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The Spherically Symmetric and Time Dependent Solution to a Classical SU(2) Gauge Theory

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Abstract

We construct a spherically symmetric and time dependent real solution to a classical SU(2) gauge theory. The coordinate transformations under which the differential equation remains invariant are discussed.

Since the discovery of the non-abelian gauge fields [1], physicists have obtained many useful results in the quantum gauge theories such as in the unified theory of the weak and the electro-magnetic interactions or in the quantum chromodynamics (QCD). More ambitious people of them now attempt to unify all the types of the particle interactions on the gauge principle (grand unification theory, GUT) and they have got many interesting consequences. In spite of such great success, however, in the quantum gauge theories are there several difficulties which are not yet resolved: problems of the Gribov ambiguity, the quark confinement and U(1) etc.. It seems that non-perturbative approaches are needed in order to solve these problems. Usually the quantum fluctuations around the classical solutions are discussed in such non-perturbative treatments. Thus the construction of classical solutions in the gauge theories is the first step of these investigations.

The present author have found axially symmetric and static solutions to a classical SU(2) gauge theory [2], but these solutions are complex numbers and are not acceptable as physical ones though they may be mathematically interesting.

In this paper we construct a real solution to the classical SU(2) gauge theory, which is spherically symmetric and has time dependence.

The classical field equation of the SU(2) gauge fields is given by

$$\nabla_\mu^{ab} G_b^{\mu\nu} = 0, \tag{1}$$

where

$$\nabla_\mu^{ab} = \delta_{ab}\partial_\mu + g\varepsilon_{acb}A_\mu^c, \tag{2}$$

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\varepsilon_{abc}A_b^\mu A_c^\nu. \tag{3}$$

Here we use the same notations as in ref. [2].

Let us impose an ansatz for the spatial components of the potentials A_k^a as follows

$$A_k^a=\frac{1}{g}\varepsilon_{akl}x_l f(r,t), \tag{4}$$

where $r=\sqrt{x^2+y^2+z^2}$. $f(r,t)$ is a function of r and time coordinate t. The time component of the potentials A_0^a should be determined from (1) and (4). Putting $\nu=0$, we get from Eq. (1)

$$\nabla_k^{ab}\nabla_k^{bc}A_c^0=0, \tag{5}$$

where we have used the antisymmetric property of ε_{abc}. $A_a^0=0$ is a solution of Eq. (5). Substituting $A_a^0=0$ into Eq. (1) we have

$$\ddot{f}-f''-\frac{4}{r}f'-3f^2+r^2f^3=0, \tag{6}$$

where $\dot{f}=\dfrac{\partial f}{\partial t}$ and $f'=\dfrac{\partial f}{\partial r}$. This equation is rewritten as

$$r^2(h''-\ddot{h})+(1-h^2)h=0, \tag{7}$$

where

$$f=\frac{1-h(r,t)}{r^2}. \tag{8}$$

The trivial solutions of Eq. (7) are $h=0$, ± 1 or $f=r^{-2}$, 0, $2r^{-2}$, respectively, which are all static. Another solution is found by Actor [3], which is given by

$$h(r,t)=\pm\frac{1}{2}\frac{\lambda^2-x^2}{\left\{\frac{1}{4}(\lambda^2-x^2)^2+\lambda^2r^2\right\}^{\frac{1}{2}}}, \tag{9}$$

where $x^2=-t^2+r^2$ and λ is a parameter which has the dimension of length.

Now let us construct a solution of (7) which does not have any dimensional parameters. If h does not have such parameters, it must be a function of t/r ($\equiv\tau$), because h is a dimensionless quantity. Then Eq. (7) is rewritten as

$$\sinh^2 y\,\frac{d^2\mathscr{G}}{dy^2}+\mathscr{G}(1-\mathscr{G}^2)=0, \tag{10}$$

where we put

$$h(r,t)=\mathscr{G}(y), \tag{11}$$

$$y=\frac{1}{2}\ln\left(\frac{\tau-1}{\tau+1}\right). \tag{12}$$

A solution of Eq. (10) is given by

$$\mathscr{G} = \pm \cosh y$$

$$= \pm \frac{t}{\sqrt{t^2 - r^2}} \,. \tag{13}$$

From Eq. (4) we have the potentials

$$A_k^a = \frac{1}{g} \varepsilon_{akl} x_l \frac{\sqrt{t^2 - r^2} \mp t}{r^2 \sqrt{t^2 - r^2}} \,. \tag{14}$$

It is seen that A_k^a is complex for $t^2 < r^2$ and is singular at $t^2 = r^2$. This means that Eq. (14) is available as a physical solution only inside the light cone, and that on the light cone there must be a charged matter which sustains the gauge field and moves with the light velocity. The charge density must be expressed as a distribution which depends on $x^2 = -t^2 + r^2$. It is difficult, however, to obtain the concrete expression of this density and we don't discuss this subject here.

Now coming back to Eq. (7), let us discuss the coordinate transformations under which Eq. (7) remains invariant. If such a transformation $(r, t) \to (r', t')$ is found, from a solution of Eq. (7) we can obtain another one by substituting (r', t') for its arguments (r, t).

The most trivial one of such transformations is the scale transformation:

$$r' = \rho r,$$

$$t' = \rho t,$$

where ρ is a constant. But under this transformation (13) is not altered and only the parameter λ changes in (9).

Another simple one is the time translation:

$$t \longrightarrow t' = t + \alpha.$$

The parameter α is a real number. Otherwise the transformed solution is not real and not a physical solution. Putting α imaginary, however, we can connect Eq. (13) with Eq. (9). Substituting $t + i\lambda$ for t into (9) we get

$$h(r, t + i\lambda) \equiv h'(r, t) = \pm \frac{1}{2} \frac{2i\lambda t + t^2 - r^2}{\left\{\frac{1}{4}(t^2 + 2i\lambda t - r^2)^2 + \lambda^2 r^2\right\}^{\frac{1}{2}}} \,.$$

Then putting $\lambda \to \infty$ we have

$$\lim_{\lambda \to \infty} h'(r, t) = \pm \lim_{\lambda \to \infty} \frac{1}{2} \frac{2i\lambda t}{\{-\lambda^2 t^2 + \lambda^2 r^2\}^{\frac{1}{2}}}$$

$$= \pm \frac{t}{\sqrt{t^2 - r^2}} \,.$$

$h'(r, t)$ is an unphysical solution of Eq. (7) because it is a complex number, but it becomes physical again by taking $\lambda \to \infty$.

In order to make Eq. (7) invariant the transformation must satisfy

$$x_1^2 \frac{\partial^2}{\partial x_\mu \partial x^\mu} = x_1'^2 \frac{\partial^2}{\partial x'_\mu \partial x'^\mu}, \tag{15}$$

where we put $t = x^0$, $r = x^1$ and $x_\mu x^\mu = -t^2 + r^2$. The left hand side of (15) is written as

$$x_1^2 \frac{\partial^2}{\partial x_\mu \partial x^\mu} = x_1^2 \left(\frac{\partial^2 x'_\nu}{\partial x_\mu \partial x^\mu} \frac{\partial}{\partial x'_\nu} + \frac{\partial x'_\nu}{\partial x_\mu} \frac{\partial x'_\lambda}{\partial x^\mu} \frac{\partial^2}{\partial x'_\nu \partial x'_\lambda} \right).$$

Thus the necessity condition for (15) is

$$\frac{\partial^2 x'_\nu}{\partial x_\mu \partial x^\mu} = 0. \tag{16}$$

This condition leads to

$$\begin{cases} t' = x'^0 = A(t+r) + B(t-r), \\ r' = x'^1 = C(t+r) + D(t-r), \end{cases} \tag{17}$$

where A, B, C and D are arbitrary functions. Then Eq. (15) gives

$$\frac{\partial x'_\nu}{\partial x_\mu} \frac{\partial x'_\lambda}{\partial x^\mu} = \frac{r'^2}{r^2} g_{\nu\lambda}. \tag{18}$$

Putting $\nu = 0$ and $\lambda = 1$ or $\nu = 1$ and $\lambda = 0$ in (18) we get

$$A'D' + B'C' = 0, \tag{19}$$

and when $\nu = \lambda = 0$ or $\nu = \lambda = 1$, we have

$$4A'B' = \frac{(C+D)^2}{r^2}, \tag{20}$$

and

$$4C'D' = -\frac{(C+D)^2}{r^2}, \tag{21}$$

respectively, where A' stands for the differential of A with respect to its argument. Eqs. (19), (20) and (21) lead to

$$\begin{cases} A = C, \\ B = -D, \end{cases} \tag{23}$$

up to the space-time inversion and the time translation. Then Eq. (20) or (21) is rewritten into

$$4A'(t+r)B'(t-r)=\frac{\{A(t+r)-B(t-r)\}^2}{r^2}. \tag{24}$$

This equation is easily integrated as

$$A(x_+)-B(x_-)=\alpha(x_+)\beta(x_-)(x_+-x_-), \tag{25}$$

where α and β are arbitrary functions and we put $x_\pm=t\pm r$. It seems that the functions α and β which satisfy Eq. (25) are only given either by constants or by

$$\begin{cases} \alpha(x_+)=\dfrac{a}{x_+}=\dfrac{a}{t+r}, \\ \beta(x_-)=\dfrac{b}{x_-}=\dfrac{b}{t-r}. \end{cases} \tag{26}$$

When α and β are constants, the transformation is the scale one, which has no effect to the solutions (9) and (14). In the second case we can see that

$$\begin{cases} A(t+r)=-\dfrac{ab}{t+r}, \\ B(t-r)=-\dfrac{ab}{t-r}, \end{cases}$$

or

$$\begin{cases} t'=\dfrac{-2abt}{t^2-r^2}, \\ r'=\dfrac{-2abr}{t^2-r^2}. \end{cases} \tag{27}$$

But unfortunately this transformation is also uneffective, because it changes λ into $2ab/\lambda$ in Eq. (9) and does not cause any changes in Eq. (13).

The physical meaning of the solution (13) will be discussed in some other place.

References

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