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The Radicals of Malcev Algebras

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In this paper, we introduce the notion of semi-prime radicals and prime radicals in Malcev algebras and investigate their properties.

§0. Introduction

The notion of radicals plays an important role in the theory of associative algebras. It seems to be interesting for us to know how the corresponding notion behaves in Malcev algebras which generalize the class of Lie algebras. In this paper we shall investigate the semi-prime radicals, prime radicals for Malcev algebras.

Let Φ be a field of characteristic 0. A *Malcev algebra* M over Φ is an anti-commutative algebra satisfying the identity

$$(xy)(xz)=((xy)z)x+((yz)x)x+((zx)x)y \quad \text{for all} \quad x,\ y,\ z \quad \text{in} \quad M.$$

Throughout the paper we shall be concerned with a finite dimensional Malcev algebra M over Φ. For an ideal A of M, we put $A^{(1)}=AA$ and $A^{(k)}=A^{(k-1)}A^{(k-1)}$ $(k\geqq 2)$. An ideal A is called *solvable* if there is a positive integer k such that $A^{(k)}=0$. Since M is finite dimensional, it contains a unique maximal solvable ideal $R_1(M)$, which is called the *solvable radical* of M. We mainly employ the terminology and notation in [6] and [9].

§1. Preliminaries

Recall that a *Lie triple algebra* (general Lie triple system) T over a field of characteristic 0 is a vector space with a bilinear composition xy and a trilinear composition $[xyz]$ satisfying

(1) $xx=0$

(2) $[xyz]=-[yxz]$

(3) $[xyz]+[yzx]+[zxy]+(xy)z+(yz)x+(zx)y=0$

(4) $[(xy)zw]+[(yz)xw]+[(zx)yw]=0$

(5) $[xy(zw)]=[xyz]w+z[xyw]$

(6) $[xy[zvw]]=[[xyz]vw]+[z[xyv]w]+[zv[xyw]]$

for all x, y, z, v, w in T (cf. [3], [11]). Any Lie algebra is a Lie triple algebra relative to $xy=[x, y]$ and $[xyz]=[[x, y], z]$. If $[xyz]=0$ for all x, y, z in T, the axioms stated above are reduced to those of Lie algebras and if $xy=0$ for all x, y in T, the axioms are reduced to those of Lie triple system.

In [6, Satz. 1], Loos proved that a Malcev algebra M becomes a Lie triple system with respect to a ternary composition $[xyz]=x(yz)-y(xz)+2(xy)z$, which is called the Lie triple system associated with M and is denoted by T_M.

Also in [11, Theorem 1.1], Yamaguti proved that a Malcev algbera M becomes a Lie triple algebra with respect to the composition xy and $[xyz]_1=(xy)z-y(xz)+x(yz)$, which is called the Lie triple algebra associated with M.

If A is an ideal of a Malcev algebra M, so are $A^{\langle 1\rangle}=AA+M(AA)$ and $A^{[1]}=AA+[MAA]_1$. Obviously, $A^{\langle 1\rangle}\supseteqq A^{[1]}$. Conversely, since $A^{\langle 1\rangle}=AA+M(AA)\subseteqq AA+[MAA]_1+[MAA]_1=A^{[1]}$, we get $A^{\langle 1\rangle}=A^{[1]}$. The notion of solvability arising from descending chains of these ideals are called *L-solvability* [4] and *Y-solvability* [11] respectively. Thus L-solvability and Y-solvability are coincident for Malcev algebras. Let $R_2(M)$ (resp. $R_3(M)$) denote the unique maximal L-solvable ideal (resp. Y-solvable ideal), which is called the *L-solvable radical* (resp. *Y-solvable radical*).

REMARK 1. In [8, Theorem 1.1], Ravisanker proved that for an ideal A of a Malcev algebra M, the following statements are equivalent:

(1) A is solvable.

(2) A is L-solvable.

(3) A is Y-solvable.

Therefore we have $R_1(M)=R_2(M)=R_3(M)$.

§2. Semi-prime radicals and prime radicals

Let M be a Malcev algebra. As in associative rings (cf. [7]), we say that an ideal Q of M is *semi-prime* if the following condition is satisfied: If $H^{\langle 1\rangle}\subseteqq Q$ for an ideal H of M, then $H\subseteqq Q$.

REMARK 2. (1) A Malcev algebra M is a semi-prime ideal of M.

(2) Let Q be a semi-prime ideal. If there is a positive integer k such that $I^{\langle k\rangle}\subseteqq Q$ for an ideal I of M; then $I\subseteqq Q$.

(3) The intersection of all the semi-prime ideals of M is a semi-prime ideal.

REMARK 3. ([11, Lemma 2.1]) If A and B be ideals of a Malcev algebra M, then $AB+[MAB]_1+[MBA]_1$ is an ideal of M.

An ideal P of M is called *prime* if the relation $HK+[MHK]_1+[MKH]_1\subseteqq P$ for ideals H, K of M implies $H\subseteqq P$ or $K\subseteqq P$.

As in commutative rings, we define the irreducibility of ideals as follows: An ideal N of M is said to be *irreducible* if $N=H\cap K$ with ideals H, K of M implies $N=H$ or $N=K$.

PROPOSITION 1. *Let M be a Malcev algebra.*

(1) *Any prime ideal is semi-prime.*

(2) *Any prime ideal is irreducible.*

(3) *Any maximal ideal is irreducible.*

(4) *Among prime, semi-prime, irreducible and maximal ideals, there are no implications except* (1), (2) *and* (3).

PROOF. (1) From $[xyz]_1=(xy)z-y(xz)+x(yz)$, it follows that the identity $2(xy)z+[zxy]_1-[zyx]_1=0$ holds.
Hence

$$M(AB)\subseteqq[MAB]_1+[MBA]_1.$$

Therefore if P is a prime ideal then it is also semi-prime.
(2) Suppose that $H\cap N$ is a prime ideal. Then (2) is immediate since

$$HN+[MHN]_1+[MNH]_1\subseteqq H\cap N.$$

(3) This is clear.
(4) Let M_0 be a 2-dimensional non abelian Malcev algebra, that is, $M_0=\langle x, y\rangle$ with $xy=x$. Then the ideals of M_0 are (0), $\langle x\rangle$ and M_0. The ideal (0) is irreducible but neither prime nor semi-prime, for $xx=0$. Clearly (0) is not maximal. Since $M_0M_0=(M_0M_0)M_0=\langle x\rangle$, $\langle x\rangle$ is maximal, but neither prime nor semi-prime. By the definition of M_0, it is prime but not maximal. Let M_1, M_2 and M_3 be simple Malcev algebras. Let $M'=M_1\oplus M_2\oplus M_3$. Then the ideals containing M_1 properly are $M_1\oplus M_2$, $M_1\oplus M_3$ and M'. Therefore M_1 is semi-prime, and since

$$(M_1\oplus M_2)(M_1\oplus M_3)\subseteqq(M_1\oplus M_2)\cap(M_1\oplus M_3)=M_1,$$

$$[(M_1\oplus M_2\oplus M_3)(M_1\oplus M_2)(M_1\oplus M_3)]_1\subseteqq(M_1\oplus M_2)\cap(M_1\oplus M_3)=M_1$$

and

$$[(M_1\oplus M_2\oplus M_3)(M_1\oplus M_3)(M_1\oplus M_2)]_1\subseteqq(M_1\oplus M_3)\cap(M_1\oplus M_2)=M_1,$$

M_1 is neither prime nor irreducible. Obviously, M_1 is not maximal. This completes the proof.

PROPOSITION 2. *If M is a Malcev algebra then P is a prime ideal of M if and only if P is irreducible and semi-prime.*

PROOF. Suppose that P is irreducible and semi-prime, and let H, K be ideals of M satisfying

$$HK+[MHK]_1+[MKH]_1 \subseteqq P.$$

If we put $N=(H+P)\cap(K+P)$, then

$$NN+M(NN)\subseteqq(H+P)(K+P)+[M(H+P)(K+P)]_1+[M(K+P)(H+P)]_1\subseteqq P.$$

Hence $N\subseteqq P$ and $P=(H+P)\cap(K+P)$. Then

$$P=H+P \quad \text{or} \quad P=K+P,$$

that is,

$$H\subseteqq P \quad \text{or} \quad K\subseteqq P.$$

Therefore P is prime. The converse was shown in Proposition 1. This completes the proof.

We denote by $R_4(M)$ (resp. $R_5(M)$) the intersection of all the semi-prime ideals of M (resp. prime ideals of M) and call it the *semi-prime radical* of M (resp. *prime radical* of M).

THEOREM 3. *For a Malcev algebra M, the semi-prime raidcal $R_4(M)$ is equal to the L-solvable radical $R_2(M)$.*

PROOF. It is obvious by (2) of Remark 2 in this section that

$$R_4(M)\supseteqq R_2(M).$$

Conversely, if $H^{\langle 1\rangle}\subseteqq R_2(M)$ for an ideal H of M, then H is L-solvable, since $H^{\langle n+1\rangle}=(H^{\langle 1\rangle})^{\langle n\rangle}=(0)$ for some positive integer n. Hence $H\subseteqq R_2(M)$, i.e. $R_2(M)$ is semi-prime. Therefore

$$R_4(M)\subseteqq R_2(M).$$

This completes the proof.

THEOREM 4. *The prime radical $R_5(M)$ of a Malcev algebra M is equal to the Y-solvable radical $R_3(M)$.*

PROOF. Let H be a Y-solvable ideal of M. Then there is an integer $n\geqq 0$ such that $H^{[n]}=(0)$. For any prime ideal P of M we have $H\subseteqq P$, since $H^{[n]}=(0)\subseteqq P$. Therefore $R_3(M)\subseteqq R_5(M)$. Assume that $R_5(M)$ is not Y-solvable. Let $\varDelta$ be a collection of ideals H such that $R_5(M)^{[n]}\subsetneqq H$ for all $n\geqq 0$. Then $\varDelta$ is not empty because $(0)\in\varDelta$. From finite dimensionality of M, it follows that $\varDelta$ has a maximal element P. If there are ideals H, K of M such that $H\supsetneqq P$, $K\supsetneqq P$ and $HK+[MHK]_1+[MKH]_1\subseteqq P$, then no one of $H+P$ and $K+P$ is contained in $\varDelta$. Hence $R_5(M)^{[h]}\subseteqq H$

$+P$ and $R_5(M)^{[k]}\subseteqq K+P$ for some integers $h, k\geqq 0$. Let $s=\max\{h, k\}$. Then

$$\begin{aligned}R_5(M)^{[s+1]} &\subseteqq (H+P)(K+P)+[M(H+P)(K+P)]_1+[M(K+P)(H+P)]_1\\ &\subseteqq HK+[MHK]_1+[MKH]_1+P\\ &\subseteqq P.\end{aligned}$$

This contradicts $P\in\Delta$. Hence P is prime and $R_5(M)\subsetneqq P$, which contradicts the definition of $R_5(M)$. Therefore $R_5(M)$ is Y-solvable and $R_5(M)\subseteqq R_3(M)$. This completes the proof.

From the theorems above and Remark 1 in §1, we have the following theorem:

THEOREM 5. *For a Malcev algebra* M, $R_1(M)=R_2(M)=R_3(M)=R_4(M)=R_5(M)$.

3. Radicals of a Malcev algebra M and its Lie triple system T_M

In this section, we shall consider the correspondence between semi-prime ideals (resp. prime ideals) in a Malcev algebra M and semi-prime ideals (resp. prime ideals) in the Lie triple system T_M associated with M. For the notion of semi-prime and prime ideals of Lie triple systems, see [2].

PROPOSITION 6. *Let Q be an ideal of M which is semi-prime as an ideal of T_M. Then Q is semi-prime as an ideal of M.*

PROOF. Let Q be a semi-prime ideal of T_M and A be an ideal of M satisfying $A^{\langle 1\rangle}\subseteqq Q$. From the definition of the ternary composition of the Lie triple system T_M associated with M,

$$\begin{aligned}[T_MAA] &\subseteqq T_M(AA)+A(T_MA)+(T_MA)A\\ &\subseteqq AA+M(AA)\\ &=A^{\langle 1\rangle}.\end{aligned}$$

Therefore we obtain that A is contained in Q. This completes the proof.

PROPOSITION 7. *Let P be an ideal of M which is prime as an ideal of T_M. Then P is prime as an ideal of M.*

PROOF. Let P be a prime ideal of T_M and H, K ideals of M satisfying

$$HK+[MHK]_1+[MKH]_1\subseteqq P.$$

As in the proof of Proposition 6,

$$
\begin{aligned}
[T_M H K] &\subseteqq T_M(HK)+H(T_M K)+(T_M H)K \\
&\subseteqq HK+M(HK) \\
&\subseteqq HK+[MHK]_1+[MKH]_1.
\end{aligned}
$$

From the definition of prime ideals of Lie triple systems in [2], we deduce $H \subseteqq P$ or $K \subseteqq P$. This completes the proof.

In [6, Satz 2], Loos proved that the solvable radical $R_1(M)$ of Malcev algebra M is equal to the solvable radical $R_1(T_M)$ of the Lie triple system T_M. From $R_1(T_M)=R_2(T_M)=R_3(T_M)=R_4(T_M)$ ([2, Theorem 4]) and $R_1(M)=R_2(M)=R_3(M)=R_4(M)=R_5(M)$, the next theorem follows.

THEOREM 8. *Let M be a Malcev algbera and T_M the Lie triple system associated with M. Then the semi-prime radical (resp. prime radical) of M is equal to the semi-prime radical (resp. prime radical) of T_M.*

4. Appendix (Radicals of generalized standard algebras)

In [1] Albert defined the class of standard algebras, including all associative and commutative Jordan algebras. Now we recall that the class of *generalized standard algebras* A over a field of characteristic 0 is defined by the conditions (i)–(iv) below (cf. [10]).

(i) A is flexible, i.e. $(x, y, x)=0$ for all x, y in A.
(ii) $H(x, y, z)x=H(x, y, xz)$ for all x, y, z in A.
(iii) $(x, y, wz)+(w, y, xz)+(z, y, xw)=[x, (w, z, y)]+(x, w, [y, z])$ for all x, y, z, w in A.
(iv) $D_{x,y}=[L_x, L_y]+[L_x, R_y]+[R_x, R_y]$ is a derivation of A for all x, y in A.

Here $(x, y, z)=(xy)z-x(yz)$
$[x, y]=xy-yx$
$H(x, y, z)=(x, y, z)+(y, z, x)+(z, x, y)$
$L_x(y)=xy \quad R_x(y)=yx$

All alternative algebras and standard algebras are generalized standard algebras.

For an ideal B of A, we put $B^{\langle 0\rangle}=B$, $B^{\langle 1\rangle}=BB+A(BB)$ and $B^{\langle k\rangle}=B^{\langle k-1\rangle}B^{\langle k-1\rangle}+A(B^{\langle k-1\rangle}B^{\langle k-1\rangle})$ $(k \geqq 2)$. An ideal B is called L-solvable (it is called *Penico solvable* in [10]) if there is a positive integer k such that $B^{\langle k\rangle}=0$.

As in the Malcev algebras of §2, we say that an ideal Q of A is *semi-prime* if the following condition is satisfied: If $H^{\langle 1\rangle} \subseteqq Q$ for an ideal H of A, then $H \subseteqq Q$.

REMARK 4. (1) A generalized standard algebra A is a semi-prime ideal of A.
(2) Let Q be a semi-prime ideal. If there is a positive integer k such that $C^{\langle k\rangle} \subseteqq Q$

for an ideal C of A; then $C \subseteqq Q$.

(3) The intersection of all the semi-prime ideals of A is a semi-prime ideal.

We also denote by $R_4(A)$ the intersection of all the semi-prime ideals of a generalized standard algebra A and call it the *semi-prime radical* of A.

THEOREM 9. *The semi-prime radical $R_4(A)$ is equal to the L-solvable radical $R_2(A)$.*

PROOF. The proof is similar to the one for Malcev algebras, and we omit it.

In [10], Schafer proved that the solvable radical is equal to the Penico solvable radical. Therefore, the following three radicals are equal: (1) Solvable radical $R_1(A)$, (2) L-solvable radical (Penico solvable radical) $R_2(A)$ and (3) Semi-prime radical $R_4(A)$.

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