultant composition. Now, let Y be a semilattice, and L_α a left zero semigroup for each $\alpha\in Y$. Let $\mathscr{L}=\{L_\alpha: \alpha\in Y\}$. For each pair $(\alpha,\beta)\in Y\times Y$ with $\alpha\geq\beta$, let ϕ_β^α be a mapping (hence, a homomorphism) of L_α into L_β. If the collection $\Phi=\{\phi_\beta^\alpha: \alpha\geq\beta, \alpha,\beta\in Y\}$ satisfies the conditions (C.3), (1) and (2), then it is well known that $L=\Sigma\{L_\alpha: \alpha\in Y\}$ (disjoint sum) becomes a left normal band under the multiplication $\cdot$ defined by

$$x\cdot y=x\phi_\beta^\alpha \qquad \text{for} \quad x\in L_\alpha,\ y\in L_\beta.$$

Of course, $L(\cdot)$ is a strong semilattice Y of $\{L_\alpha: \alpha\in Y\}$ and has $\{\phi_\beta^\alpha: \alpha\geq\beta, \alpha,\beta\in Y\}$ as its characteristic family. This $L(\cdot)$ is called *the left normal s-composition of* $\{L_\alpha: \alpha\in Y\}$ *determined by* Φ, and denoted by $L(\Phi)$. The system Φ above is called *a transitive system of homomorphisms for* $\mathscr{L}=\{L_\alpha: \alpha\in Y\}$. Let $T(\mathscr{L})$ be the set of all transitive systems of homomorphisms for $\mathscr{L}=\{L_\alpha: \alpha\in Y\}$. For any $\tau\in G(\mathscr{L})$ and any $\Phi\in T(\mathscr{L})$, where $\Phi=\{\Phi_\beta^\alpha: \alpha\geq\beta, \alpha,\beta\in Y\}$, let $\Phi^\tau=\{\tau^{-1}\phi_\beta^\alpha\tau: \alpha\geq\beta, \alpha,\beta\in Y\}$ (where $x\,\tau^{-1}\phi_\beta^\alpha\tau=x(\tau\mid L_\alpha)^{-1}\phi_\beta^\alpha(\tau\mid L_\beta)$ ($\tau\mid L_\xi$ means the restriction of τ to L_ξ)).

THEOREM 5. *Let Y be a semilattice, and $\mathscr{L}=\{L_\alpha: \alpha\in Y\}$ a collection of left zero semigroups L_α. Let $\Phi=\{\phi_\beta^\alpha: \alpha\geq\beta, \alpha,\beta\in Y\}$ be a transitive system of homomorphisms for $\mathscr{L}$, and τ a restricted bijection on $L=\Sigma\{L_\alpha: \alpha\in Y\}$ such that $\Phi^\tau=\Phi$, then L becomes a left normal band under the multiplication $\cdot$ defined by $x\cdot y=x\phi_{\alpha\beta}^\alpha$ for $x\in L_\alpha$, $y\in L_\alpha$, and $L\otimes L^d$ is a special $*$-regular semigroup with respect to the operation $*$ defined by $(i,j)^*=(j\tau^{-1}, i\tau)$. Further, every specially involutive normal band $L\otimes L^d$ and every special involution $*$ in $L\otimes L^d$ can be obtained in this way.*

PROOF. Obvious from the discussion above.

For any $\tau\in G(\mathscr{L})$ above and for any $\Phi\in T(\mathscr{L})$,

LEMMA 6. *Φ^τ is a transitive system of homomorphisms for $\{L_\alpha: \alpha\in Y\}$.*

PROOF. For any $x\in L_\alpha$, $x(\tau^{-1}\phi_\alpha^\alpha\tau)=(x\tau^{-1})\tau=x$. Hence, $\tau^{-1}\phi_\alpha^\alpha\tau$ is the identity mapping on L_α. Next, for any $\alpha,\beta,\gamma\in Y$ with $\alpha\geq\beta\geq\gamma$ and for any $x\in L_\alpha$, $x(\tau^{-1}\phi_\beta^\alpha\tau)(\tau^{-1}\phi_\gamma^\beta\tau)=x\tau^{-1}\phi_\beta^\alpha\phi_\gamma^\beta\tau=x(\tau^{-1}\phi_\gamma^\alpha\tau)$. Hence, $(\tau^{-1}\phi_\beta^\alpha\tau)(\tau^{-1}\phi_\gamma^\beta\tau)=\tau^{-1}\phi_\gamma^\alpha\tau$. Thus $\Phi^\tau=\{\tau^{-1}\phi_\beta^\alpha\tau: \alpha\geq\beta, \alpha,\beta\in Y\}$ is a transitive system of homomorphisms for $\{L_\alpha: \alpha\in Y\}$. For some $\tau\in G(\mathscr{L})$, we have $\Phi^\tau=\Phi$ for all $\Phi\in T(\mathscr{L})$. (Of course, if τ is the identity mapping on L then $\tau\in G(\mathscr{L})$ and $\Phi^\tau=\Phi$). $H(\mathscr{L})=\{\tau\in G(\mathscr{L}): \Phi^\tau=\Phi$ for all $\Phi\in T(\mathscr{L})\}$ is a subgroup of $G(\mathscr{L})$.

LEMMA 7. *$H(\mathscr{L})$ is a normal subgroup of $G(\mathscr{L})$.*

PROOF. Let $\eta \in H(\mathscr{L})$ and $\tau \in G(\mathscr{L})$. For any $\Phi \in T(\mathscr{L})$, $\Phi^{\tau^{-1}\eta\tau} = ((\Phi^{\tau^{-1}})^{\eta})^{\tau} = \Phi^{\tau^{-1}\tau} = \Phi$. Hence, $\tau^{-1}\eta\tau \in H(\mathscr{L})$. Therefore, we can consider the factor group $\overline{G(\mathscr{L})} = G(\mathscr{L})/H(\mathscr{L})$. We shall denote the coset containing $\tau \in G(\mathscr{L})$ by $\bar{\tau}$. For $\Phi \in T(\mathscr{L})$, we define $\Phi^{\bar{\tau}}$ by $\Phi^{\bar{\tau}} = \Phi^{\tau}$. It is obvious that this is well defined. If $\Phi^{\bar{\tau}} = \Phi^{\bar{\delta}}$ for all $\Phi \in T(\mathscr{L})$, then $\bar{\tau} = \bar{\delta}$. Therefore, we can regard $\overline{G(\mathscr{L})}$ as a permutation group on $T(\mathscr{L})$. For any $\Phi \in T(\mathscr{L})$, let $\overline{F_{\Phi}(\mathscr{L})}$ be the fixed group of Φ. Hereafter, let Y be a semilattice, $\mathscr{L} = \{L_\alpha : \alpha \in Y\}$ a collection of left zero semigroups and $L = \Sigma\{L_\alpha : \alpha \in Y\}$ (disjoint sum).

Obviously,

THEOREM 8. *If $|H(\mathscr{L})| > 1$, then $L \otimes L^d$ has at least two p-systems for any left normal s-composition L of $\{L_\alpha : \alpha \in Y\}$.*

COROLLARY 9. *If $|H(\mathscr{L})|\,|\overline{F_{\Phi}(\mathscr{L})}| > 1$, then $L(\Phi) \otimes L(\Phi)^d$ has at least two p-systems. In this case, $|\mathscr{P}(L(\Phi) \otimes L(\Phi)^d)| = |H(\mathscr{L})|\,|\overline{F_{\Phi}(\mathscr{L})}|$.*

Hereafter, we consider the case $|\overline{G(\mathscr{L})}| < \infty$ and $|T(\mathscr{L})| < \infty$. Decompose $T(\mathscr{L})$ into the systems of transitivity with respect to $\overline{G(\mathscr{L})}$: $T(\mathscr{L}) = \Delta_1 \dotplus \Delta_2 \dotplus \cdots \dotplus \Delta_r$, where each Δ_i is a system of transitivity and $\dotplus$ denotes "disjoint sum". Hence, if $\Phi_i \in \Delta_i$, then $\Delta_i = \{\Phi_i^{\bar{\tau}} : \bar{\tau} \in \overline{G(\mathscr{L})}\}$. The length ℓ_i of Δ_i is given by $\ell_i = |\Delta_i| = |\overline{G(\mathscr{L})} : \overline{F_{\Phi_i}(\mathscr{L})}|$ (the index of $\overline{F_{\Phi_i}(\mathscr{L})}$ in $\overline{G(\mathscr{L})}$). Hence, $\ell_i \,|\, |\overline{G(\mathscr{L})}|$ for all $i = 1, 2, \dots, r$. If $\ell_i = |\overline{G(\mathscr{L})}|$ for all i, then $r|\overline{G(\mathscr{L})}| = |T(\mathscr{L})|$. Therefore, $|\overline{G(\mathscr{L})}| \,|\, |T(\mathscr{L})|$. Hence, we have the following:

THEOREM 10. *If $|G(\mathscr{L})| \nmid |T(\mathscr{L})|$, then there exists $\Phi \in T(\mathscr{L})$ such that $L(\Phi) \otimes L(\Phi)^d$ has at least two p-systems.*

PROOF. If $|H(\mathscr{L})| > 1$, then this theorem follows from Theorem 8. Suppose that $|H(\mathscr{L})| = 1$. Then, $\overline{G(\mathscr{L})} = G(\mathscr{L})$. Hence, $|\overline{F_{\Phi_i}(\mathscr{L})}| > 1$ for some $\Phi_i \in \Delta_i$. Therefore, $L(\Phi_i) \otimes L(\Phi_i)^d$ has at least two p-systems.

In particular, let us consider the case $|T(\mathscr{L})| < |G(\mathscr{L})|$. In this case, for any $\Phi \in T(\mathscr{L})$ there exist at least two-different $\tau_1, \tau_2 \in G(\mathscr{L})$ such that $\Phi^{\tau_1} = \Phi^{\tau_2}$. Then, $\Phi^{\tau_1\tau_2^{-1}} = \Phi$ and $\tau_1\tau_2^{-1} \neq 1$ (the identity of $G(\mathscr{L})$). Therefore, $L(\Phi) \otimes L(\Phi)^d$ has at least two p-systems. Thus, we have the following results:

COROLLARY 11. *If $|T(\mathscr{L})| < |G(\mathscr{L})|$, then $L(\Phi) \otimes L(\Phi)^d$ has at least two p-systems for any $\Phi \in T(\mathscr{L})$.*

COROLLARY 12. *If $L(\Phi) \otimes L(\Phi)^d$ has a unique p-system for every $\Phi \in T(\mathscr{L})$, then $|G(\mathscr{L})| \,|\, |T(\mathscr{L})|$.*

COROLLARY 13. *If $|G(\mathscr{L})| \,|\, |T(\mathscr{L})|$, $|H(\mathscr{L})| = 1$ and $G(\mathscr{L})$ is a transitive group, then $L(\Phi) \otimes L(\Phi)^d$ has a unique p-system for every $\Phi \in T(\mathscr{L})$.*

Remark. It is obvious that $|G(\mathscr{L})|$ can be evaluated as follows: Let $|L_\alpha|=n_\alpha$ for $\alpha\in Y$. Then, $|G(\mathscr{L})|=\prod_{\alpha\in Y}(n_\alpha!)$.

Examples. Let $Y=\{\alpha, \beta, 0\}$ be a semilattice such that $\alpha\beta=\beta\alpha=0$ and 0 is the zero element.

1. Let $L_\alpha=\{a, b\}$, $L_\beta=\{c, d\}$ and $L_0=\{e, f\}$ be left zero semigroups. Put $\mathscr{L}=\{L_\xi: \xi\in Y\}$, and $L=\Sigma\{L_\xi: \xi\in Y\}$. In this case, $|T(\mathscr{L})|=16$ and $|G(\mathscr{L})|=8$. Hence $|G(\mathscr{L})|\,|\,|T(\mathscr{L})|$. Now, consider the transitive system $\Phi=\{\phi_\alpha^\alpha, \phi_\beta^\beta, \phi_0^0, \phi_0^\alpha, \phi_0^\beta\}$ such that ϕ_0^α: $\{a, b\}\to e$, ϕ_0^β: $\{c, d\}\to e$. Then, for $\tau\in G(\mathscr{L})$ such that $a\tau=b$, $b\tau=a$, $c\tau=d$, $d\tau=c$, $e\tau=e$ and $f\tau=f$, $\Phi^\tau=\Phi$. The bijection τ is clearly not an identity mapping. Hence, $L(\Phi)\otimes L(\Phi)^d$ has at least two p-systems.

2. Let $L_\alpha=\{a\}$, $L_\beta=\{b\}$ and $L_0=\{e, f\}$ be left zero semigroups. Put $\mathscr{L}=\{L_\alpha, L_\beta, L_0\}$, and $L=\Sigma\{L_\xi: \xi\in Y\}$. Then, $|T(\mathscr{L})|=4$ and $|G(\mathscr{L})|=2$. Hence, $|G(\mathscr{L})|\,|\,|T(\mathscr{L})|$. In this case, it is easy to see that $\tau\in G(\mathscr{L})$ and $\Phi^\tau=\Phi$ for $\Phi\in T(\mathscr{L})$ imply $\tau=1$ (the identity of $G(\mathscr{L})$). Therefore, $L(\Phi)\otimes L(\Phi)^d$ has a unique p-system for every $\Phi\in T(\mathscr{L})$.

3. Let $L_\alpha=\{a, b\}$, $L_\beta=\{c\}$ and $L_0=\{e, f\}$ be left zero semigroups. Put $\mathscr{L}=\{L_\alpha, L_\beta, L_0\}$, and $L=\Sigma\{L_\xi: \xi\in Y\}$. Then, $|T(\mathscr{L})|=8$ and $|G(\mathscr{L})|=4$. Hence, $|G(\mathscr{L})|\,|\,|T(\mathscr{L})|$. Now, consider the transitive system $\Phi_1=\{\phi_\alpha^\alpha, \phi_\beta^\beta, \phi_0^0, \phi_0^\alpha, \phi_0^\beta\}$ such that $a\phi_0^\alpha=e$, $b\phi_0^\alpha=f$ and $c\phi_0^\beta=e$. Then, $L(\Phi_1)\otimes L(\Phi_2)$ has a unique p-system. On the other hand, consider the transitive system $\Phi_2=\{\psi_\alpha^\alpha, \psi_\beta^\beta, \psi_0^0, \psi_0^\alpha, \psi_0^\beta\}$ such that a$\psi_0^\alpha=e$, $b\psi_0^\alpha=e$ and $c\psi_0^\beta=e$. Take the bijection $\tau\in G(\mathscr{L})$ such that $a\tau=b$, $b\tau=a$, $e\tau=e$ and $f\tau=f$. Then, $\Phi_2^\tau=\Phi_2$ and τ is not the identity mapping on L. Therefore, $L(\Phi_2)\otimes L(\Phi_2)^d$ has at least two p-systems.

As was seen in the examples above, in case where $|G(\mathscr{L})|\,|\,|T(\mathscr{L})|$, there exist the following both cases:

1. For some $\Phi\in T(\mathscr{L})$, there exists $\tau\in G(\mathscr{L})$ such that $\tau\neq 1$ and $\Phi^\tau=\Phi$.
2. $\Phi^\tau=\Phi$, $\tau\in G(\mathscr{L})$, $\Phi\in T(\mathscr{L})$ imply $\tau=1$.

Now, we easily obtain the following from the group theory:

THEOREM 14. *Let $|G(\mathscr{L})|\,|\,|T(\mathscr{L})|$, and $T_\Phi(\mathscr{L})$ the system of transitivity (of $T(\mathscr{L})$ with respect to $G(\mathscr{L})$) which contains Φ. Then, $H(\mathscr{L})=1$ and $|T_\Phi(\mathscr{L})|=|G(\mathscr{L})|$ for all $\Phi\in T(\mathscr{L})$ if and only if $L(\Phi)\otimes L(\Phi)^d$ has a unique p-system for all $\Phi\in T(\mathscr{L})$.*

Further,

THEOREM 15. *If $\Phi_1^\tau=\Phi_2$ for $\Phi_1, \Phi_2\in T(\mathscr{L})$ and for $\tau\in G(\mathscr{L})$, $L(\Phi_1)\otimes L(\Phi_1)^d$ and $L(\Phi_2)\otimes L(\Phi_2)^d$ have the same number of p-systems.*

PROOF. Both Φ_1 and Φ_2 are contained in the same system of transitivity. Hence, $T_{\Phi_1}(\mathscr{L})=T_{\Phi_2}(\mathscr{L})$. Therefore, $|T_{\Phi_1}(\mathscr{L})|=|\overline{G(\mathscr{L})}: \overline{F_{\Phi_1}(\mathscr{L})}|=|\overline{G(\mathscr{L})}: \overline{F_{\Phi_2}(\mathscr{L})}|=|T_{\Phi_2}(\mathscr{L})|$. Thus, $|\overline{F_{\Phi_1}(\mathscr{L})}|=|\overline{F_{\Phi_2}(\mathscr{L})}|$. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the sets of p-systems in $L(\Phi_1)\otimes L(\Phi_1)^d$

and in $L(\Phi_2)\otimes L(\Phi_2)^d$ respectively. Then, it follows from Theorem 8 that $|\mathscr{P}_1|=|H(\mathscr{L})|\,|\overline{F_{\Phi_1}(\mathscr{L})}|=|H(\mathscr{L})|\,|\overline{F_{\Phi_2}(\mathscr{L})}|=|\mathscr{P}_2|$.

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