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On Homogeneous Systems V

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Abstract homogeneous systems are characterized as subsets of their enveloping groups. The results are applied to construction of analytic homogeneous systems tangent to the given Lie triple algebras.

Introduction

As a slight generalization of homogeneous loops in [1], we have introduced the concept of homogeneous systems and investigated in [3] several fundamental properties of analytic homogeneous systems and their tangent Lie triple algebras. Let $\mathfrak{G}$ be a real Lie triple algebra of finite dimension, and $\mathfrak{A}=\mathfrak{G}\oplus\mathfrak{K}$ an enveloping Lie algebra by a Lie algebra $\mathfrak{K}$ of derivations of $\mathfrak{G}$. To construct an analytic homogeneous system (G, η) with its tangent Lie triple algebra $\mathfrak{G}$, it is natural to consider a Lie group A and its subgroup K with their Lie algebras $\mathfrak{A}$ and $\mathfrak{K}$, respectively, and to set $G=A/K$. In this paper, we investigate how an abstract homogeneous system (G, η) is embedded into its enveloping group $A=G\times K_e$ at $e\in G$. We apply the results to the case of analytic homogeneous systems and find some conditions for $G=A/K$ to admit an analytic homogeneous system whose tangent Lie triple algebra is isomorphic to a given Lie triple algebra $\mathfrak{G}$ (Theorem 2). The terminology used in this paper are referred to [3].

§1. Abstract homogeneous systems embedded in their enveloping groups

In this section we are concerned with abstract homogeneous systems. Let (G, η) be an abstract homogeneous system on a set G with a fixed element e. We recall the enveloping groups of (G, η) (cf. [3–IV]). Let μ be the binary operation on G given by $\mu(x, y)=\eta(e, x, y)$ for $x, y\in G$. Under this multiplication, e is a two-sided unit and the element $x^{-1}=\eta(x, e, e)$ is the unique two-sided inverse of x. If we denote by $L_x y=\mu(x, y)$, the set of all left inner mappings $L_{x,y}=L^{-1}_{\mu(x,y)}L_xL_y=\eta(x^{-1}, e)\eta(y, x^{-1})\eta(e, y)$, $x, y\in G$, generates a subgroup Λ_e (*left inner mapping group*) of the isotropy subgroup A_e of $\mathrm{Aut}(\eta)$. Let K_e be a subgroup of A_e containing Λ_e. The set $A=G\times K_e$ forms a group under the group multiplication

$$(x, \alpha)(y, \beta)=(\mu(x, \alpha y), L_{x,\alpha y}\alpha\beta) \tag{1.1}$$

for $x, y \in G$ and $\alpha, \beta \in K_e$. The group A is called the *enveloping group* of (G, η) by the group K_e. The unit of A is $(e, 1_G)$ and the element $(\alpha^{-1}(x^{-1}), \alpha^{-1})$ is the inverse of $(x, \alpha) \in A$. This group has been given originally for homogeneous loops [1]. By identifying G with the subset $G \times \{1_G\}$ of the enveloping group A we can characterize homogeneous systems as subsets of groups, that is;

PROPOSITION 1. *Let (G, η) be an abstract homogeneous system with a fixed element e, $A = G \times K_e$ an enveloping group of (G, η) by a group K_e as above. Then, the subgroup $K = \{e\} \times K_e$ and the subset $G = G \times \{1_G\}$, $x = (x, 1_G)$, satisfy the following conditions* (i)–(iv):

(i) $A = GK$ *(uniquely factored).*

(ii) $G \cap K = \{e\}$, *where $e = (e, 1_G)$ is the unit of A.*

(iii) $G^{-1} = G$.

(iv) $(\mathrm{ad}k)G = G$ *for $k \in K$.*

Conversely, for a subgroup K of an abstract group A, if a subset G of A satisfies (i)–(iv) *above, then G admits a homogeneous system η. In this case, for the normal subgroup $K_0 = \{k \in K \mid kx = xk, x \in G\}$ of A, the quotient group A/K_0 is isomorphic to an envenloping group $G \times (K/K_0)$ of (G, η).*

PROOF. Let $A = G \times K_e$ be an enveloping group of (G, η). Then, (i)–(iv) follow directly from the definition (1.1) of the group multiplication of A. Conversely, assume that a subset G and a subgroup K of an abstract group A satisfy the conditions (i)–(iv). Denote by $p: A \to G$ and $q: A \to K$ the projections to G-factor and K-factor, respectively, of the elements of A in the factorization (i). We consider two operations $\mu: G \times G \to G$ and $\alpha: G \times G \to K$ given by $\mu(x, y) = p(xy)$ and $\alpha(x, y) = q(xy)$ for $x, y \in G$. The following are checked easily for μ and α: (a) $\mu(e, x) = \mu(x, e) = x$, $\alpha(e, x) = \alpha(x, e) = e$; (b) $\mu(x^{-1}, x) = \mu(x, x^{-1}) = e$, $\alpha(x^{-1}, x) = \alpha(x, x^{-1}) = e$; (c) $\mu(x^{-1}, \mu(x, y)) = y$, $\alpha(x, y)^{-1} = \alpha(x^{-1}, \mu(x, y))$. Moreover, the condition (iv) implies (d) $\mu((\mathrm{ad}\,k)x, (\mathrm{ad}\,k)y) = (\mathrm{ad}\,k)\mu(x, y)$, $\alpha((\mathrm{ad}\,k)x, (\mathrm{ad}\,k)y) = (\mathrm{ad}\,k)\alpha(x, y)$ and (e)$L_{x,y} = \mathrm{ad}\,\alpha(x, y)$, where $L_{x,y} = L^{-1}_{\mu(x,y)} L_x L_y$ denotes the left inner mapping of the multiplication $L_x y = \mu(x, y)$ in G. Now, we define a ternary operation $\eta: G \times G \times G \to G$ by

$$\eta(x, y, z) = L_x \mu(L_x^{-1} y, L_x^{-1} z) \qquad \text{for} \quad x, y, z \in G. \tag{1.2}$$

In the same way as the proof of Theorem 1 in [2], we can show that η satisfies the axiom of homogeneous systems. Since $\eta(e, x, y) = \mu(x, y)$ holds in (1.2), (d) and (e) imply that $K_e = (\mathrm{ad}\,K)|_G$ is a subgroup of $\mathrm{Aut}(\eta)$ leaving e fixed and containing the left inner mapping group A_e of μ. Thus the group A/K_0 is isomorphic to the enveloping group $G \times K_e$ of (G, η) for $K_e = K/K_0$. q.e.d.

In the following, we describe the conditions (i)–(iv) in Proposition 1 in terms of the projections $p: A \to G$ and $q: A \to K$. Let A be an abstract group, K a subgroup of A and $G = A/K$ the left cosets of A modulo K. Denote by $i: K \to A$ the inclusion map and by $p\colon A\to G$ the natural projection.

PROPOSITION 2. *For the sequence*

$$K \xrightarrow{i} A \xrightarrow{p} G = A/K$$

the following (1) *and* (2) *are mutually equivalent*:

(1) *There exists a map* $j\colon G\to A$ *such that* (i) $pj=1_G$, (ii) $jp(e)=e$, (iii) $(jp(a))^{-1}=jp((jp(a))^{-1})$ *and* (iv) $jp(ka)=(\mathrm{ad}\, k)jp(a)$ *for* $a\in A$ *and* $k\in K$, *where* e *denotes the unit of* A.

(2) *There exists a map* $q\colon A\to K$ *such that* (i) $qi=1_K$, (ii) $q(q(a)a^{-1})=e$, (iii) $q(ak)=q(a)k$ *and* $q(ka)=kq(a)$, *for* $a\in A$ *and* $k\in K$.

If either of (1) *and* (2) *occurs, then each of the maps* j *and* q *determines the other by the following relation*

$$(jp(a))(iq(a))=a,\ a\in A, \tag{1.3}$$

which gives $q^{-1}(e)=j(G)$.

PROOF. By comparing the conditions in (1) and (2) under the relation (1.3), we get the proposition directly. q.e.d.

Propositions 1 and 2 imply the following theorem:

THEOREM 1. *A set* G *admits a homogeneous system* $\eta\colon G\times G\times G\to G$ *if and only if any one of the following is satisfied for a group* A *and a subgroup* K *of* A:

(1) *Under an injection* $j\colon G\to A$, G *can be identified with a subset of* A *satisfying* (i)–(iv) *of* Proposition 1.

(2) G *can be regarded as the left cosets* A/K *and there exists a map* $j\colon G\to A$ *which satisfies the condition* (1) *of* Proposition 2.

(3) G *can be regarded as the left cosets* A/K *and there exists a map* $q\colon A\to K$ *which satisfies the condition* (2) *of* Proposition 2.

PROOF. By virtue of Propositions 1 and 2, it is sufficient to show that the subset $j(G)$ of A satisfies (i)–(iv) of Proposition 1 if and only if a surjection $q\colon A\to K$ with $q^{-1}(e)=j(G)$ satisfies the conditions (i)–(iii) in (2) of Proposition 2. For an injection $j\colon G\to A$, suppose that the subset $j(G)$ of A satisfies (i)–(iv) of Proposition 1. Then, the factorization $A=j(G)K$ induces the natural projection $q\colon A\to K$ of A into the K-factor. For any $a\in A$ let x and $k=q(a)$ be the $j(G)$-factor and K-factor of a, that is, $a=xk$, $x\in j(G)$, $k\in K$. We have $q(q(a)a^{-1})=q(kk^{-1}x^{-1})=e$ since $x^{-1}\in j(G)$. For any $k_1\in K$, $q(ak_1)=q(xkk_1)=kk_1=q(a)k_1$ and $q(k_1ak_1^{-1})=q((k_1xk_1^{-1})(k_1kk_1^{-1}))=k_1q(a)k_1^{-1}$ holds since $(\mathrm{ad}\, k_1)j(G)=j(G)$. The condition $qi=1_K$ is clearly satisfied and we see that the map q satisfies (2) of Proposition 2 with $j(G)=q^{-1}(e)$. Conversely, suppose that there exists a surjection $q\colon A\to K$ satisfying the conditions (i)–(iii) in (2) of Proposition 2 and that the given set G is identified with A/K, for a group A and its subgroup K. Then, the relation (1.3) implies $q^{-1}(e)=j(G)$ and the unique factorization $A=j(G)K$ with $j(G)\cap K=e$ follows. If $x\in j(G)$, then $q(x)=e$ and $q(q(x)x^{-1})=e$, that is, $x^{-1}\in q^{-1}(e)=j(G)$. Moreover, for any $k\in K$, $q((\mathrm{ad}\,k)x)=(\mathrm{ad}\,k)q(x)=(\mathrm{ad}\,k)e=e$ and we get $(\mathrm{ad}\,k)G=G$. q.e.d.

§2. Construction of analytic homogeneous systems from their tangent Lie triple algebras

Let $\mathfrak{G}$ be a finite dimensional real Lie triple algebra with the bilinear multiplication XY and the trilinear multiplication $D(X, Y)Z$ for $X, Y, Z\in\mathfrak{G}$. Let $\mathrm{Der}(\mathfrak{G})$ denote the derivation algebra of $\mathfrak{G}$, and $\mathfrak{A}=\mathfrak{G}\oplus\mathrm{Der}(\mathfrak{G})$ be the enveloping Lie algebra of G by $\mathrm{Der}(\mathfrak{G})$. Let A be the connected and simply connected Lie group whose Lie algebra is $\mathfrak{A}$, and K the connected Lie subgroup of A whose Lie algebra is $\mathrm{Der}(\mathfrak{G})$. Then, K is a closed subgroup of A and $G=A/K$ is a reductive homogeneous space of K. Nomizu [4]. Under the natural projection $p\colon A\to G$, the tangent space $T_e(G)$ at the origin $e=p(K)$ is identified with $\mathfrak{G}$. Indeed, the Lie triple algebra $\mathfrak{G}$ is isomorphic to the Lie triple algebra $T_e(G)$ whose multiplications are given by $XY=S_e(X, Y)$ and $D(X, Y)Z=R_e(X, Y)Z$ for $X, Y, Z\in T_e(G)$, where S (resp. R) is the torsion (resp. curvature) of the canonical connection of the reductive homogeneous space A/K.

THEOREM 2. *Let $\mathfrak{G}$ be a finite dimensional real Lie triple algebra, $\mathfrak{A}=\mathfrak{G}\oplus\mathrm{Der}(\mathfrak{G})$ the enveloping Lie algebra of $\mathfrak{G}$ by the derivation algebra $\mathrm{Der}(\mathfrak{G})$ of $\mathfrak{G}$. The simply connected homogeneous space $G=A/K$ obtained above admits an analytic homogeneous system η if there exists an analytic map $j\colon G\to A$ satisfying the conditions* (i)–(iv) *in* (1) *of* Proposition 2, *or an analytic map $q\colon A\to K$ satisfying* (i)–(iii) *in* (2) *of* Proposition 2. *In this case, the tangent Lie triple algebra of (G, η) is isomorphic to the given Lie triple algebra $\mathfrak{G}$.*

PROOF. Suppose that there exists an analytic map $j\colon G\to A$ satisfying (i) $pj=1_G$, (ii) $jp(e)=e$, (iii) $(jp(a))^{-1}=jp((jp(a)^{-1})$ and (iv) $jp(ka)=(\mathrm{ad}\,k)jp(a)$ for $a\in A$ and $k\in K$. By (i) we see that j is an immersion and $j(G)$ is an analytic submanifold of A. On the other hand, Proposition 2 implies that there exists a map $q\colon A\to K$ determined by the relation (1.3) for the given map j, i.e., $iq(a)=(jp(a))^{-1}a$, $a\in A$. This relation shows that the map q is analytic. Conversely, if the map q in (2) of Proposition 2 is given and analytic, then the map j determined by (1.3) is analytic since the map jp is so.

Now, assume that there exists such a map j (or q). By Theorem 1, there exists a homogeneous system η on G given by (1.2), and it is analytic since $\mu(x, y)=p(xy)$ is analytic in (x, y). The left inner map of μ is given by $L_{x,y}=ad\,\alpha(x, y)$, where $\alpha(x, y)=$

$q(xy)$ is analytic in (x, y). Let $K_e=(\mathrm{ad}\,K)|_G$ be the group of automorphisms of (G, η) generated by the inner automorphisms $\mathrm{ad}\,k$ $(k\in K)$ restricted on G. Then, K_e is a connected Lie group containing the left inner mapping group A_e of (G, η). Let $\tilde{A}=G\times K_e$ be the enveloping Lie group of (G, η) by K_e. The map $\psi: A\to\tilde{A}$ defined by $\psi(xk)=(x, 1_G)(e, \mathrm{ad}\,k|_G)$ for $a=xk$, $x\in G$, $k\in K$, is an analytic homomorphism of A onto $\tilde{A}$. The kernel of ψ consisting of $K_0=\{k\in K \mid kx=xk,\ x\in G\}$ is a discrete subgroup of K since the Lie algebra of K is the derivation algebra $\mathrm{Der}(\mathfrak{G})$ of $\mathfrak{G}$. Thus, we have an isomorphism $d\psi$ of the Lie algebra $\mathfrak{A}=\mathfrak{G}\oplus\mathrm{Der}(\mathfrak{G})$ onto the enveloping Lie algebra $\tilde{\mathfrak{A}}=\tilde{\mathfrak{G}}\oplus\tilde{\mathfrak{K}}$ of the tangent Lie triple algebra $\tilde{\mathfrak{G}}$ of (G, η), where $\tilde{\mathfrak{K}}$ denotes the Lie algebra of the Lie subgroup $\{e\}\times K_e$ of $\tilde{A}$. Since ψ maps the Lie subgroup K onto $\{e\}\times K_e$ and the submanifold G of A onto $G\times\{1_G\}$, $d\psi$ maps $\mathrm{Der}(\mathfrak{G})$ isomorphically onto $\tilde{\mathfrak{K}}$ and $d\psi(\mathfrak{G})=\tilde{\mathfrak{G}}$. Hence, $(d\psi X)(d\psi Y)=[d\psi X, d\psi Y]_{\tilde{\mathfrak{G}}}=d\psi[x, y]_{\mathfrak{G}}=d\psi(XY)$ and $\tilde{D}(d\psi X, d\psi Y)d\psi Z=[[d\psi X, d\psi Y]_{\tilde{\mathfrak{K}}}, d\psi Z]=[d\psi[X, Y]_{\mathrm{Der}(\mathfrak{G})}, d\psi Z]=d\psi(D(X, Y)Z)$ hold for $X, Y, Z\in\mathfrak{G}$, that is, the Lie triple algebra $\mathfrak{G}$ is isomorphic to the tangent Lie triple algebra $\tilde{\mathfrak{G}}$ of (G, η). q.e.d.

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