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SPACE OF GEODESICS IN HYPERBOLIC SPACES AND LORENTZ NUMBERS

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In this note, we will study about the space of oriented geodesics in hyperbolic spaces $\mathbb{H}^n$. It is well-known that the space of oriented geodesics (i.e., oriented great circles) in spheres $\mathbb{S}^n$ is identified with oriented real 2-plane Grassmannian $\widetilde{G}_2(\mathbb{R}^{n+1})$ and complex quadric Q^n. We will show that the space of oriented geodesics in $\mathbb{H}^n$ is also given similarly by using *Lorentz numbers*. Oriented real 2-plane Grassmannian plays important roles among differential geometry of submanifolds. For example, let f be an immersion from a Riemann surface Σ to the Euclidean space $\mathbb{R}^{n+1}$. Then the Gauss map γ from Σ to the Grassmannian $\widetilde{G}_2(\mathbb{R}^{n+1})$ of oriented 2-plane in $\mathbb{R}^{n+1}$ of f is *anti-holomorphic* (resp. *holomorphic*) if and only if the immersion f is minimal (resp. totally umbilical). Here we will remark that similar results valid for timelike surfaces in Lorentz space $\mathbb{R}^{n+1}_1$ without proof.

1. Complex numbers and Lorentz numbers

According to [8] (section 4), we review the complex numbers $\mathbb{C}$ and the Lorentz numbers $\mathbb{L}$. Let $\mathbb{R}(2,0)$ be the vector space $\mathbb{R}^2$ with an inner product $\varepsilon_{2,0}(x,y) = x_1y_1 + x_2y_2$. The *square norm* associated with $\varepsilon_{2,0}$ is defined by $\|x\| = \varepsilon_{2,0}(x,x)$. Then the complex numbers $\mathbb{C}$ are defined to be $\mathbb{R}(2,0)$ with the multiplication, given by $(a,b)(c,d) := (ac-bd, ad+bc)$. Let $1 := (1,0)$ and $i := (0,1)$, so that $(a,b) = a+bi$ and $i^2 = -1$. *Conjugation* is defined by $\bar{z} = a-ib$ for $z = a+ib$. Note that $\overline{zw} = \bar{z}\bar{w}$, $z\bar{z} = \|z\|$, and hence $\|zw\| = \|z\|\|w\|$. If $z \neq 0$, then $z^{-1} = \bar{z}/\|z\|$, so that $\mathbb{C}$ is a (commutative) field.

Let $e^{i\theta} = \cos\theta + i\sin\theta$ denote a point on the unit circle and note that $M_{e^{i\theta}}$, multiplication by $e^{i\theta}$, is an orthogonal transformation since $\|e^{i\theta}\| = 1$. As a 2×2 real matrix,

$$M_{e^{i\theta}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

so that $M_{e^{i\theta}} \in SO(2)$. Since $M_{e^{i\theta}e^{i\psi}} = M_{e^{i(\theta+\psi)}}$, the map $\theta \mapsto M_{e^{i\theta}}$ induces the group isomorphism, $\mathbb{R}/2\pi\mathbb{Z} \cong SO(2)$.

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Let $\mathbb{R}(1,1)$ be the vector space $\mathbb{R}^2$ with an inner product $\varepsilon_{1,1}(x,y) = x_1y_1 - x_2y_2$. The *square norm* associated with $\varepsilon_{1,1}$ is defined by $\|x\| = \varepsilon_{1,1}(x,x)$. Then the Lorentz numbers $\mathbb{L}$ are defined to be $\mathbb{R}(1,1)$ with the multiplication, given by $(a,b)(c,d) := (ac+bd, ad+bc)$. Let $1 := (1,0)$ and $\tau := (0,1)$, so that $(a,b) = a+b\tau$ and $\tau^2 = 1$. *Conjugation* is defined by $\bar{z} = a - b\tau$ for $z = a + b\tau$. Note that $\overline{zw} = \bar{z}\bar{w}$, $z\bar{z} = \|z\|$, and hence $\|zw\| = \|z\|\|w\|$. Thus if $\|z\| \neq 0$ (z non null), then $z^{-1} = \bar{z}/\|z\|$ exists, while for $\|z\| = 0$ (z null) z can not have an inverse.

Let $e^{\tau\theta} = \cosh\theta + \tau\sinh\theta$ (calculate the formal power series for $e^{\tau\theta}$ to see that this definition is appropriate). Note that $M_{e^{\tau\theta}}$, multiplication by $e^{i\theta}$, is an orthogonal transformation since $\|e^{\tau\theta}\| = 1$. As a 2×2 real matrix,

$$M_{e^{\tau\theta}} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix},$$

so that $\det M_{e^{\tau\theta}} = 1$. Define a timelike vector $z = a + b\tau$ to be *future timelike* if $b > 0$. Since $M_{e^{\tau\theta}} = \sinh\theta + \tau\cosh\theta$, multiplication by $e^{\tau\theta}$ preserves the futurelike time cone. Thus, $M_{e^{\tau\theta}} \in SO^+(1,1)$ (the connected component of the identity of the Lorentz group $O(1,1)$). In fact, since $M_{e^{\tau\theta}e^{\tau\psi}} = M_{e^{\tau(\theta+\psi)}}$, the map $\theta \mapsto M_{e^{\tau\theta}}$ determines the group isomorphism, $\mathbb{R} \cong SO^+(1,1)$.

2. Space of oriented geodesics in spheres

In this section we recall (cf. [9] and [11]) that space of oriented geodesics (i.e., oriented great circles) in the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is identified with complex quadric Q^{n-1} in complex projective space $\mathbb{CP}^n$ and oriented 2-plane Grassmannian $\widetilde{G}_2(\mathbb{R}^{n+1})$. Let $\mathbb{R}^{n+1}$ be the *Euclidean* $(n+1)$*-space*, that is the set of all $(n+1)$-tuples $\mathbf{p} = (p_1, \cdots, p_{n+1})$, with the dot product $\mathbf{p}\cdot\mathbf{q} = \sum p_j q_j$. Then $\mathbb{S}^n = \{\mathbf{p} \in \mathbb{R}^{n+1} | \ \mathbf{p}\cdot\mathbf{p} = 1\}$ is the unit sphere. The geodesic γ in $\mathbb{S}^n$ of unit speed with $\gamma(0) = \mathbf{p}$ and $\gamma'(0) = \mathbf{x}$ ($\|\mathbf{x}\| = 1$) is written as $\gamma(\theta) = \cos\theta\mathbf{p} + \sin\theta\mathbf{x}$.

Let

$$V_2^{n+1} = \{(\mathbf{e}_1, \mathbf{e}_2) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \mid \mathbf{e}_\alpha\cdot\mathbf{e}_\beta = \delta_{\alpha\beta}\ (\alpha,\beta = 1,2)\} \tag{1}$$

be a *Stiefel manifold* of orthonormal 2-vectors in $\mathbb{R}^{n+1}$. As a homogeneous space, $V_2^{n+1} = SO(n+1)/SO(n-1)$ and $\dim_{\mathbb{R}} V_2^{n+1} = 2n-1$. We consider the action of $SO(2)$ on V_2^{n+1} as

$$(\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = (\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2). \tag{2}$$

Then each orbit $\{(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2) | \theta \in \mathbb{R}\}$ of the action (2) is identified with a pair (γ, γ') of unit speed geodesic γ on $\mathbb{S}^n$ and its unit tangent vector field γ' with $\gamma(0) = \mathbf{e}_1$ and $\gamma'(0) = \mathbf{e}_2$. Note that orbit space of the action (2) is nothing but the oriented 2-plane Grassmannian

$$\widetilde{G}_2(\mathbb{R}^{n+1}) = \{\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\} | \ \mathbf{e}_\alpha\cdot\mathbf{e}_\beta = \delta_{\alpha\beta}\ (\alpha,\beta = 1,2)\}.$$

Then V_2^{n+1} is a principal fiber bundle over $\widetilde{G}_2(\mathbb{R}^{n+1})$ with structure group $\mathbb{S}^1$ and projection map $\pi : V_2^{n+1} \to \widetilde{G}_2(\mathbb{R}^{n+1})$ defined by

$$\pi((\mathbf{e}_1, \mathbf{e}_2)) = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}.$$

The tangent space $T_{(\mathbf{e}_1,\mathbf{e}_2)}V_2^{n+1}$ is

$$\mathbb{R}(-\mathbf{e}_2,\mathbf{e}_1)\oplus\{(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\mid \mathbf{x}_1,\mathbf{x}_2\perp \operatorname{span}\{\mathbf{e}_1,\mathbf{e}_2\}\}.$$

The inner product on $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ defined by

$$\begin{aligned}&\langle(\mathbf{x}_1,\mathbf{x}_2),\ (\mathbf{y}_1,\mathbf{y}_2)\rangle=\langle\mathbf{x}_1,\mathbf{y}_1\rangle+\langle\mathbf{x}_2,\mathbf{y}_2\rangle\\ &\text{for}\quad(\mathbf{x}_1,\mathbf{x}_2),(\mathbf{y}_1,\mathbf{y}_2)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\end{aligned}$$

induces a Riemannian metric $\widetilde{g}$ on V_2^{n+1}. Since $\widetilde{g}$ on V_2^{n+1} is invariant by the structure group, we may define a Riemannian metric g on $\widetilde{G}_2(\mathbb{R}^{n+1})$ such that π is a Riemannian submersion.

The distribution given by

$$T'_{(\mathbf{e}_1,\mathbf{e}_2)}(V_2^{n+1})=\{(\mathbf{x}_1,\mathbf{x}_2)\in T_{(\mathbf{e}_1,\mathbf{e}_2)}(V_2^{n+1})\mid \mathbf{x}_1,\mathbf{x}_2\perp\operatorname{span}\{(\mathbf{e}_1,\mathbf{e}_2)\}\}. \tag{3}$$

defines a connection in the principal fiber bundle $V_2^{n+1}(\widetilde{G}_2(\mathbb{R}^{n+1}),S^1)$, because $T'_{(\mathbf{e}_1,\mathbf{e}_2)}$ is complementary to the subspace $\mathbb{R}(-\mathbf{e}_2,\mathbf{e}_1)$ tangent to the fiber through $(\mathbf{e}_1,\mathbf{e}_2)$, and invariant under the S^1-action. The natural projection $\pi: V_2^{n+1}\to\widetilde{G}_2(\mathbb{R}^{n+1})$ induces a linear isomorphism of $T'_{(\mathbf{e}_1,\mathbf{e}_2)}(V_2^{n+1})$ onto $T_p(\widetilde{G}_2(\mathbb{R}^{n+1}))$, where $\pi((\mathbf{e}_1,\mathbf{e}_2))=p$. The complex structure $\widetilde{J}$ on $T'_{(\mathbf{e}_1,\mathbf{e}_2)}(V_2^{n+1})$ defined by

$$(\mathbf{x}_1,\mathbf{x}_2)\mapsto(-\mathbf{x}_2,\mathbf{x}_1) \tag{4}$$

induces a canonical complex structure J on $\widetilde{G}_2(\mathbb{R}^{n+1})$ through $d\pi$. Then it can be seen that

$$J^2=-1,\quad \langle JX_1,X_2\rangle+\langle X_1,JX_2\rangle=0,\quad \nabla J=0,$$

where ∇ denotes the Levi-Civita connection of $(\widetilde{G}_2(\mathbb{R}^{n+1}),g)$, so $\widetilde{G}_2(\mathbb{R}^{n+1})$ is a Kähler manifold.

Let $\mathbb{C}^{n+1}=\{\mathbf{z}=\mathbf{x}+i\mathbf{y}|\mathbf{x},\mathbf{y}\in\mathbb{R}^{n+1}\}$ be the *complex Euclidean space*, and define the dot product on $\mathbb{C}^{n+1}$ as

$$(\mathbf{x}+i\mathbf{y})\cdot(\mathbf{u}+i\mathbf{v})=(\mathbf{x}\cdot\mathbf{u}-\mathbf{y}\cdot\mathbf{v})+i(\mathbf{x}\cdot\mathbf{v}+\mathbf{y}\cdot\mathbf{u}),$$

where $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}\in\mathbb{R}^{n+1}$. The submanifold $V_{\mathbb{C}}^{2n-1}$ of $\mathbb{C}^{n+1}$ is defined by

$$V_{\mathbb{C}}^{2n-1}=\{\mathbf{z}\in\mathbb{C}^{n+1}|\ \mathbf{z}\cdot\bar{\mathbf{z}}=2,\ \mathbf{z}\cdot\mathbf{z}=0\}, \tag{5}$$

where $\bar{\mathbf{z}}=\mathbf{x}-i\mathbf{y}$ for $\mathbf{z}=\mathbf{x}+i\mathbf{y}\in\mathbb{C}^{n+1}$. Then the map

$$V_2^{n+1}\ni(\mathbf{e}_1,\mathbf{e}_2)\mapsto\mathbf{e}_1+i\mathbf{e}_2\in V_{\mathbb{C}}^{2n-1}$$

is a diffeomorphism. Moreover $V_{\mathbb{C}}^{2n-1}$ is a submanifold of $\mathbb{S}^{2n+1}(\sqrt{2})$ with radius $\sqrt{2}$ and is invariant under the action of unit complex numbers $\{e^{i\theta}\}$ on $\mathbb{S}^{2n+1}(\sqrt{2})$ defined by $\mathbf{z}\mapsto e^{i\theta}\mathbf{z}$. Hence if we denote $\pi:\mathbb{S}^{2n+1}(\sqrt{2})\to\mathbb{CP}^n$ the Hopf fibration, then $\mathbb{Q}^{n-1}:=\pi(V_{\mathbb{C}}^{2n-1})$ is nothing but the complex quadric in $\mathbb{CP}^n$ defined by the quadratic equation $z_0^2 + \cdots + z_n^2 = 0$, and is diffeomorphic to $\widetilde{G}_2(\mathbb{R}^{n+1})$ such that the following diagram is commutative:

$$\begin{array}{ccc} V_2^{n+1} & \xrightarrow{\sim} & V_{\mathbb{C}}^{2n-1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{G}_2(\mathbb{R}^{m+1}) & \xrightarrow{\sim} & \mathbb{Q}^{n-1}. \end{array}$$

3. Space of geodesics in hyperbolic spaces

In this section, we will see that space of oriented geodesics in the hyperbolic space $\mathbb{H}^n$ is identified with some *indefinite* Grassmannian and given by using Lorentz numbers. Let $\mathbb{R}_1^{n+1}$ be the Minkowski $(n+1)$-space with the scalar product $p \cdot q = -p_0 q_0 + \sum_{j=1}^n p_j q_j$ of signature $(1, n)$. Then

$$\mathbb{H}^n = \{p = (p_0, p_1, \ldots, p_n) \in \mathbb{R}_1^{n+1} |\ p \cdot p = -1,\ p_0 > 0\}$$

is the hyperbolic space with constant sectional curvature -1 . The tangent space $T_p(\mathbb{H}^n)$ at $p \in \mathbb{H}^n$ is

$$T_p\mathbb{H}^n = \{X \in \mathbb{R}_1^{n+1} |\ X \cdot p = 0\}.$$

Then the geodesic γ of unit speed in $\mathbb{H}^n$ with $\gamma(0) = p \in \mathbb{H}^n$ and $\gamma'(0) = X \in T_{\mathbf{e}}(\mathbb{H}^n)$ ($\|X\| = 1$) is written as

$$\gamma(t) = \cosh tp + \sinh tX.$$

Let

$$V_{1,1}^{n+1} = \{(\mathbf{e}, \mathbf{f}) \mid \mathbf{e} = (e_0, e_1, \ldots, e_n),\ \mathbf{f} \in \mathbb{R}_1^{n+1},\ e_0 > 0,\\ \mathbf{e} \cdot \mathbf{e} = -1,\ \mathbf{f} \cdot \mathbf{f} = 1,\ \mathbf{e} \cdot \mathbf{f} = 0\}.$$

Note that as a homogeneous space, $V_{1,1}^{n+1} = SO^+(1,n)/SO(n-1)$ and $\dim_{\mathbb{R}} V_{1,1}^{n+1} = 2n-1$, where $SO^+(1,n)$ is the proper Lorentz group (cf. [12]). We consider the action of $SO^+(1,1)$ on $V_{1,1}^{n+1}$ as

$$(6) \qquad (\mathbf{e}, \mathbf{f}) \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} = (\cosh\theta\mathbf{e} + \sinh\theta\mathbf{f}, \sinh\theta\mathbf{e} + \cosh\theta\mathbf{f}).$$

Then each orbit $\{(\cosh\theta\mathbf{e} + \sinh\theta\mathbf{f}, \sinh\theta\mathbf{e} + \cosh\theta\mathbf{f}) | \theta \in \mathbb{R}\}$ of the action (6) is identified with a pair (γ, γ') of unit speed geodesic γ on $\mathbb{H}^n$ and its unit tangent vector field γ' with $\gamma(0) = \mathbf{e}$ and $\gamma'(0) = \mathbf{f}$. The orbit space of the above action is identified with the space of oriented geodesics in $\mathbb{H}^n$. We also identify $[(\mathbf{e}, \mathbf{f})] \in V_{1,1}^{n+1}/SO^+(1,1)$ with the oriented 2-plane with a signature $(1,1)$ in $\mathbb{R}_1^{n+1}$ spanned by $\mathbf{e}$and $\mathbf{f}$. Hence the space of oriented geodesics in $\mathbb{H}^n$ is the oriented indefinite 2-plane Grassmannian $\widetilde{G}_{1,1}^+(\mathbb{R}_1^{n+1})$.

Let $\pi : V_{1,1}^{n+1} \to G_{1,1}^+(1,n)$ be the natural projection. Tangent space of $V_{1,1}^{n+1}$ at the point $(\mathbf{e}, \mathbf{f})$ is

$$T_{(\mathbf{e},\mathbf{f})} V_{1,1}^{n+1} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x},\ \mathbf{y} \in \mathbb{R}_1^{n+1},\ \mathbf{x} \cdot \mathbf{e} = \mathbf{y} \cdot \mathbf{f} = \mathbf{x} \cdot \mathbf{f} + \mathbf{e} \cdot \mathbf{y} = 0\}.$$

Put

$$T'_{(\mathbf{e},\mathbf{f})} = \{(\mathbf{x}, \mathbf{y}) \in T_{(\mathbf{e},\mathbf{f})} V_{1,1}^{n+1} \mid \mathbf{x} \cdot \mathbf{f} = \mathbf{y} \cdot \mathbf{e}\}.$$

Then the distribution $T'_{(\mathbf{e},\mathbf{f})}$ gives a connection on the principal fiber bundle $V^{n+1}_{1,1}(G^+_{1,1}(1,n), SO^+(1,1))$, and the projection π induces the linear isomorphism $\pi_* : T'_{(\mathbf{e},\mathbf{f})} \to T_{\pi(\mathbf{e},\mathbf{f})}G^+_{1,1}(1,n)$. For tangent vectors X_1, X_2 at $p \in G^+_{1,1}(1,n)$ and their horizontal lifts $(\mathbf{x}_1,\mathbf{y}_1), (\mathbf{x}_2,\mathbf{y}_2) \in T_{(\mathbf{e},\mathbf{f})}V^{n+1}_{1,1}$ with $\pi(\mathbf{e},\mathbf{f}) = p$, we put

$$\langle X_1, X_2\rangle = -\mathbf{x}_1\cdot\mathbf{x}_2 + \mathbf{y}_1\cdot\mathbf{y}_2. \tag{7}$$

Then $\langle\ ,\ \rangle$ gives a semi-Riemannian metric g of signature $(n-1, n-1)$ on $G^+_{1,1}(1,n)$. Note that these indefinite Grassmannian and semi-Riemannian metric are constructed by Ejiri [7].

Let $P : T_pG^+_{1,1}(1,n) \to T_pG^+_{1,1}(1,n)$ be the linear endomorphism defined by

$$P\pi_*(\mathbf{x},\mathbf{y}) = \pi_*(\mathbf{y},\mathbf{x}), \qquad (\mathbf{x},\mathbf{y}) \in T'_{(\mathbf{e},\mathbf{f})},\quad \pi(\mathbf{e},\mathbf{f}) = p. \tag{8}$$

Then

$$P^2 = 1, \tag{9}$$

$$\dim_{\mathbb{R}}\{X|PX = \pm X\} = \dim_{\mathbb{R}} M/2, \tag{10}$$

$$\langle PX_1, X_2\rangle + \langle X_1, PX_2\rangle = 0, \tag{11}$$

$$\nabla P = 0, \tag{12}$$

where ∇ denotes the Levi-Civita connection of $(G^+_{1,1}(1,n), g)$.

Definition 3.1. [4, 10] A tensor field P of type $(1,1)$ on a differentiable manifold M is called *almost product structure* (resp. *almost para-complex structure*) if (9) (resp. (9,10)) valid. A tensor field P of type $(1,1)$ on a semi-Riemannian manifold $(M, \langle\ ,\ \rangle, \nabla)$ is called *almost para-Hermitian structure* (resp. *para-Kähler structure*) if (9,10,11) (resp. (9,10,11,12)) hold.

Note that on a para-Kähler manifold $(M, P, \langle\ ,\ \rangle)$, a 2-form defined by $\omega(X,Y) = \langle PX, Y\rangle$ gives a symplectic form. $(G^+_{1,1}(1,n), P, \langle\ ,\ \rangle)$ is a para-Hermitian symmetric space [4, 10], especially is a symplectic affine symmetric space.

Proposition 3.2. For an oriented 2-dimensional semi-Riemannian manifold $(\Sigma^2_1, \cdot)$ with signature $(1,1)$, there is a canonical para-Kähler structure P on Σ^2_1.

In fact, let (u,v) be an isothermal coordinate, which is compatible with the orientation of Σ^2_1, i.e., $\partial_u\cdot\partial_u + \partial_v\cdot\partial_v = \partial_u\cdot\partial_v = 0$ and $\partial_u\cdot\partial_u < 0$. Then the canonical para-Kähler structure on Σ^2_1 is defined by

$$P\partial_u = \partial_v, \quad P\partial_v = \partial_u. \tag{13}$$

Let $\mathbb{L}^{n+1} = \{\mathbf{z} = \mathbf{x} + \tau\mathbf{y} | \mathbf{x},\mathbf{y} \in \mathbb{R}^{n+1}\}$ be the space of *Lorentz numbers*, and define the dot product on $\mathbb{L}^{n+1}$ as

$$(\mathbf{x}+\tau\mathbf{y})\cdot(\mathbf{u}+\tau\mathbf{v}) = (\mathbf{x}\cdot\mathbf{u} + \mathbf{y}\cdot\mathbf{v}) + \tau(\mathbf{x}\cdot\mathbf{v} + \mathbf{y}\cdot\mathbf{u}),$$

where $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v} \in \mathbb{R}^{n+1}$. Then $\mathbb{L}^{n+1}$ is naturally identified with the semi-Euclidean space $\mathbb{R}^{2n+2}_{n+1}$ (cf. [12], pp.55) with the scalar product

$$\langle \mathbf{x}+\tau\mathbf{y}, \mathbf{u}+\tau\mathbf{v}\rangle = \mathrm{Re}((\mathbf{x}+\tau\mathbf{y})\cdot(\mathbf{u}-\tau\mathbf{v})).$$

The submanifold $V_{\mathbb{L}}^{2n-1}$ of $\mathbb{L}^{n+1}$ is defined by

$$V_{\mathbb{L}}^{2n-1} = \{\mathbf{z} \in \mathbb{L}^{n+1} |\ \mathbf{z} \cdot \bar{\mathbf{z}} = -2,\ \mathbf{z} \cdot \mathbf{z} = 0\}, \tag{14}$$

where $\bar{\mathbf{z}} = \mathbf{x} - \tau \mathbf{y}$ for $\mathbf{z} = \mathbf{x} + \tau \mathbf{y} \in \mathbb{L}^{n+1}$. Then the map

$$V_{1,1}^{n+1} \ni (\mathbf{e}, \mathbf{f}) \mapsto \mathbf{e} + \tau \mathbf{f} \in V_{\mathbb{L}}^{2n-1}$$

is a diffeomorphism. Moreover $V_{\mathbb{L}}^{2n-1}$ is a submanifold of the *pseudohyperbolic space* $\mathbb{H}_n^{2n+1}(\sqrt{2})$ (cf. [12], pp.110) with radius $\sqrt{2}$ and is invariant under the action of unit Lorentz numbers $\{e^{\tau\theta}\}$ on $\mathbb{H}_n^{2n+1}(\sqrt{2})$ defined by $\mathbf{z} \mapsto e^{\tau\theta}\mathbf{z}$. From these facts, we may consider the space of oriented geodesics in $\mathbb{H}^n$ as "*Lorentz quadric*".

4. Gauss maps for timelike surfaces in the Lorentz spaces

Let $\varphi : \Sigma_1^2 \to \mathbb{R}_1^{n+1}$ be an immersion from an oriented timelike surface Σ_1^2 to the Lorentz space $\mathbb{R}_1^{n+1}$. Then its Gauss map τ is defined as

$$\tau : \Sigma_1^2 \to G_{1,1}^+(1, n), \quad \tau(p) = \varphi_*(T_p\Sigma_1^2).$$

Proposition 4.1. (i) τ is conformal $\Leftrightarrow$ φ is pseudo umbilical.
(ii) τ is harmonic $\Leftrightarrow$ mean curvature vector of φ is parallel with respect to the normal connection.

With respect to the almost product structure P (resp. $\bar{P}$) on Σ_1^2 (resp. $G_{1,1}^+(1, n)$) defined by (13) (resp. (8)), the following hold:

Proposition 4.2. (i) $\tau_* \circ P = \bar{P} \circ \tau_*$ $\Leftrightarrow$ the mean curvature vector of φ vanishes.
(ii) $\tau_* \circ P = -\bar{P} \circ \tau_*$ $\Leftrightarrow$ φ is totally umbilical.

References

[1] R. L. Bryant, *A duality theorem for Willmore surfaces*, J. Diff. Geom. **20** (1984), 23-53.

[2] R. L. Bryant, *Bochner-Kähler metrics*, J. Amer. Math. Soc. **14** (2001), 623-715.

[3] A. A. Borisenko, *Extrinsic geometry of strongly parabolic multidimensional submanifolds*, Uspekhi Mat. Nauk **52** (1997), no.6(318); translation in Russian Math. Surveys **52** (1997), no. 6, 1141-1190.

[4] V. Cruceanu, P. Fortuny and P. M. Gadea, *A survey on paracomplex geometry*, Rocky Mountain J. **26** (1996), 83-115.

[5] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385-524.

[6] N. Ejiri, *Isotropic harmonic maps of Riemann surfaces into the de Sitter space time*, Quart. J. Math. Oxford **39** (1988), 291-306.

[7] N. Ejiri, *Willmore surfaces with a duality in* $S^N(1)$, Proc. London Math. Soc. **57** (1988), 383-416.

[8] F. R. Harvey, *Spinors and Calibrations*, Academic Press (1990).

[9] G. Ishikawa, M. Kimura and R. Miyaoka, *Submanifolds with degenerate Gauss mappings in spheres*, Advanced Studies in Pure Mathematics, **37** (2002), 115-149.

[10] S. Kaneyuki and M. Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81-98.

[11] M. Kimura, *Minimal immersions of some circle bundles over holomorphic curves in complex quadric to sphere*, Osaka J. Math., 37 (2000) 883-903.

[12] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press (1983).
[13] T. Weinstein, *Lorentz surfaces*, Walter de Gruyter (1996).

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