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Remarks on Invariant Forms of Lie Triple Algebras

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Invariant forms of Lie triple algebras have been introduced in [3] as generalizations of those of Lie algebras and Lie triple systems. In this paper, the meaning of the definition ((2.1) and (2.2)) of invariant forms is clarified from a viewpoint of invariance under endomorphisms of the Lie triple algebra (Proposition 3). The main theorem shows that there exists a one-to-one correspondence between the set of all invariant forms of a Lie triple algebra $\mathfrak{g}$ and the set of invariant forms of its standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ satisfying the orthogonal condition $\mathfrak{g}\perp D(\mathfrak{g}, \mathfrak{g})$.

§1. Lie algebra generated by $L(\mathfrak{g})$ and $D(\mathfrak{g}, \mathfrak{g})$

Let $\mathfrak{g}$ be an n-dimensional Lie triple algebra over a field $\mathbf{F}$ of characteristic zero (cf. [2], [3], [4], [8]). For X, Y, Z in $\mathfrak{g}$ denote by $L(X)$: $Y\mapsto XY$ and $D(X, Y)$: $Z\mapsto D(X, Y)Z$ the left multiplication of the anti-commutative algebra and the inner derivation of the trilinear operation of $\mathfrak{g}$, respectively. These endomorphisms satisfy the following axioms; (i) $D(X, X)=0$, (ii) $\mathfrak{S}\{(XY)Z+D(X, Y)Z\}=0$, (iii) $\mathfrak{S}D(XY, Z)$ 0, (iv) $[D(X, Y), L(Z)]=L(D(X, Y)Z)$ and (v) $[D(X, Y), D(Z, W)]=D(D(X, Y)Z, W)+D(Z, D(X, Y)W)$. Here, $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. Let $K(X, Y)$ be the endomorphism of $\mathfrak{g}$ given by $K(X, Y)Z=D(X, Z)Y-D(Y, Z)X$. Then the axiom (ii) is written as follows:

$$L(XY)-[L(X), L(Y)]+D(X, Y)-K(X, Y)=0, \quad \text{for} \quad X, Y\in\mathfrak{g}. \tag{1.1}$$

The axiom (v) implies that the linear subspace $D(\mathfrak{g}, \mathfrak{g})$ of End $(\mathfrak{g})$ spanned by all inner derivations $\{D(X, Y) \mid X, Y\in\mathfrak{g}\}$ is a Lie subalgebra of End $(\mathfrak{g})$. Let $L(\mathfrak{g})$ denote the Lie subalgebra of End $(\mathfrak{g})$ generated by all left multiplications $\{L(X) \mid X\in\mathfrak{g}\}$, and let $A(\mathfrak{g})$ be the linear subspace of End $(\mathfrak{g})$ spanned by $L(\mathfrak{g})$ and $D(\mathfrak{g}, \mathfrak{g})$.

PROPOSITION 1. (1) *$A(\mathfrak{g})=L(\mathfrak{g})+D(\mathfrak{g}, \mathfrak{g})$ is a Lie subalgebra of* End $(\mathfrak{g})$, *and $L(\mathfrak{g})$ is an ideal of $A(\mathfrak{g})$.*
(2) *The endomorphism $K(X, Y)$ belongs to $A(\mathfrak{g})$ for $X, Y\in\mathfrak{g}$.*

PROOF. The axiom (iv) implies $[D(\mathfrak{g}, \mathfrak{g}), L(\mathfrak{g})]\subset L(\mathfrak{g})$, which shows (1). (2) is an immediate consequence of (1.1). q.e.d.

Set $K_*(X)Y=D(X, Y)X$ for X, Y in $\mathfrak{g}$. The endomorphism $K_*(X)$ is quadratic in

X and satisfies the following;

$$(1.2) \qquad 2D(X, Y)Z = K(X, Y)Z + K_*(X+Y)Z - K_*(X)Z - K_*(Y)Z.$$

PROPOSITION 2. *A subspace $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if and only if it is invariant under $A(\mathfrak{g}) = L(\mathfrak{g}) + D(\mathfrak{g}, \mathfrak{g})$ and $K_*(\mathfrak{g}) = \{K_*(X) \mid X \in \mathfrak{g}\}$.*

PROOF. The subspace $\mathfrak{h}$ is, by definition, an ideal of $\mathfrak{g}$ if $\mathfrak{g}\mathfrak{h} \subset \mathfrak{h}$ and $D(\mathfrak{g}, \mathfrak{h})\mathfrak{g} \subset \mathfrak{h}$. If $\mathfrak{h}$ is an ideal, then the axiom (ii) implies $D(\mathfrak{g}, \mathfrak{g})\mathfrak{h} \subset \mathfrak{h}$. Hence $\mathfrak{h}$ is invariant under $L(X)$, $D(X, Y)$ and $K_*(X)$ for any X, Y in $\mathfrak{g}$. Conversely, if a subspace $\mathfrak{h}$ is $A(\mathfrak{g})$-invariant, then it is invariant under $L(X)$, that is, $\mathfrak{g}\mathfrak{h} \subset \mathfrak{h}$. On account of Proposition 1, $\mathfrak{h}$ is also invariant by $K(X, Y)$. Moreover, if $\mathfrak{h}$ is invariant by $K_*(\mathfrak{g})$, then (1.2) implies $2D(X, \mathfrak{h})Y = K(X, Y)\mathfrak{h} + K_*(X+Y)\mathfrak{h} - K_*(X)\mathfrak{h} - K_*(Y)\mathfrak{h} \subset \mathfrak{h}$ for X, Y in $\mathfrak{g}$. Thus $\mathfrak{h}$ satisfies $\mathfrak{g}\mathfrak{h} \subset \mathfrak{h}$ and $D(\mathfrak{g}, \mathfrak{h})\mathfrak{g} \subset \mathfrak{h}$. q.e.d.

§ 2. Invariant forms of $\mathfrak{g}$

An *invariant form* of $\mathfrak{g}$ is a symmetric bilinear form $b: \mathfrak{g} \times \mathfrak{g} \to \mathbf{F}$ on $\mathfrak{g}$ satisfying

$$(2.1) \qquad b(XY, Z) + b(Y, XZ) = 0, \quad \text{and}$$

$$(2.2) \qquad b(D(X, Y)Z, W) - b(D(Z, W)X, Y) = 0 \quad \text{(cf. [3])}.$$

This is a generalized concept of invariant forms of Lie algebras and of Lie triple systems (cf. [7]).

PROPOSITION 3. *A symmetric bilinear form b on $\mathfrak{g}$ is an invariant form if and only if the following* (1) *and* (2) *are satisfied.*

(1) *b is $A(\mathfrak{g})$-invariant, i.e., $b(TX, Y) + b(X, TY) = 0$ for $T \in A(\mathfrak{g})$ and X, $Y \in \mathfrak{g}$.*

(2) *b is $K_*(\mathfrak{g})$-symmetric, i.e., $b(K_*(X)Y, Z) - b(Y, K_*(X)Z) = 0$ for X, Y, $Z \in \mathfrak{g}$.*

PROOF. Suppose that b is an invariant form of $\mathfrak{g}$. Then b is $L(\mathfrak{g})$-invariant by (2.1). Replacing $W = Z$ in (2.2) we get $b(D(X, Y)Z, Z) = 0$, which implies $b(D(X, Y)Z, W) + b(Z, D(X, Y)W) = 0$. Hence b is invariant by $A(\mathfrak{g}) = L(\mathfrak{g}) + D(\mathfrak{g}, \mathfrak{g})$. On the other hand, we get $b(D(X, Y)X, W) - b(D(X, W)X, Y) = 0$ by putting $X = Z$ in (2.2), that is, b is $K_*(\mathfrak{g})$-symmetric. Conversely, let b be a symmetric bilinear form which is $A(\mathfrak{g})$-invariant and $K_*(\mathfrak{g})$-symmetric. Then, (2.1) is clear, and since b is invariant by $K(X, Y)$, we get

$$(2.3) \quad b(D(X, Z)Y, W) - b(D(Y, Z)X, W) + b(Z, D(X, W)Y) - b(Z, D(Y, W)X) = 0.$$

Since b is $K_*(X)$-symmetric, we have

$$(2.3) \quad b(D(X, Z)Y, W) + b(D(Y, Z)X, W) - b(Z, D(X, W)Y) - b(Z, D(Y, W)X) = 0.$$

From (2.3) and (2.4) we obtain (2.2). Therefore, b is an invariant form of $\mathfrak{g}$.

q.e.d.

PROPOSITION 4. (Cf. Prop. 3 in [3]) *Let b be an invariant form of* $\mathfrak{g}$. *For any ideal* $\mathfrak{h}$ *of* $\mathfrak{g}$, $\mathfrak{h}^{\perp}=\{X\in\mathfrak{g} \mid b(X,\mathfrak{h})=0\}$ *is an ideal of* $\mathfrak{g}$.

PROOF. Since b is $A(\mathfrak{g})$-invariant and $K_{*}(\mathfrak{g})$-symmetric by Proposition 3, and since any ideal $\mathfrak{h}$ is invariant under $A(\mathfrak{g})$ and $K_{*}(\mathfrak{g})$ by Proposition 2, it is easy to see that $\mathfrak{h}^{\perp}$ is also invariant under $A(\mathfrak{g})$ and $K_{*}(\mathfrak{g})$. q.e.d.

Let $\mathfrak{g}^{(1)}=\mathfrak{g}\mathfrak{g}+D(\mathfrak{g},\mathfrak{g})\mathfrak{g}$ be a subspace of $\mathfrak{g}$ spanned by $\mathfrak{g}\mathfrak{g}$ and $D(\mathfrak{g},\mathfrak{g})\mathfrak{g}$. For an invariant form b of $\mathfrak{g}$, denote by R_b the orthogonal complement $(\mathfrak{g}^{(1)})^{\perp}$ of $\mathfrak{g}^{(1)}$ with respect to b.

PROPOSITION 5. *Let b be an invariant form of* $\mathfrak{g}$.

(1) *R_b is an ideal of* $\mathfrak{g}$.

(2) *The center* $\mathfrak{z}$ *of* $\mathfrak{g}$ *is contained in* R_b, *where* $\mathfrak{z}=\{X\in\mathfrak{g} \mid X\mathfrak{g}=\{0\}$ *and* $D(\mathfrak{g},X)\mathfrak{g}=\{0\}\}$ (cf. [3]).

(3) *If b is nodegenerate, then* $\mathfrak{z}=R_b$.

PROOF. (1) Since $\mathfrak{g}^{(1)}$ is an ideal of $\mathfrak{g}$ (cf. [2]), R_b is also an ideal by Proposition 4 above.

(2) If $X\in\mathfrak{z}$, then $L(\mathfrak{g})X=0$, $K(\mathfrak{g},\mathfrak{g})X=0$ and $K_{*}(\mathfrak{g})X=0$. Since b is $A(\mathfrak{g})$-invariant and $K_{*}(\mathfrak{g})$-symmetric by Proposition 3, we get $b(X, L(\mathfrak{g})\mathfrak{g})=0$, $b(X, K(\mathfrak{g},\mathfrak{g})\mathfrak{g})=0$ and $b(X, K_{*}(\mathfrak{g})\mathfrak{g})=0$. (1.1) and (1.2) imply $\mathfrak{g}^{(1)}=L(\mathfrak{g})\mathfrak{g}+K(\mathfrak{g},\mathfrak{g})\mathfrak{g}+K_{*}(\mathfrak{g})\,\mathfrak{g}$, and $X\in(\mathfrak{g}^{(1)})^{\perp}=R_b$ is shown.

(3) Suppose that X_0 is an element of R_b. Then $X_0\mathfrak{g}=0$ is obtained from $0=b(X_0,\mathfrak{g}\mathfrak{g})=b(X_0\mathfrak{g},\mathfrak{g})$. On the other hand, $b(D(\mathfrak{g},\mathfrak{g})X_0,\mathfrak{g})=b(X_0, D(\mathfrak{g},\mathfrak{g})\mathfrak{g})=0$ implies $D(\mathfrak{g},\mathfrak{g})X_0=0$. Thus $A(\mathfrak{g})X_0=0$ holds, and especially $K(Y,Z)X_0=0$ holds for any Y,Z in $\mathfrak{g}$. The relations $b(X_0, K_{*}(Y)\mathfrak{g})=b(X_0, D(Y,\mathfrak{g})Y)\subset b(X_0,\mathfrak{g}^{(1)})=\{0\}$ and $b(X_0, K_{*}(Y)\mathfrak{g})=b(X_{*}(Y)X_0,\mathfrak{g})$ imply $K_{*}(Y)X_0=0$. From $K(Y,Z)X_0=0$ and $K_{*}(Y)X_0=0$ we get $D(\mathfrak{g},X_0)\mathfrak{g}=0$. Therefore, X_0 must belong to $\mathfrak{z}$, and we obtain $\mathfrak{z}=R_b$ from (2). q.e.d.

§3. Associated invariant forms of the standard enveloping Lie algebra

Let $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g},\mathfrak{g})$ be the standard enveloping Lie algebra of the Lie triple algebra $\mathfrak{g}$. We consider now invariant forms of the Lie algebra $\mathfrak{A}$ whose restrictions on $\mathfrak{g}$ are invariant forms of $\mathfrak{g}$.

THEOREM. *Let a be an invariant form of the standard enveloping Lie algebra* $\mathfrak{A}$ *of a Lie triple algebra* $\mathfrak{g}$. *If a satisfies* $a(\mathfrak{g}, D(\mathfrak{g},\mathfrak{g}))=0$, *then the restriction* $b=a|_{\mathfrak{g}\times\mathfrak{g}}$ *is an invariant form of* $\mathfrak{g}$. *Conversely, if b is an invariant form of* $\mathfrak{g}$, *then there exists a unique invariant form a of the Lie algebra* $\mathfrak{A}$ *such that* $a(\mathfrak{g}, D(\mathfrak{g},\mathfrak{g}))=0$ *and* $b=a|_{\mathfrak{g}\times\mathfrak{g}}$.

PROOF. Assume that a is an invariant form of $\mathfrak{A}$ satisfying $a(\mathfrak{g}, D(\mathfrak{g}, \mathfrak{g}))=0$, and let $b=a|_{\mathfrak{g}\times\mathfrak{g}}$. The formulas $a([X, Y], Z)+a(Y, [X, Z])=0$ and $a([D(X, Y), Z], W)+a(Z, [D(X, Y), W])=0$ for X, Y, Z, W in $\mathfrak{g}$ imply that b is $A(\mathfrak{g})$-invariant. Also, we get $a([[Y, X], X], Z]+a(Y, [[X, Z], X])=0$, which shows that b is $K_*(\mathfrak{g})$-symmetric. Thus, from Proposition 3 it follows that $b=a|_{\mathfrak{g}\times\mathfrak{g}}$ is an invariant form of $\mathfrak{g}$.

Conversely, suppose that b is an invariant form of $\mathfrak{g}$. A bilinear form $a: \mathfrak{A}\times\mathfrak{A}\rightarrow \mathbf{F}$ can be defined in the following way: $a(X, Y)=b(X, Y)$, $a(X, D(Y, Z))=a(D(Y, Z), X)=0$ and $a(D(U, V), D(X, Y))=b(D(U, V)X, Y)$ for X, Y, Z, U, V in $\mathfrak{g}$. This bilinear form a is symmetric. In fact, since b is invariant, $b(D(X, Y)Z, W)-b(D(Z, W)X, Y)=0$ holds, so that $a(D(Z, W), D(X, Y))=a(D(X, Y), D(Z, W))$. By definition, $a(\mathfrak{g}, D(\mathfrak{g}, \mathfrak{g})=0$. To prove that a is an invariant form of $\mathfrak{A}$ it is sufficient to show the following (1)–(5): (1) $a([X, Y], Z)+a(Y, [X, Z])=0$. This is equivalent to $b(XY, Z)+b(Y, XZ)=0$ under the condition $a(\mathfrak{g}, D(\mathfrak{g}, \mathfrak{g}))=0$. (2) $a([X, Y], D(Z, W))+a(Y, [X, D(Z, W)])=0$, which is reduced to the definition $a(D(X, Y), D(Z, W))=b(Y, D(Z, W)X)$. (3) $a([D(X, Y), Z], W)+a(Z, [D(X, Y), W])=0$, which is obtained from $b(D(X, Y)Z, W)+b(Z, D(X, Y)W)=0$. (4) $a([D(X, Y), Z], D(U, V))+a(Z, [D(X, Y), D(U, V)])=0$, in which each term of the left hand side vanishes by definition. Finally, (5) $a([D(X, Y), D(Z, W)], D(U, V))+a(D(Z, W), [D(X, Y), D(U, V)])=0$. In fact, since $a(D(Z, W), [D(X, Y), D(U, V)])=a(D(Z, W), D(D(X, Y)U, V))+a(D(Z, W), D(U, D(X, Y)V))=b(D(Z, W)D(X, Y)U, V)+b(D(Z, W)U, D(X, Y)V)$, the left hand side of (5) is equal to $b([D(X, Y), D(Z, W)]U, V)+b(D(Z, W)D(X, Y)U, V)+b(D(Z, W)U, D(X, Y)V)=b(D(X, Y)D(Z, W)U, V)+b(D(Z, W)U, D(X, Y)V)=0$. The uniqueness of such a is shown by $a([Z, W], [X, Y])+a([X, [Z, W]], Y)=0$ and $b(ZW, XY)+b(X(ZW), Y)=0$. q.e.d.

In this theorem, the invariant form a of the standard enveloping Lie algebra of $\mathfrak{g}$ will be said to be *associated* with the invariant form b of $\mathfrak{g}$.

REMARK 1. In [3] the Killing-Ricci form β of a Lie triple algebra $\mathfrak{g}$ is treated. The Killing form α of $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ is associated with β if and only if $\gamma=0$, where $\gamma(X, Y, Z)=\alpha(D(X, Y), Z)$ for X, Y, Z in $\mathfrak{g}$. If $\mathfrak{g}$ is reduced to a Lie triple system, then α is associated with the Killing form β of $\mathfrak{g}$ (cf. [5], [6]). In general, α is not always associated with the Killing-Ricci form β. For instance, let $\mathfrak{g}$ be a Malcev algebra (cf. [5]). K. Yamaguti [9] has shown that $\mathfrak{g}$ becomes a Lie triple algebra (general Lie triple system) under the operations $L(X)=\lambda(X)$ and $D(X, Y)=\lambda(XY)+[\lambda(X), \lambda(Y)]$, where $\lambda(X): Y\mapsto XY$ is the left multiplication of the Malcev algebra. The Killing form α of the standard enveloping Lie algebra of this Lie triple algebra satisfies $\alpha(D(X, Y), Z)=-\theta(XY, Z)$ which does not always vanish, where $\theta(X, Y)=\mathrm{tr}\,\lambda(X)\lambda(Y)$ denotes the Killing form of the Malcev algebra.

In [4] the concept of K-radical of $\mathfrak{g}$ has been introduced as the orthogonal complement $R_\beta=(\mathfrak{g}^{(1)})^\perp$ with respect to the Killing-Ricci form β under the condition $\gamma=0$.

It is considered as a generalization of radicals of Lie algebras, by virtue of the theorem on p. 73 in [1]. Some analogous properties for an invariant form b and $R_b=(\mathfrak{g}^{(1)\perp})$ with respect to b are mentioned in the following:

Let b be an invariant form of $\mathfrak{g}$ and a the invariant form of $\mathfrak{A}=\mathfrak{g}\oplus D(\mathfrak{g}, \mathfrak{g})$ associated with b.

Proposition 7. *The form b is nondegenerate if and only if a is nondegenerate.*

Proof. If a is nondegenerate, so is b, since $b(X, \mathfrak{g})=0$ implies $a(X, \mathfrak{A})=0$ for X in $\mathfrak{g}$. Conversely, assume that b is nondegenerate and that $a(X_0+D_0, \mathfrak{A})=0$ for some $X_0\in\mathfrak{g}$ and $D_0\in D(\mathfrak{g}, \mathfrak{g})$. Then, $a(X_0+D_0, \mathfrak{g})=b(X_0, \mathfrak{g})=0$ implies $X_0=0$. On the other hand, $a(D_0, D(X, Y))=b(D_0X, Y)=0$ for any $X, Y\in\mathfrak{g}$. Hence $D_0X=0$, that is, $D_0=0$ as an endomorphism of $\mathfrak{g}$. q.e.d.

Proposition 8. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and $\mathfrak{B}=\mathfrak{h}\oplus D(\mathfrak{g}, \mathfrak{h})$ an ideal of $\mathfrak{A}$ generated by $\mathfrak{h}$* (cf. [2]). *Then, $b(\mathfrak{h}, \mathfrak{g}^{(1)})=0$ if and only if $a(\mathfrak{B}, [\mathfrak{A}, \mathfrak{A}])=0$.*

Proof. Assume that $b(\mathfrak{h}, \mathfrak{g}^{(1)})=0$. Then $\mathfrak{h}$ is contained in the orthogonal complement of $[\mathfrak{A}, \mathfrak{A}]$ with respect to a. In fact, $a(\mathfrak{h}, [\mathfrak{A}, \mathfrak{A}])\subset a(\mathfrak{h}, \mathfrak{g}\mathfrak{g}+D(\mathfrak{g}, \mathfrak{g})\mathfrak{g})=b(\mathfrak{h}, \mathfrak{g}^{(1)})=\{0\}$. Hence $a(\mathfrak{B}, [\mathfrak{A}, \mathfrak{A}])=a(\mathfrak{h}\oplus D(\mathfrak{g}, \mathfrak{h}), [\mathfrak{A}, \mathfrak{A}])\subset a(\mathfrak{h}+[\mathfrak{A}, \mathfrak{h}], [\mathfrak{A}, \mathfrak{A}])=\{0\}$. The converse is clear since $a(\mathfrak{h}, [X, Y])=b(\mathfrak{h}, XY)$ and $a(\mathfrak{h}, [D(X, Y), Z])=b(\mathfrak{h}, D(X, Y)Z)$ for X, Y, Z in $\mathfrak{g}$. q.e.d.

Remark 2. In the case of the Killing-Ricci form β, the proposition obtained above is reduced to Proposition 2 in [4].

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