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Gamonic Functionals on a Product Space

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Dedicated to Professor Makoto Ohtsuka on his 60th birthday

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This paper is concerned with a gamonic functional on the space of continuous functions defined on the product of two compact Hausdorff spaces. Some componentwise properties of the gamonic functional are discussed with its projective gamonic functionals. Symmetric properties of the gamonic functional are also studied from the viewpoint of componentwise sum and product. A new gamonic functional is generated by means of projective gamonic functionals and a set-to-set mapping.

Introduction

The value of a game may be regarded as a functional of its pay-off function. N. J. Kalton [3] called this a gamonic functional and established a minimax representation of it. This paper is concerned with a gamonic functional on a product space. Such a gamonic functional often appears not only in mathematical programming problems but in the theory of capacities (cf. [6]).

In this paper, some properties of a gamonic functional V on a product space are discussed with its projective gamonic functionals V_1 and V_2 in §3. Symmetric properties of V are studied in §4. For a further study of V, we introduce in §5 a new gamonic functional W_A generated by the projective functionals V_1 and V_2 of V and a set-to-set mapping A. In case $W_A = V$, some properties of V may be expressed in terms of some properties of the mapping A. Our theory is applied to some useful gamonic functionals in §6.

§1. Preliminaries

Let S be a compact Hausdorff space, let R be the set of all real numbers and let R^+ be the set of all non-negative real numbers. Denote by $C(S)$ the set of all real valued continuous functions on S, by $C^+(S)$ the subset of $C(S)$ which consists of non-negative functions, by $M(S)$ the set of all real Radon measures on S of any sign, by $M^+(S)$ the subset of $M(S)$ which consists of non-negative measures. Let $P(S) = \{\lambda \in M^+(S); \lambda(S) = 1\}$ and let $PP_c(S)$ be the class of all convex subsets of $P(S)$ which are nonempty and vaguely compact.

DEFINITION 1.1. We say that a real valued functional V on $C(S)$ is gamonic if the following conditions are fulfilled:

(1.1) $V(f)\geq V(g)$ whenever $f\geq g$ with $f, g\in C(S)$,

(1.2) $V(f+a)=V(f)+a$ whenever $f\in C(S)$ and $a\in R$,

(1.3) $V(af)=aV(f)$ whenever $f\in C(S)$ and $a\in R^{+}$.

Denote by $ad[V]$ the class of all admissible sets C for V, i.e., $C\in PP_c(S)$ such that

$$V(f)\leq \max_{\lambda\in C}\int fd\lambda \qquad \text{for all} \quad f\in C(S). \tag{1.4}$$

Let $m[V]$ be the subclass of $ad[V]$ which consists of minimal subsets with respect to the inclusion relation. Kalton [3] proved

THEOREM K. $V(f)=\min_{C\in m[V]}\max_{\lambda\in C}\int fd\lambda$ *for all* $f\in C(S)$.

We say that a subclass $e[V]$ of $ad[V]$ is essential for V if

$$V(f)=\inf_{C\in e[V]}\max_{\lambda\in C}\int fd\lambda \qquad \text{for all} \quad f\in C(S). \tag{1.5}$$

Clearly $ad[V]$ and $m[V]$ are essential for V.

We have

THEOREM 1.1. *The gamonic functional V is sublinear if and only if $ad[V]$ is a singleton.*

PROOF. If $ad[V]$ is a singleton, then $m[V]$ is a singleton and V is sublinear by Theorem K. Assume that V is sublinear and let $f\in C(S)$. Then we see by (1.1) that $V(f)$ is equal to the value of the following programming problem:

$$\text{Find} \quad \inf\{V(g);\ g\in C(S),\ g\geq f \text{ on } S\}. \tag{1.6}$$

The dual problem to (1.6) can be written as follows:

$$\text{Find} \quad \sup\{\int fd\lambda;\ \lambda\in C_V\}, \tag{1.7}$$

where $C_V=\{\lambda\in M^{+}(S);\ \int gd\lambda\leq V(g)$ for all $g\in C(S)\}$. By a duality theorem [7; Theorem 4], we have

$$V(f)=\max\{\int fd\lambda;\ \lambda\in C_V\}. \tag{1.8}$$

Since $V(1)=1$ and $V(-1)=-1$ by (1.2), we see that $C_V \in PP_c(S)$, and hence $C_V \in ad[V]$. Suppose that there exists $C \in ad[V]$ such that $C \neq C_V$. Then we see by a separation theorem [1; p. 73, Proposition 4] and by (1.8) that there exists $f_0 \in C(S)$ such that

$$V(f_0)=\max_{\lambda \in C_V}\int f_0 d\lambda > \max_{\lambda \in C}\int f_0 d\lambda \geq V(f_0).$$

This is a contradiction. Thus $ad[V]=\{C_V\}$.

The adjoint gamonic functional V^* of V is defined by

$$V^*(f)=-V(-f) \qquad \text{for all } f \in C(S). \tag{1.9}$$

Note that V is sublinear if and only if V^* is superlinear.

THEOREM 1.2. *The gamonic functional V is superlinear if and only if there exists a unique $C_0 \in PP_c(S)$ such that*

$$V(f)=\min_{\lambda \in C_0}\int f d\lambda \qquad \textit{for all } f \in C(S). \tag{1.10}$$

In this case $m[V]=\{\{\lambda\};\ \lambda \in C_0\}$.

PROOF. Assume that V is superlinear. Then the adjoint gamonic functional V^* is sublinear. By Theorem 1.1, there exists a unique $C_0 \in PP_c(S)$ such that $ad[V^*]=\{C_0\}$, or equivalently (1.10) holds. The "if" part is clear. Let $C \in m[V]$. If $C \cap C_0 = \emptyset$, then there exists $f_0 \in C(S)$ such that

$$V(f_0) \leq \max_{\lambda \in C}\int f_0 d\lambda < \min_{\lambda \in C_0}\int f_0 d\lambda = V(f_0).$$

This is a contradiction. Therefore $C \cap C_0 \neq \emptyset$. Since $\{\lambda\} \in ad[V]$ for every $\lambda \in C_0$ and C is minimal, we see that C is a singleton.

§ 2. Projective gamonic functionals

For a further study of gamonic functionals, we prepare some fundamental notion on a product space.

Let K_1 and K_2 be compact Hausdorff spaces and let S be the product space $K_1 \times K_2$. For $p_i \in C(K_i)\,(i=1, 2)$, let us define the componentwise sum $p_1 \oplus p_2$ and the componentwise product $p_1 \otimes p_2$ by

$$(p_1 \oplus p_2)(x, y)=p_1(x)+p_2(y),$$

$$(p_1 \otimes p_2)(x, y)=p_1(x)p_2(y)$$

for all $(x, y) \in K_1 \times K_2$. Then $p_1 \oplus p_2$ and $p_1 \otimes p_2$ belong to $C(S)$.

Hereafter we always assume that each of the spaces $M(S)$ and $M(K_i)(i=1, 2)$ is equipped with the vague topology. Let $T_i (i=1, 2)$ be the continuous linear transformation from $M(S)$ into $M(K_i)$ defined by

$$\int p_1 d(T_1\lambda) = \int p_1 \otimes 1 d\lambda, \quad \int p_2 d(T_2\lambda) = \int 1 \otimes p_2 d\lambda$$

for all $p_i \in C(K_i)(i=1, 2)$ and $\lambda \in M(S)$. The measure $T_i\lambda$ is the projection of λ onto K_i. Let T be the linear transformation from $M(S)$ into $M(K_1) \times M(K_2)$ defined by $T\lambda = (T_1\lambda, T_2\lambda)$ for $\lambda \in M(S)$. Note that T is continuous and that $T_i(C) \in PP_c(K_i)$ for every $C \in PP_c(S)$.

For $\mu_i \in P(K_i)(i=1, 2)$, let us define $\mu_1 \otimes \mu_2$ by

$$\int f d(\mu_1 \otimes \mu_2) = \int f(x, y) d\mu_1(x) d\mu_2(y)$$

for every $f \in C(S)$ (cf. [2]). For $C_i \in PP_c(K_i)(i=1, 2)$, denote by $C_1 \otimes C_2$ the closed convex hull of the set $\{\mu_1 \otimes \mu_2; \mu_i \in C_i \ (i=1, 2)\}$.

We prepare

LEMMA 2.1. *Let $C_i \in PP_c(K_i)(i=1, 2)$. Then $T(C_1 \otimes C_2) = C_1 \times C_2$.*

PROOF. Let $\lambda \in C_1 \otimes C_2$. In case λ belongs to the convex hull D of the set $\{\mu_1 \otimes \mu_2; \mu_i \in C_i (i=1, 2)\}$, there exist $\mu_i^{(j)} \in C_i$ and $a_j \in R^+$ $(i=1, 2; j=1,\dots, n)$ such that

$$\lambda = \sum_{j=1}^{n} a_j(\mu_1^{(j)} \otimes \mu_2^{(j)}) \quad \text{and} \quad \sum_{j=1}^{n} a_j = 1. \tag{2.1}$$

Since C_i is convex, we have

$$T\lambda = (\sum_{j=1}^{n} a_j \mu_1^{(j)}, \sum_{j=1}^{n} a_j \mu_2^{(j)}) \in C_1 \times C_2.$$

In the general case, there is a net $\{\lambda_\alpha\}$ in D which converges vaguely to λ. Since T is continuous and $C_1 \times C_2$ is vaguely compact, we have $T\lambda \in C_1 \times C_2$. Therefore $T(C_1 \otimes C_2) \subset C_1 \times C_2$. On the other hand, for every $(\mu_1, \mu_2) \in C_1 \times C_2$, we have $\mu_1 \otimes \mu_2 \in C_1 \otimes C_2$ and $T(\mu_1 \otimes \mu_2) = (\mu_1, \mu_2)$. Therefore $C_1 \times C_2 \subset T(C_1 \otimes C_2)$.

LEMMA 2.2. *Let $C_i \in PP_c(K_i)$ and $p_i \in C^+(K_i)(i=1, 2)$. Then*

$$\max \{\int p_1 \otimes p_2 d\lambda; \lambda \in C_1 \otimes C_2\} = (\max_{\mu \in C_1} \int p_1 d\mu)(\max_{\nu \in C_2} \int p_2 d\nu).$$

PROOF. Let $\lambda \in C_1 \otimes C_2$. In case λ belongs to the set D defined in the above proof, it can be written in the form (2.1), so that

$$\int p_1\otimes p_2 d\lambda=\sum_{j=1}^{n} a_j \int p_1\otimes p_2 d\mu_1^{(j)} d\mu_2^{(j)}=\sum_{j=1}^{n} a_j \int p_1 d\mu_1^{(j)} \int p_2 d\mu_2^{(j)}$$

$$\leq (\max_{\mu\in C_1}\int p_1 d\mu)(\max_{\nu\in C_2}\int p_2 d\nu).$$

In the general case, there exists a net $\{\lambda_\alpha\}$ in D which converges vaguely to λ. Then we have

$$\int p_1\otimes p_2 d\lambda=\lim \int p_1\otimes p_2 d\lambda_\alpha \leq (\max_{\mu\in C_1}\int p_1 d\mu)(\max_{\nu\in C_2}\int p_2 d\nu),$$

and hence

$$\max\{\int p_1\otimes p_2 d\lambda;\ \lambda\in C_1\otimes C_2\}\leq (\max_{\mu\in C_1}\int p_1 d\mu)(\max_{\nu\in C_2}\int p_2 d\nu).$$

The converse inequality follows from the fact that $C_1\otimes C_2$ contains the set $\{\mu_1\otimes\mu_2;\ \mu_i\in C_i\ (i=1, 2)\}$.

We can easily prove

THEOREM 2.1. *Let V be a gamonic functional on $C(S)$ and define functionals V_i on $C(K_i)$ by*

$$V_1(p_1)=V(p_1\otimes 1) \quad \text{and} \quad V_2(p_2)=V(1\otimes p_2) \tag{2.2}$$

for $p_i\in C(K_i)(i=1, 2)$. Then each V_i is gamonic on $C(K_i)$.

We call V_1 and V_2 the projective gamonic functionals obtained from V. Note that the class $\{T_i(C);\ C\in m[V_i]\}$ is essential for V_i.

§3. Componentwise sum and product

Hereafter we shall be concerned with some relations between a gamonic functional V on $C(S)$ and its projective gamonic functionals V_1 and V_2 from the viewpoint of the componentwise sum and product. Let us consider the following relations:

(CS) $V(p_1\oplus p_2)=V_1(p_1)+V_2(p_2)$ whenever $p_i\in C(K_i)\ (i=1, 2)$,
(CSL) $V(p_1\oplus p_2)\geq V_1(p_1)+V_2(p_2)$ whenever $p_i\in C(K_i)\ (i=1, 2)$,
(CSR) $V(p_1\oplus p_2)\leq V_1(p_1)+V_2(p_2)$ whenever $p_i\in C(K_i)\ (i=1, 2)$,
(CP) $V(p_1\otimes p_2)=V_1(p_1)V_2(p_2)$ whenever $p_i\in C^+(K_i)\ (i=1, 2)$,
(CPL) $V(p_1\otimes p_2)\geq V_1(p_1)V_2(p_2)$ whenever $p_i\in C^+(K_i)\ (i=1, 2)$,
(CPR) $V(p_1\otimes p_2)\leq V_1(p_1)V_2(p_2)$ whenever $p_i\in C^+(K_i)\ (i=1, 2)$.

First we show by examples that any one of the above relations does not hold in general.

EXAMPLE 3.1. Let $K_1=K_2=\{x_1, x_2\}$ and define p_1 and p_2 by $p_1(x_1)=p_2(x_2)=1$ and $p_1(x_2)=p_2(x_1)=0$. For $C\in PP_c(S)$, we consider the gamonic functional:

$$V(f)=\max_{\lambda\in C}\int f d\lambda \qquad \text{for} \quad f\in C(S).$$

This V is sublinear by Theorem 1.1. In case $C=\{\lambda\in P(S);\ \lambda(\{x_1, x_2\})=1/4\}$, we have $V_1(p_1)=V_2(p_2)=1$, $V(p_1\oplus p_2)=5/4$ and $V(p_1\otimes p_2)=1/4$, and hence (CSL) and (CPL) do not hold. In case $C=\{\lambda\in P(S);\ \lambda(\{(x_2, x_1)\})=1/2\}$, we have $V_1(p_1)=V_2(p_2)=1/2$ and $V(p_1\otimes p_2)=1/2$, and hence (CPR) does not hold.

EXAMPLE 3.2. Let K_i and p_i be the same as in Example 3.1 and define $V(f)$ for $f\in C(S)$ by

$$V(f)=\min\{(f_{11}+f_{12})/2,\ (f_{12}+f_{22})/2\},$$

where $f_{ij}=f(x_i, x_j)$. Then $V(f)$ is gamonic and $V_1(p_1)=V_2(p_2)=1/2$ and $V(p_1\oplus p_2)=3/2$, and hence (CSR) does not hold.

We begin with the relation between (CS) and (CP).

THEOREM 3.1. (CPL)(*resp.* (CPR)) *implies* (CSL)(*resp.* (CSR)).

PROOF. Assume that (CPL) holds and let $p_i\in C(K_i)\,(i=1, 2)$. First we consider the case where $p_i\in C^+(K_i)\,(i=1, 2)$. For any $a\in R^+$ with $a\neq 0$, we have by (CPL)

$$V((ap_1+1)\otimes(ap_2+1))\geq V_1(ap_1+1)V_2(ap_2+1).$$

We have by (1.2) and (1.3)

$$V((ap_1+1)\otimes(ap_2+1))=aV(a(p_1\otimes p_2)+(p_1\oplus p_2))+1,$$

$$V_1(ap_1+1)V_2(ap_2+1)=a^2V_1(p_1)V_2(p_2)+a[V_1(p_1)+V_2(p_2)]+1,$$

so that

$$V(ap_1\otimes p_2)+(p_1\oplus p_2))\geq aV_1(p_1)V_2(p_2)+V_1(p_1)+V_2(p_2). \tag{3.1}$$

Let $b_i=\max\{p_i(s);\ s\in K_i\}\,(i=1, 2)$. Then we have by (1.1)

$$V(p_1\oplus p_2)\leq V(ap_1\otimes p_2+(p_1\oplus p_2))\leq ab_1b_2+V(p_1\oplus p_2),$$

and hence $V(ap_1\otimes p_2+(p_1\oplus p_2))\to V(p_1\oplus p_2)$ as $a\to 0$. By letting $a\to 0$ in (3.1), we obtain $V(p_1\oplus p_2)\geq V_1(p_1)+V_2(p_2)$. Next we consider the general case. Let $a_i=\min\{p_i(s);\ s\in K_i\}\,(i=1, 2)$. Then $p_i-a_i\in C^+(K_i)$ and

$$V((p_1-a_1)\oplus(p_2-a_2))\geq V_1(p_1-a_1)+V_2(p_2-a_2)$$

by the above observation. Since $(p_1-a_1)\oplus(p_2-a_2)=p_1\oplus p_2-(a_1+a_2)$, we see by

(1.2) that $V(p_1 \oplus p_2) \geq V_1(p_1) + V_2(p_2)$. Namely (CSL) holds. Similarly we can prove that (CPR) implies (CSR).

COROLLARY. (CP) *implies* (CS).

If V is linear, then (CS) holds. If V is sublinear (resp. superlinear), then (CSR) (resp. (CSL)) holds. More generally we have

THEOREM 3.2. *The following condition implies* (CSR):

(3.2) *For every* $C_i \in m[V_i]$ $(i=1, 2)$, $C_1 \times C_2$ *contains an element of* $T(ad[V])$.

PROOF. Let $p_i \in C(K_i)$ $(i=1, 2)$. For any $\varepsilon > 0$, we can find $C_i \in m[V_i]$ such that

$$V_i(p_i) + \varepsilon > \max_{\mu \in C_i} \int p_i d\mu \tag{3.3}$$

for $i=1, 2$. By (3.2), there exists $C' \in ad[V]$ such that $C_1 \times C_2 \supset T(C')$, so that

$$\begin{aligned} V_1(p_1) + V_2(p_2) + 2\varepsilon &> \max_{\mu \in C_1} \int p_1 d\mu + \max_{\nu \in C_2} \int p_2 d\nu \\ &= \max \left\{ \int p_1 \oplus p_2 d\mu d\nu;\ (\mu, \nu) \in C_1 \times C_2 \right\} \\ &\geq \max \left\{ \int p_1 \oplus p_2 d\mu d\nu;\ (\mu, \nu) \in T(C') \right\} \\ &\geq \max_{\lambda \in C'} \int p_1 \oplus p_2 d\lambda \geq V(p_1 \oplus p_2). \end{aligned}$$

Thus $V_1(p_1) + V_2(p_2) + 2\varepsilon > V(p_1 \oplus p_2)$, and hence (CSR) holds.

REMARK 3.1. If V is sublinear, then condition (3.2) is fulfilled. In fact, any one of $ad[V]$ and $m[V_i]$ $(i=1, 2)$ is a singleton by Theorem 1.1.

It is easily verified that neither (CPR) nor (CPL) holds even if V is linear. We have

THEOREM 3.3. *If V is linear, then any one of* (CPR) *and* (CPL) *implies* (CP).

PROOF. If V is linear, then $ad[V] = \{\{\lambda\}\}$ with $\lambda \in P(S)$. Let $T\lambda = (\mu, \nu)$. Suppose that (CPR) holds. Then

$$\int p \otimes q d\lambda \leq \int p \otimes q d(\mu \otimes \nu)$$

for all $p \in C^+(K_1)$ and $q \in C^+(K_2)$. Let $f \in C(S)$. For any $\varepsilon > 0$, there exist $p_i \in C(K_1)$ and $q_i \in C(K_2)$ $(i=1, \ldots, n)$ such that $|f(x, y) - \sum_{i=1}^{n} (p_i \otimes q_i)(x, y)| < \varepsilon$ on S (cf. [2]). We can find a number M such that $p_i + M \in C^+(K_1)$ and $q_i + M \in C^+(K_2)$ for all i.

We have

$$\int p_i\otimes q_i d\lambda=\int \{(p_i+M)\otimes(q_i+M)-M(p_i\oplus q_i)-M^2\}d\lambda$$

$$=\int (p_i+M)\otimes(q_i+M)d\lambda - M\int p_i d\mu - M\int q_i d\nu - M^2$$

$$\leq\int (p_i+M)\otimes(q_i+M)d(\mu\otimes\nu) - M\int (p_i\oplus q_i)d(\mu\otimes\nu) - M^2$$

$$=\int p_i\otimes q_i d(\mu\otimes\nu).$$

It follows that

$$\int fd\lambda-\varepsilon<\sum_{i=1}^{n}\int p_i\otimes q_i d\lambda\leq\sum_{i=1}^{n}\int p_i\otimes q_i d(\mu\otimes\nu)<\int fd(\mu\otimes\nu)+\varepsilon,$$

so that

$$\int fd\lambda\leq\int fd(\mu\otimes\nu).$$

Since f is arbitrary, we conclude that $\lambda=\mu\otimes\nu$. Thus V satisfies (CP). Similarly we can prove that (CPL) implies (CP).

COROLLARY. *Let V be linear and let $ad[V]=\{\{\lambda\}\}$ and $T\lambda=(\mu, \nu)$. Then V satisfies* (CP) *if and only if $\lambda=\mu\otimes\nu$.*

THEOREM 3.4. *The following condition implies* (CPR):

(3.4) $$C_1\otimes C_2\in ad[V] \quad \text{for each} \quad C_i\in m[V_i]\,(i=1, 2).$$

PROOF. Let $p_i\in C^+(K_i)(i=1, 2)$. For any $\varepsilon>0$, we can find $C_i\in m[V_i]\,(i=1, 2)$ which satisfy (3.3). We have by Lemma 2.2 and (3.4)

$$(V_1(p_1)+\varepsilon)(V_2(p_2)+\varepsilon)>(\max_{\mu\in C_1}\int p_1 d\mu)(\max_{\nu\in C_2}\int p_2 d\nu)$$

$$=\max\{\int p_1\otimes p_2 d\lambda;\ \lambda\in C_1\otimes C_2\}$$

$$\geq V(p_1\otimes p_2).$$

Thus (CPR) holds.

We see by Lemma 2.1 that (3.4) implies (3.2).

THEOREM 3.5. *The following condition implies* (CPL):

(3.5) *For every* $C \in m[V]$, *there exist* $C'_i \in ad[V_i]$ $(i=1, 2)$ *such that* $C'_1 \otimes C'_2 \subset C$.

PROOF. Let $p_i \in C^+(K_i)$ $(i=1, 2)$. For any $\varepsilon > 0$, there exists $C \in m[V]$ such that

$$V(p_1 \otimes p_2) + \varepsilon > \max_{\lambda \in C} \int p_1 \otimes p_2 d\lambda.$$

We can find $C'_i \in ad[V_i]$ $(i=1, 2)$ such that $C'_1 \otimes C'_2 \subset C$ by (3.5). It follows from Lemma 2.2 that

$$\max_{\lambda \in C} \int p_1 \otimes p_2 d\lambda \geq \max \{ \int p_1 \otimes p_2 d\lambda;\ \lambda \in C'_1 \otimes C'_2 \}$$

$$= (\max_{\mu \in C'_1} \int p_1 d\mu)(\max_{\nu \in C'_2} \int p_2 d\nu) \geq V_1(p_1) V_2(p_2).$$

Thus $V(p_1 \otimes p_2) + \varepsilon > V_1(p_1) V_2(p_2)$, and hence (CPL) holds.

§4. Symmetric properties of V

In the case where $K_1 = K_2 = K$, we consider the following symmetric properties of V:

(SYS) $V(p_1 \oplus p_2) = V(p_2 \oplus p_1)$ whenever $p_i \in C(K)$ $(i=1, 2)$,
(SYP) $V(p_1 \otimes p_2) = V(p_2 \otimes p_1)$ whenever $p_i \in C(K)$ $(i=1, 2)$,
(SY) $V(\check{f}) = V(f)$ whenever $f \in C(K)$,

where $\check{f} \in C(S)$ is defined by $\check{f}(x, y) = f(y, x)$ for all $x, y \in K$.

Obviously (SY) implies (SYP). By the analogous reasoning to the proof of Theorem 3.1, we can prove

THEOREM 4.1. (SYP) *implies* (SYS).

We show by the following example that (SYS) does not imply (SYP) in general:

EXAMPLE 4.1. Let $K = \{x_1, x_2, x_3\}$. Denote by ε_{ij} the unit point measure at $(x_i, x_j) \in S$ and by ε_i the unit point measure at $x_i \in K$. Let us consider the gamonic functional V defined by

$$V(f) = \int f d\lambda \quad \text{with} \quad \lambda = (1/4)(\varepsilon_{11} + \varepsilon_{13} + \varepsilon_{21} + \varepsilon_{32}).$$

Then $T\lambda = (\mu, \nu)$ with $\mu = \nu = (1/2)\varepsilon_1 + (1/4)(\varepsilon_2 + \varepsilon_3)$. We see easily that (SYS) holds. On the other hand, let p and q be the functions defined by

$$p(x_1) = 1,\ p(x_2) = p(x_3) = 0;\ q(x_2) = 1,\ q(x_1) = q(x_3) = 0.$$

Then we have $V(p\otimes q)=0<1/4=V(q\otimes p)$. Namely (SYP) does not hold.

Since $p\otimes 1=p\oplus 0$ for all $p\in C(K)$, we have

THEOREM 4.2. *If* (SYS) *holds, then* $V_1(p)=V_2(p)$ *for all* $p\in C(K)$.

In order to obtain a sufficient condition for (SYS), let us put $p\ominus p=p\oplus(-p)$ for $p\in C(K)$. We have

THEOREM 4.3. *Assume that* $V(p\ominus p)=0$ *for all* $p\in C(K)$. *If* V *is sublinear or superlinear, then* V *satisfies* (SYS).

PROOF. Let V be sublinear and let $p_i\in C(K)\,(i=1, 2)$. Put $f=p_1\oplus p_2$, $g=p_2\oplus p_1$ and $h=f-g$. Then $h=p_1\ominus p_1+(-p_2)\ominus(-p_2)$ and

$$V(h)\leq V(p_1\ominus p_1)+V((-p_2)\ominus(-p_2))=0,$$

so that

$$V(f)=V(g+h)\leq V(g)+V(h)\leq V(g).$$

By interchanging the role of p_1 and p_2, we obtain $V(g)\leq V(f)$. Thus V satisfies (SYS). In case V is superlinear, it suffices to note that $V(h)\geq 0$.

Now we study property (SY). For $\lambda\in M(S)$, define $\check{\lambda}\in M(S)$ by

$$\int f d\check{\lambda}=\int \check{f} d\lambda \qquad \text{for all} \quad f\in C(S).$$

For $C\in PP_c(S)$, put $\check{C}=\{\check{\lambda};\ \lambda\in C\}$. Then $\check{C}\in PP_c(S)$.
We have

THEOREM 4.4. (SY) *holds if and only if* $\check{C}\in ad[V]$ *for every* $C\in m[V]$.

PROOF. Assume that $\check{C}\in ad[V]$ for every $C\in m[V]$. Let $f\in C(S)$. For any $\varepsilon>0$, there exists $C\in m[V]$ such that

$$V(f)+\varepsilon>\max_{\lambda\in C}\int f d\lambda.$$

We have

$$\max_{\lambda\in C}\int f d\lambda=\max\{\int \check{f} d\lambda;\ \lambda\in\check{C}\}\geq V(\check{f}),$$

so that $V(f)\geq V(\check{f})$. By replacing f by $\check{f}$, we have $V(\check{f})\geq V(f)$. Thus (SY) holds. On the other hand, assume that (SY) holds. Suppose that there exists $C\in m[V]$ such that $\check{C}\notin ad[V]$. Then we can find $f_0\in C(S)$ such that

$$\max\{\int f_0 d\lambda;\ \lambda\in\check{C}\}<V(f_0).$$

Since $C\in m[V]$, we have

$$\max\{\int f_0 d\lambda;\ \lambda\in\check{C}\}=\max_{\lambda\in C}\int f_0 d\check{\lambda}=\max_{\lambda\in C}\int \check{f}_0 d\lambda\geq V(\check{f}_0)=V(f_0).$$

This is a contradiction. Thus $\check{C}\in ad[V]$ for every $C\in m[V]$.

COROLLARY. *Let V be sublinear and $ad[V]=\{C\}$. Then V satisfies* (SY) *if and only if $\check{C}=C$.*

In case V is linear, (SYP) implies (SY). In the general case, we do not know whether (SYP) implies (SY) or not.

§5. A-convolution of V_1 and V_2

Let V_i be the projective gamonic functionals on $C(K_i)$ obtained from V. We shall construct a new gamonic functional on $C(S)$ from V_1 and V_2.

Let A be a set-to-set mapping from $PP_c(K_1)\times PP_c(K_2)$ into $PP_c(S)$, i.e., $A(C_1, C_2)\in PP_c(S)$ for each $C_i\in PP_c(K_i)$. Denote by $m(A)$ and $a(A)$ the images of $m[V_1]\times m[V_2]$ and $ad[V_1]\times ad[V_2]$ under A respectively, i.e.,

$$m(A)=\{A(C_1, C_2);\ C_i\in m[V_i]\,(i=1, 2)\},$$

$$a(A)=\{A(C_1, C_2);\ C_i\in ad[V_i]\,(i=1, 2)\}.$$

Let us define a functional $W_A(f)$ on $C(S)$ by

$$W_A(f)=\inf_{C\in m(A)}\max_{\lambda\in C}\int f d\lambda. \tag{5.1}$$

Then it is easily seen that W_A is gamonic on $C(S)$. We call W_A an A-convolution of V_1 and V_2.

We say that A is monotone if $C_i, C'_i\in PP_c(K_i)$ and $C'_i\subset C_i$ $(i=1, 2)$ imply $A(C'_1, C'_2)\subset A(C_1, C_2)$. If A is monotone, then we have

$$W_A(f)=\inf_{C\in a(A)}\max_{\lambda\in C}\int f d\lambda \qquad \text{for all} \quad f\in C(S). \tag{5.2}$$

Hereafter in this section, we always assume that A is monotone. Let us study the relations between $V(f)$ and $W_A(f)$.

THEOREM 5.1. *The inequality $V(f)\leq W_A(f)$ holds for all $f\in C(S)$ if and only if A satisfies the following condition:*

$$A(C_1, C_2)\in ad[V] \qquad \textit{for every} \quad C_i\in m[V_i]\,(i=1, 2). \tag{5.3}$$

PROOF. If A satisfies (5.3), then $m(A)\subset ad[V]$, so that $V(f)\leq W_A(f)$ for all $f\in C(S)$. On the other hand, assume that A does not satisfy (5.3). Then there exists $C_0\in m(A)$ such that $C_0\notin ad[V]$. We can find $f_0\in C(S)$ such that

$$V(f_0)>\max_{\lambda\in C_0}\int f_0 d\lambda\geq W_A(f_0).$$

We can easily prove

THEOREM 5.2. *The inequality* $V(f)\geq W_A(f)$ *holds for all* $f\in C(S)$ *if* A *satisfies the following condition*:

(5.4) *For every* $C\in m[V]$, *there exist* $C_i'\in ad[V_i]\,(i=1, 2)$ *such that* $A(C_1', C_2')\subset C$.

THEOREM 5.3. *The inequality* $V_1(p)\leq W_A(p\otimes 1)$ *holds for all* $p\in C(K_1)$ *if and only if* A *satisfies the following condition*:

(5.5) $$T_1(A(C_1, C_2))\in ad[V_1] \quad \text{for all} \quad C_i\in m[V_i]\,(i=1, 2).$$

THEOREM 5.4. *The inequality* $V_1(p)\geq W_A(p\otimes 1)$ *holds for all* $p\in C(K_1)$ *if* A *satisfies the following condition*:

(5.6) *For every* $C_1\in m[V_1]$, *there exist* $C_i'\in ad[V_i]\,(i=1, 2)$ *such that* $T_1(A(C_1', C_2'))\subset C_1$.

Next we are concerned with the following relations for the componentwise sum and product as in §3:

(WCS)	$W_A(p_1\oplus p_2)=V_1(p_1)+V_2(p_2)$	whenever	$p_i\in C(K_i)\,(i=1, 2)$,
(WCSL)	$W_A(p_1\oplus p_2)\geq V_1(p_1)+V_2(p_2)$	whenever	$p_i\in C(K_i)\,(i=1, 2)$,
(WCSR)	$W_A(p_1\oplus p_2)\leq V_1(p_1)+V_2(p_2)$	whenever	$p_i\in C(K_i)\,(i=1, 2)$,
(WCP)	$W_A(p_1\otimes p_2)=V_1(p_1)V_2(p_2)$	whenever	$p_i\in C^+(K_i)\,(i=1, 2)$,
(WCPL)	$W_A(p_1\otimes p_2)\geq V_1(p_1)V_2(p_2)$	whenever	$p_i\in C^+(K_i)\,(i=1, 2)$,
(WCPR)	$W_A(p_1\otimes p_2)\leq V_1(p_1)V_2(p_2)$	whenever	$p_i\in C^+(K_i)\,(i=1, 2)$.

By the same reasoning as in §3, we can prove

THEOREM 5.5. (WCPL) *(resp.* (WCPR)) *implies* (WCSL) *(resp.* (WCSR)). *In particular,* (WCP) *implies* (WCS).

THEOREM 5.6. (WCSR) *holds if* A *satisfies the following condition*:

(5.7) *For every* $C_i\in m[V_i]$, *there exist* $C_i'\in ad[V_i]\,(i=1, 2)$ *such that* $T(A(C_1', C_2'))\subset C_1\times C_2$.

THEOREM 5.7. (WCPR) *holds if* A *satisfies the following condition*:

(5.8) *For every* $C_i \in m[V_i]$, *there exist* $C'_i \in ad[V_i]$ $(i=1, 2)$ *such that* $A(C'_1, C'_2) \subset C_1 \otimes C_2$.

THEOREM 5.8. (WCPL) *holds if A satisfies the following condition*:

(5.9) *For every* $C_i \in m[V_i]$, *there exist* $C'_i \in ad[V_i]$ $(i=1, 2)$ *such that* $C'_1 \otimes C'_2 \subset A(C_1, C_2)$.

As applications of Theorems 5.5, 5.6 and 5.7 and Lemma 2.1, we have

PROPOSITION 5.1. *Let* $A(C_1, C_2) = C_1 \otimes C_2$ *for* $C_i \in PP_c(K_i)$ $(i=1, 2)$. *Then A is monotone and* (WCP) *holds.*

PROPOSITION 5.2. *Let* $A(C_1, C_2) = T^{-1}(C_1 \times C_2)$ *for* $C_i \in PP_c(K_i)$ $(i=1, 2)$. *Then A is monotone and* (WCS) *and* (WCPL) *hold.*

§ 6. Applications

In this section, we recall several useful gamonic functionals on a product space in mathematical programming or in potential theory and apply our theory to them.

APPLICATION 6.1. Transportation problem

Let $\alpha_i \in P(K_i)$ $(i=1, 2)$ be fixed. The value $V(f)$ $(f \in C(S))$ of the continuous transportation problem due to Kantorovich [4] is defined by

$$V(f) = \inf \left\{ \int f d\lambda;\ \lambda \in P(S),\ T\lambda = (\alpha_1, \alpha_2) \right\}. \tag{6.1}$$

It is easily seen that V is gamonic on $C(S)$ and its projective gamonic functionals are given by

$$V_i(p_i) = \int p_i d\alpha_i \quad \text{for} \quad p_i \in C(K_i)\ (i=1, 2). \tag{6.2}$$

Let us put $C(\alpha_1, \alpha_2) = \{\lambda \in P(S);\ T\lambda = (\alpha_1, \alpha_2)\}$. Since V is superlinear, we see by Theorem 1.2 that $m[V] = \{\{\lambda\};\ \lambda \in C(\alpha_1, \alpha_2)\}$. (CRP) holds by Theorem 3.4. However (CPR) does not hold in general. For the adjoint V^* of V, we have $ad[V^*] = \{C(\alpha_1, \alpha_2)\}$. In case $K_1 = K_2$ and $\alpha_1 = \alpha_2$, we have $\check{C}(\alpha_1, \alpha_2) = C(\alpha_1, \alpha_2)$. We obtain by Theorem 4.2 and Corollary of Theorem 4.4

PROPOSITION 6.1. *Aussme that* $K_1 = K_2$. *Then* (SY) *holds if and only if* $\alpha_1 = \alpha_2$.

APPLICATION 6.2. Continuous game

The minimax value $V(f)$ $(f \in C(S))$ of the continuous zero-sum two-person game is defined by

(6.3)
$$V(f) = \min_{\nu \in P(K_2)} \max_{\mu \in P(K_1)} \int f(x, y) d\mu(x) d\nu(y).$$

It is easy to see that V is gamonic on $C(S)$ and that

$$V_1(p_1) = \max_{\mu \in P(K_1)} \int p_1 d\mu = \max_{x \in K_1} p_1(x),$$

$$V_2(p_2) = \min_{\nu \in P(K_2)} \int p_2 d\nu = \min_{y \in K_2} p_2(y).$$

Note that the class $\{P(K_1) \otimes \{\nu\};\ \nu \in P(K_2)\}$ is essential for V and that $ad[V_1] = \{P(K_1)\}$ and $m[V_2] = \{\{\nu\};\ \nu \in P(K_2)\}$.

Define a set-to-set mapping A from $PP_c(K_1) \times PP_c(K_2)$ into $PP_c(S)$ by $A(C_1, C_2) = C_1 \otimes C_2$. Then $m(A) = \{P(K_1) \otimes \{\nu\};\ \nu \in P(K_2)\}$ and the A-convolution $W_A(f)$ of V_1 and V_2 is equal to $V(f)$ for every $f \in C(S)$.

APPLICATION 6.3. Potential-theoretic gamonic functional

Let $K_1 = K_2 = K$ and denote by $S\mu$ the support of $\mu \in P(K)$. For $f \in C(S)$, let us consider the following minimax value $V(f)$ which is related to a capacity in potential theory (cf. [6]):

(6.4)
$$V(f) = \min_{\nu \in P(K)} \max_{x \in S\nu} \int f(x, y) d\nu(y)$$

$$= \min_{\nu \in P(K)} \max_{\mu \in P(S\nu)} \int f(x, y) d\mu(x) d\nu(y).$$

It is easily seen that V is gamonic on $C(S)$ and that

$$V_1(p) = V_2(p) = \min_{\mu \in P(K)} \int p d\mu \qquad \text{for} \quad p \in C(K).$$

By Ohtsuka's theorem [5], we see that (SY) holds.

Let us consider the point-to-set mapping F from $P(K)$ into $PP_c(K)$ defined by $F(\mu) = P(S\mu)$. Then F is a convex mapping, i.e.,

$$aF(\mu) + (1-a)F(\nu) \subset F(a\mu + (1-a)\nu)$$

for all $\mu, \nu \in P(K)$ and $a \in \mathrm{R}$, $0 \leq a \leq 1$. Define a set-to-set mapping A from $PP_c(K_1) \times PP_c(K_2)$ into $PP_c(S)$ by $A(C_1, C_2) = \overline{F(C_3)} \otimes C_3$ with $C_3 = (C_1 + C_2)/2$, where $\overline{F(C_3)}$ denotes the closure of $F(C_3)$ in $M(K)$ with respect to the vague topology. Then we have $m(A) = \{F(\mu) \otimes \{\mu\};\ \mu \in P(K)\}$ and the A-convolution $W_A(f)$ of V_1 and V_2 is equal to $V(f)$ for all $f \in C(S)$. Since A satisfies condition (5.9), (CPL) holds.

APPLICATION 6.4. Quadratic gamonic functional

Let $K_1=K_2=K$ and consider the value $V(f)\,(f\in C(S))$ of the following quadratic programming problem:

$$V(f)=\min_{\mu\in P(K)}\int f(x,y)\,d\mu(x)\,d\mu(y). \tag{6.5}$$

This is also related to the energy capacity in potential theory (cf. [6]). Clearly V is gamonic on $C(S)$ and

$$V_1(p)=V_2(p)=\min_{\mu\in P(K)}\int p\,d\mu \quad \text{for} \quad p\in C(K).$$

Define a set-to-set mapping A from $PP_c(K_1)\times PP_c(K_2)$ into $PP_c(S)$ by $A(C_1, C_2)=C_3\otimes C_3$ with $C_3=(C_1+C_2)/2$. Then we see that $m(A)=\{\{\mu\otimes\mu\};\ \mu\in P(K)\}$ and $W_A(f)=V(f)$ for all $f\in C(S)$. Since A satisfies condition (5.9), (CPL) holds. It is easily seen that (CSR) does not hold.

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