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# Note on the Properties of the Runge-Lenz Vector

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A simplified treatment of the Kepler problem is developed in momentum space by the use of the Runge-Lenz vector. The vector is a constant of motion for the inverse square central force. The meaning and the construction method in momentum space is considered, and as examples of this method familiar results are derived easily without integration of differential equation.

## §1. Introduction

After the success of SU(3) in particle physics, attempts were made to reveal the symmetry in the classical three-dimensional Keplerian and isotropic oscillator systems. It has long been known that the above familiar systems in classical mechanics possess invariance under groups O(4) and SU(3) respectively. Fradkin [1] developed a general procedure to show that all dynamical problems involving central potentials, non-relativistically and relativistically, inherently possess both O(4) and SU(3) symmetry. As a consequence of an internal symmetry associated with non-relativistical Kepler problem, there exists a conserved quantity, the so-called Runge-Lenz vector. In the harmonic oscillator, there is a covserved symmetric tensor, which has some properties analogous to the Runge-Lenz vector.

The original classical discussion of the vector is due to Runge and Lenz [2]. The energy levels of the hydrogen atom were determined algebraically by Pauli [3] using this vector earlier, independently and simultaneously with Schrödinger's treatment [4] on the basis of the wave equation. Such a vector has been known for a long time. In recent years several authors have treated this vector in the papers, for example, Dahl [5] has found the Runge-Lenz vector to be a vector quantity and completely understood in the framework of special relativity. Redmond [6] has obtained the generalization of this vector in the presence of a uniform electric field.

In this article a construction of the Runge-Lenz vector in the Kepler problem is presented in momentum space instructionally and geometrically. It is based on the recognition that hodographs in velocity space for the orbit of conic sections are always circles.

## § 2. Construction of the Runge-Lenz Vector in Momentum Space

The system of a particle of mass $m$ in the central force field possess rotational symmetry and the orbital angular momentum $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$ is a constant of motion. Thus the orbit in the Keplerian system has a constant orientation to the angular momentum $\boldsymbol{L}$. If the central force is an inverse square force $\kappa/r^2$ and attractive with force constant $\kappa<0$ ($\kappa=-GmM$ for gravitational force and $\kappa=-Ze^2$ for hydrogen-like atom). The motion is bounded and periodic and the orbit of the particle is closed. There exists, therefore, two vectors with fixed direction in the plane, which are the constants of motion. As one of these, the Runge-Lenz vector $\boldsymbol{K}$ is defined and the other is $\boldsymbol{K}\times\boldsymbol{L}$. The vector $\boldsymbol{K}$ starts at the center of the force and points to a certain direction in the plane of the orbit. Since the plane is perpendicular to $\boldsymbol{L}$

$$\boldsymbol{L}\cdot\boldsymbol{K}=0 \tag{1}$$

The Hamiltonian of the system

$$H=\frac{\mathrm{p}^2}{2m}+\frac{\kappa}{r} \tag{2}$$

is also a constant of motion.

In order to determine the Runge-Lenz vector, one recall that in momentum space the terminal of the momentum vector draws a circle whose radius $p'$ is $m|\kappa|/L$ and the center of the circle is shifted from the origin by an amount $p_0=m|\kappa|e/L$, where $e$ is the eccentricity. (In velocity space, the locus of the velocity vector corresponding to the path in the coordinate space is called a hodograph)

The elliptical orbit and the corresponding path in momentum space are shown in Fig. 1(a) and (b).

In the figure the angular momentum $\boldsymbol{L}$ points out of paper. From the figure

$$\boldsymbol{p}_0+\boldsymbol{p}'=\boldsymbol{p} \tag{3}$$

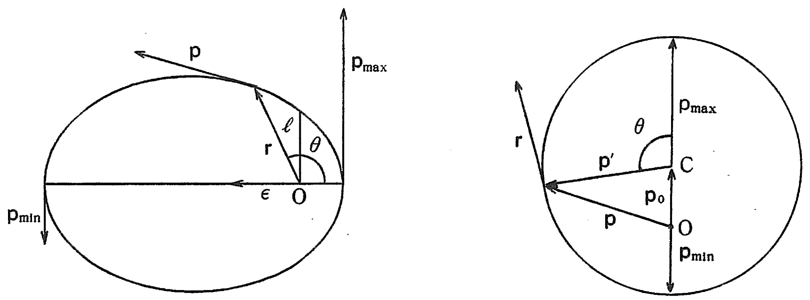


Fig. 1. (a) Elliptical orbit (b) Circular path in momentum space

As the particle moves around the orbit, momentum vector $\boldsymbol{p}$ changes in direction and magnitude in momentum space, and the end point of $\boldsymbol{p}$ draws a circle of radius $p'$, but the vector $\boldsymbol{p}_0$ does not change. Thus $\boldsymbol{p}_0$ is a fixed vector and hence the constant of motion. This is related to the Runge-Lenz vector. The cross product $\boldsymbol{K}=\boldsymbol{p}_0\times\boldsymbol{L}$ is also a constant of motion and is reduced to the relation

$$\begin{aligned}\boldsymbol{K}&=\boldsymbol{p}_0\times\boldsymbol{L}=(\boldsymbol{p}-\boldsymbol{p}')\times\boldsymbol{L}=\boldsymbol{p}\times\boldsymbol{L}+m\kappa\hat{\boldsymbol{r}}\\&=m\dot{\boldsymbol{r}}\times\boldsymbol{L}+m\kappa\hat{\boldsymbol{r}}\end{aligned}\tag{4}$$

where $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$ the unit vector along $r$. This is the definition of the Runge-Lenz vector.

To show that $\boldsymbol{K}$ is a constant of motion explicitly, we need to prove

$$\frac{d\boldsymbol{K}}{dt}=0\tag{5}$$

We perform the differentiation

$$\frac{d\boldsymbol{K}}{dt}=\frac{d\boldsymbol{p}}{dt}\times\boldsymbol{L}+\boldsymbol{p}\times\frac{d\boldsymbol{L}}{dt}+m\kappa\frac{d}{dt}\hat{\boldsymbol{r}}\tag{6}$$

Now

$$\frac{d\boldsymbol{L}}{dt}=0,\quad \frac{d\boldsymbol{p}}{dt}=\frac{\kappa}{r^2}\hat{\boldsymbol{r}}=\frac{\kappa}{r^3}\boldsymbol{r},\ \boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}=m\boldsymbol{r}\times\boldsymbol{v}\tag{7}$$

and

$$\boldsymbol{r}\cdot\boldsymbol{v}=\boldsymbol{r}\cdot\left(\frac{dr}{dt}\hat{\boldsymbol{r}}+r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right)=r\frac{dr}{dt}\tag{8}$$

where $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ are the unit vectors along $r$ and $\theta$ axis respectively.

Therefore

$$\begin{aligned}\frac{d\boldsymbol{K}}{dt}&=\frac{\kappa}{r^3}\boldsymbol{r}\times\boldsymbol{L}+m\kappa\left\{\frac{\boldsymbol{v}}{r}-\frac{1}{r^3}(\boldsymbol{v}\cdot\boldsymbol{r})\boldsymbol{r}\right\}\\&=\frac{m\kappa}{r^3}\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{v})+m\kappa\left\{\frac{r^2\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{r})\boldsymbol{r}}{r^3}\right\}\\&=\frac{m\kappa}{r^3}\{(\boldsymbol{r}\cdot\boldsymbol{v})\boldsymbol{r}-r^2\boldsymbol{v}\}+m\kappa\left\{\frac{r^2\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{r})\boldsymbol{r}}{r^3}\right\}\\&=0\end{aligned}\tag{9}$$

The magnitude of $\boldsymbol{K}$ is $m|\kappa|e$ by the definition $\boldsymbol{K}=\boldsymbol{p}_0\times\boldsymbol{L}$ and the vector

$$\boldsymbol{\varepsilon}=\frac{\boldsymbol{p}\times\boldsymbol{L}}{m\kappa}+\hat{\boldsymbol{r}}\tag{10}$$

is a dimensionless vector of magnitude $e$ and points to the aphelion along the major axis of elliptical orbit. If $e=0$ the Runge-Lenz vector is indefinite.

## §3. Equation of Orbit and Energy

The orbit equation for the Kepler problem is obtained conventionally in terms of the variable $u=1/r$ and a constant of motion $L=mr^2\dot{\theta}$ and by performing the integration of the differential equation for $r$. The labor of solving differential equation may be avoided by the use of eccentricity vector $\boldsymbol{\varepsilon}$. Taking $\boldsymbol{\varepsilon}$ to be the fixed direction and the azimuthal angle $\theta$ is measured from the direction of $-\boldsymbol{\varepsilon}$.
From the definition of $\boldsymbol{\varepsilon}$ (Eq. (10))

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{r}=\varepsilon r\cos(\pi-\theta)=\frac{1}{m\kappa}(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{r}+r$$

$$=\frac{1}{m\kappa}(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{L}+r$$

$$=\frac{1}{m\kappa}\boldsymbol{L}^2+r$$

or

$$r(1+e\cos\theta)=-\frac{L^2}{m\kappa}=\frac{L^2}{m|\kappa|} \tag{11}$$

which is just the equation of conic section and thus $\boldsymbol{\varepsilon}$ lies along the major axis of elliptical orbit and points aphelion. This is the reason why $\boldsymbol{\varepsilon}$ is called the eccentricity vector.

The energy of a particle at perihelion or aphelion where $\dot{r}=0$ and

$$E=\frac{1}{2}mr^2\dot{\theta}^2+\frac{\kappa}{r}=\frac{L^2}{2mr^2}+\frac{\kappa}{r} \tag{12}$$

This give the equation

$$r^2-\frac{\kappa}{E}r-\frac{L^2}{2mE}=0 \tag{13}$$

and could be solved for the both turning points for r. Thus leads to

$$r_{\max}=-\frac{\kappa}{2E}(1+e)$$
$$r_{\min}=-\frac{\kappa}{2E}(1-e) \tag{14}$$

For the elliptical orbit of semi-major axis $a$

$$E=\frac{\kappa}{2a} \quad \text{and} \quad L^2=-m\kappa a(1-e^2) \tag{15}$$

and thus

$$\begin{aligned} r_{\max}&=a(1+e)=-\frac{L^2/m\kappa}{1-e} \\ r_{\min}&=a(1-e)=-\frac{L^2/m\kappa}{1+e} \end{aligned} \tag{16}$$

From Eq. (16) the eccentricity $e$ is given by

$$e=\left(1+\frac{2EL^2}{m\kappa^2}\right)^{\frac{1}{2}} \tag{17}$$

which for $E<0$ obeys

$$0\leq e<1$$

To verify the conservation of energy geometrically, we apply the cosine formula of trigonometry to the triangle in momentum space

$$\begin{aligned} p^2&=p_0^2+p'^2+2p_0p'\cos(\pi-\theta) \\ &=p_0^2+p'^2-2p_0p'\cos\theta \end{aligned} \tag{18}$$

We remind $p_0=m|\kappa|e/L$ and the orbit equation $r=L^2/m|\kappa|(1+e\cos\theta)^{-1}$, the following relation is obtained

$$p_0\cos\theta=\frac{m\kappa}{L}+L/r \tag{19}$$

Substituting Eq. (19) in Eq. (18), we obtain

$$p^2=\left(\frac{m\kappa}{L^2}\right)(e^2-1)-\frac{2m\kappa}{r} \tag{20}$$

and hence

$$E=\frac{1}{2m}p^2+\frac{\kappa}{r}=\left(\frac{m\kappa}{L}\right)^2(e^2-1)=\text{const.} \tag{21}$$

The relation between the magnitude of $\boldsymbol{K}$, $E$ and $\boldsymbol{L}$ is

$$\boldsymbol{K}^2=2E\boldsymbol{L}^2+m^2\kappa^2 \tag{22}$$

## §4. Scattering in the Repulsive Inverse Square Central Force Field

Next we consider the scattering of a particle due to the inverse square repulsive

force. Now $\kappa = ZZ'e^2 > 0$.
Key point to this problem is to obtain the relation between the collision parameter $b$ and the angle of scattering $\varphi$. The angle of scattering is easily obtained for the hyperbolic orbit considering the corresponding path in momentum space. Path of the end point of momentum vector draws a circular arc of radius $p' = m\kappa/L$ also in the repulsive case. Hyperbolic orbit and part of a circle are shown in Fig. 2(a), (b).

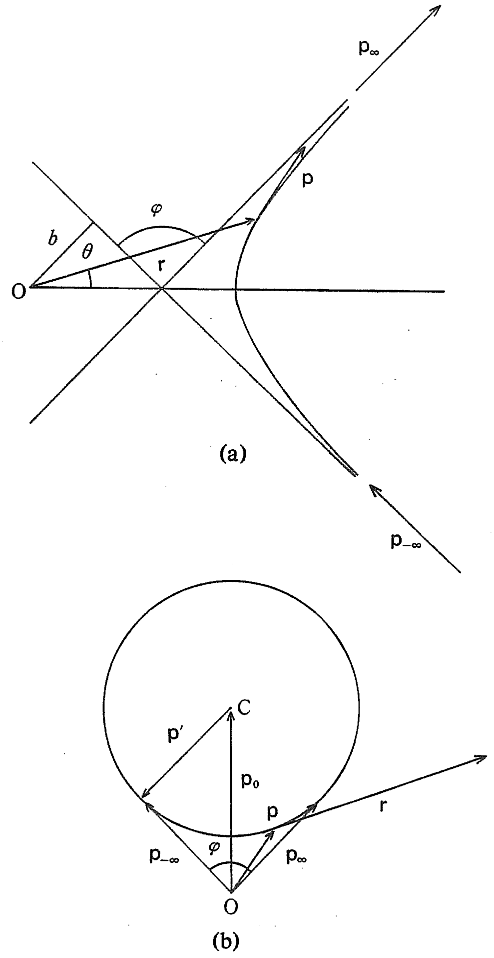


Fig. 2. (a) Hyperbolic Orbit (b) Circular arc in momentum space corresponding to the hyperbolic orbit

In this repulsive case the origin O in momentum space is outside the circle. The center of the circle is shifted also $p_0 = m\kappa e/L$. Now we denote the momentum of incoming and outgoing particle at infinity by $\boldsymbol{p}_{-\infty}$ and $\boldsymbol{p}_{\infty}$, both tangent to the circle and $p_{-\infty}=p_{\infty}$ from Fig. 2(b) $\tan\frac{\varphi}{2}=p'/p_{\infty}$ and using the relations

$$p'=\frac{m\kappa}{L},\quad E=\frac{1}{2}mv_{\infty}^2=\frac{1}{2}mp_{\infty}^2,\ L=bp_{\infty} \tag{23}$$

the following result is obtained

$$\tan\frac{\varphi}{2}=\frac{p'}{p_{\infty}}=\frac{m\kappa}{Lp_{\infty}}=\frac{m\kappa}{bp_{\infty}^2}=\frac{\kappa}{2bE} \tag{24}$$

or

$$b=\frac{\kappa}{mv^2\tan\frac{\varphi}{2}}=\frac{\kappa}{2E}\cot\frac{\varphi}{2} \tag{25}$$

The same result can also be found by the conservation of the Runge-Lenz vector for the incident and scattered particles: $\boldsymbol{K}_i=\boldsymbol{K}_s$.
where

$$\boldsymbol{K}_{\mathrm{i}}=\boldsymbol{p}_{-\infty}\times\boldsymbol{L}_{\mathrm{i}}+m\kappa\hat{r}_{\mathrm{i}},\quad \boldsymbol{K}_{\mathrm{s}}=\boldsymbol{p}_{\infty}\times\boldsymbol{L}_{\mathrm{s}}+m\kappa\hat{r}_{\mathrm{s}} \tag{26}$$

and

$$\boldsymbol{L}_{\mathrm{i}}=bp_{\infty}\hat{z}=\boldsymbol{L}_{\mathrm{s}} \tag{27}$$

$\hat{z}$ is a unit vector which points out of page. The angle between the direction $\hat{r}_{\mathrm{i}}$ and $\hat{r}_{\mathrm{s}}$ and the angle between the vector $\boldsymbol{p}_{-\infty}\times\boldsymbol{L}_{\mathrm{i}}$ and $\boldsymbol{p}_{\infty}\times\boldsymbol{L}_{\mathrm{s}}$ are both the angle of scattering. The particle having a collision parameter between $b$ and $b+db$ will be scattered through an angle between $\varphi$ and $\varphi+d\varphi$.
From Eq. (25)

$$\left|\frac{db}{d\varphi}\right|=\frac{\kappa}{2mv^2\sin^2\frac{\varphi}{2}}$$

Thus, the expression for the differential cross section of the particles incident on the area $d\sigma=2\pi bdb$ are scattered in the solid angle $d\Omega=2\pi\sin\varphi d\varphi$ is immediately written as

$$\frac{d\sigma}{d\Omega}=\frac{\kappa^2}{4m^2v^4\sin^4\frac{\varphi}{2}} \tag{28}$$

which is the famous Rutherford scattering formula.

## §5. Concluding Remarks

The Runge-Lenz vector is a useful tool to deal with the Kepler problem in classical mechanics and its elementary derivation is obtained by considering momentum space. However, it has a deep implication concerning the dynamical symmetry in central force field. In order to treat the hydrogen atom quantum-mechanically by this operator, it must be symmetralized to the form

$$\boldsymbol{K}=\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p}+m\kappa\hat{\boldsymbol{r}} \tag{29}$$

Constants of motion may be shown from the commutation relation for $\boldsymbol{r}$ and $\boldsymbol{p}$, and the following relations can then be shown [7] after a considerable amount of computation,

$$[\boldsymbol{K}, H]=0 \quad \boldsymbol{L}\cdot\boldsymbol{K}=\boldsymbol{K}\cdot\boldsymbol{L}=0$$

$$\boldsymbol{K}^2=2H(\boldsymbol{L}^2+\hbar^2)+m^2\kappa^2 \tag{30}$$

These are the quantum-mechanical analogues of the constancy of $\boldsymbol{K}$ and of Eqs(1) and (22).

Hydrogen atom in the presence of uniform electric and magnetic fields will be treated by this operator.

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