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AN ENUMERATIVE PROBLEM CONCERNING LOOPS ON A GRAPH

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Abstract. Correspondence between a set of loops on a graph and a set consisting of perfect matchings in complete multipartite graphs assigned to vertices is considered. The latter is introduced to investigate some enumerative problem concerning to loops on a graph.

1. Acknowledgement

In this article, a part of results obtained in Ref.[1] is announced. A motivation is described in [2]; thus it is not repeated here. More detailed account is planned to be published in the future.

2. Admissible closed walks and loops on a graph

Let G be a triple (V, E, ϕ) where V and E are nonempty finite sets and $\phi : E \to V \times V$. We write $\phi(e) = (i(e), t(e))$. We shall refer to this triple simply as a *graph* G. Formally let $-E = \{-e \mid e \in E\}$ and let $\mathcal{I} = E \sqcup (-E)$ (disjoint union). Define $- : \mathcal{I} \to \mathcal{I}$, $i : \mathcal{I} \to V$, $t : \mathcal{I} \to V$, $\epsilon : \mathcal{I} \to E$ and $o : \mathcal{I} \to \{+1, -1\}$ as follows: Let $m \in \mathcal{I}$; if $m = e$ for some $e \in E$ then $-m = -e$, $i(m) = i(e)$, $t(m) = t(e)$, $\epsilon(m) = e$, $o(m) = +1$; otherwise, if $m = -e$ for some $e \in E$ then $-m = e$, $i(m) = t(e)$, $t(m) = i(e)$, $\epsilon(m) = e$, $o(m) = -1$.

Let p be a positive integer, and let $\mathcal{W}_p$ be the set consisting of all p-tuples $(m_0, m_1, \ldots, m_{p-1})$ of members of $\mathcal{I}$ satisfying $t(m_i) = i(m_{i+1})$ for $0 \le i \le p-2$, $t(m_{p-1}) = i(m_0)$, $m_{i+1} \ne -m_i$ for $0 \le i \le p-2$ and $m_0 \ne -m_{p-1}$. We shall call an element of $\mathcal{W}_p$ an *admissible closed walk* (on the graph G) of length p. Let $\mathbf{D}_p = \langle S, R \mid S^p = R^2 = 1, RSR^{-1} = S^{-1}\rangle$ be the dihedral group of order $2p$. Define an action of $\mathbf{D}_p$ on $\mathcal{W}_p$ by $S((m_0, m_1, \ldots, m_{p-1})) = (m_1, \ldots, m_{p-1}, m_0)$ and $R((m_0, \ldots, m_{p-1})) = (-m_{p-1}, \ldots, -m_0)$. Let $w \in \mathcal{W}_p$, and let $\mathbf{D}_{p,w} = \{X \in \mathbf{D}_p \mid X \text{ fixes } w\}$. ($\mathbf{D}_{p,w}$ is the stabilizer of w.) We call the number $d(w) = |\mathbf{D}_{p,w}|$ the *degeneracy* of w. An orbit of $\mathbf{D}_p$ on $\mathcal{W}_p$ (a $\mathbf{D}_p$-equivalence class of $\mathcal{W}_p$) is said to be a *loop* of length p (on the graph G). Let $\mathcal{L}_p$ be the set of all loops of length

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p. The degeneracy of $L \in \mathcal{L}_p$ is defined to be that of $w \in \mathcal{W}_p$ which belongs to the class L.

Let $\mathcal{L} = \sqcup_{p=1}^{\infty} \mathcal{L}_p$; the element of $\mathcal{L}$ will be called a loop (on G). Consider formally an associative and commutative product of loops; let $\mathcal{Z}$ be the set of all such products: $\mathcal{Z} = \{L_1 \cdots L_n \,|\, n \geq 1, L_i \in \mathcal{L}\}$. Note that each element of $\mathcal{Z}$ is factorized as $Z = L_1^{j_1} \cdots L_k^{j_k}$ where the factors $L_1, \ldots, L_k$ are distint loops and $j_1, \ldots, j_k$ are positive integers; this factorization is unique except for the order of the factors.

Let $\mathbf{x} = \{x_e \,|\, e \in E\}$ be a set of commutative indeterminates. Define $\mathbf{x}^w = \prod_{i=0}^{p-1} x_{\epsilon(m_i)}$ for $w = (m_0, \ldots, m_{p-1}) \in \mathcal{W}_p$. For $L \in \mathcal{L}_p$ define $\mathbf{x}^L = \mathbf{x}^w$ where $w \in \mathcal{W}_p$ is a representative of L. For $Z = L_1 \cdots L_n \in \mathcal{Z}$ define $\mathbf{x}^Z = \mathbf{x}^{L_1} \cdots \mathbf{x}^{L_n}$. For $\mathbf{r} = (r_e \,|\, e \in E) \in \mathbb{Z}_{\geq 0} \times \cdots \times \mathbb{Z}_{\geq 0}$ write $\mathbf{x}^{\mathbf{r}} = \prod_{e \in E} x_e^{r_e}$. Define $\mathcal{A} = \{\mathbf{r} \in \mathbb{Z}_{\geq 0} \times \cdots \times \mathbb{Z}_{\geq 0} \,|\, \mathbf{x}^{\mathbf{r}} = \mathbf{x}^Z$ for some $Z \in \mathcal{Z}\}$, and $\mathcal{Z}(\mathbf{r}) = \{Z \in \mathcal{Z} \,|\, \mathbf{x}^Z = \mathbf{x}^{\mathbf{r}}\}$ for $\mathbf{r} \in \mathcal{A}$.

3. Complete multipartite graphs assigned to vertices and perfect matchings in them

Let $\mathbf{r} = (r_e \,|\, e \in E) \in \mathcal{A}$.

Define $\mathrm{Part}(\alpha, \mathbf{r}) = \{(e, +) \,|\, e \in E, i(e) = \alpha, r_e \geq 1\} \sqcup \{(e, -) \,|\, e \in E, t(e) = \alpha, r_e \geq 1\}$ for $\alpha \in V$, and define $V(\mathbf{r}) = \{\alpha \in V \,|\, \mathrm{Part}(\alpha, \mathbf{r})$ is not empty$\}$.

Let $\alpha \in V(\mathbf{r})$. For each $(e, s) \in \mathrm{Part}(\alpha, \mathbf{r})$ write $X_{e,s} = \{(e, s, \rho) \,|\, \rho = 1, \ldots, r_e\}$; consider a complete $|\mathrm{Part}(\alpha, \mathbf{r})|$-partite graph,[1] denoted $K(\alpha, \mathbf{r})$, with vertex classes $X_{e,s}$, $(e, s) \in \mathrm{Part}(\alpha, \mathbf{r})$. Let $\mathcal{M}(\alpha, \mathbf{r})$ be the set consisting of all perfect matchings[1] in the graph $K(\alpha, \mathbf{r})$, and let $\mathcal{M}(\mathbf{r}) = \prod_{\alpha \in V(\mathbf{r})} \mathcal{M}(\alpha, \mathbf{r})$.

Define $E(\mathbf{r}) = \{e \in E \,|\, r_e \geq 1\}$. Let $\mathbf{S}(\mathbf{r}) = \prod_{e \in E(\mathbf{r})} \mathbf{S}_{r_e}$, where $\mathbf{S}_r$ is the symmetric group of order r. Define an action of the group $\mathbf{S}(\mathbf{r})$ on $\mathcal{M}(\mathbf{r})$ as follows: For $\sigma = (\sigma_e \,|\, e \in E(\mathbf{r}))$ and $M = (M_\alpha \,|\, \alpha \in V(\mathbf{r}))$, $M \cdot \sigma = (M_\alpha \cdot \sigma \,|\, \alpha \in V(\mathbf{r}))$, and $M_\alpha \cdot \sigma$ is defined to be $\{\{(e, s, \rho \cdot \sigma_e), (e', s', \rho' \cdot \sigma_{e'})\}, \ldots\}$ for $M_\alpha = \{\{(e, s, \rho), (e', s', \rho')\}, \ldots\}$.

4. Correspondence between $\mathbf{S}(\mathbf{r})$-orbits and products of loops

Proposition. *Let $\mathbf{r} \in \mathcal{A}$. There exists a one-to-one correspondence between $\mathcal{Z}(\mathbf{r})$ and the set of all $\mathbf{S}(\mathbf{r})$-orbits on $\mathcal{M}(\mathbf{r})$.*

The proof is in a similar way to that of the next theorem; it is omitted here.

[1] A *simple graph* H is a pair (V, E) where V (the vertex set) is a nonempty finite set and E (the edge set) is a set whose element is a set consisting of two members in V. A simple graph H is a *complete k-partite graph with vertex classes* $V_1, \ldots, V_k$ if the vertex set is $V_1 \cup \cdots \cup V_k$, where $V_i \cap V_j = \emptyset$ whenever $i < j$, and the edge set consists of all edges joining vertices in distinct classes. A subset M of the edge set of a simple graph H is called a *matching* in H if $e \cap e' = \emptyset$ whenever $e \neq e'$, $e, e' \in M$; a matching M in H is *perfect* if $\cup_{e \in M} e$ is the vertex set of H. (For graph theory terminology, see, e.g., [3].)

Theorem. *Let $Z \in \mathcal{Z}$, $Z = L_1^{j_1} \cdots L_n^{j_n}$, where $L_1, \dots, L_n$ are distinct loops and $j_1 \geq 1, \dots, j_n \geq 1$. Let $\mathbf{r}$ be an element of $\mathcal{A}$ such that $\mathbf{x}^\mathbf{r} = \mathbf{x}^Z$. Let M be an element of $\mathcal{M}(\mathbf{r})$ which corresponds to Z. Then,*

$$|M \cdot \mathbf{S}(\mathbf{r})| = \frac{\prod_{e \in E(\mathbf{r})} r_e!}{j_1! d(L_1)^{j_1} \cdots j_n! d(L_n)^{j_n}}$$

where $M \cdot \mathbf{S}(\mathbf{r})$ is an orbit on $\mathcal{M}(\mathbf{r})$.

Outline of proof. Write $Z = L_1^{j_1} \cdots L_n^{j_n} = L'_1 \cdots L'_k$ $(k = j_1 + \cdots + j_n)$. We explain how to make $M \in \mathcal{M}(\mathbf{r})$ (perfect matchings at vertices) from this Z. Explanation is algorithmic; a phrase '$A \leftarrow B$' means 'A is (re)defined to be B'.

Set $M_\alpha = \emptyset$ for $\alpha \in V(\mathbf{r})$, and set $c = 1$ (used as a counter). Let B_e be, initially, a set $\{1, 2, \dots, r_e\}$.

(STEP i) If p is the length of L'_c, choose arbitrary a representative $w = (m_0, \dots, m_{p-1})$ $(\in \mathcal{W}_p)$ of the class L'_c.

(STEP ii) Choose an element ρ from $B_{\epsilon(m_0)}$. Let $B_{\epsilon(m_0)} \leftarrow B_{\epsilon(m_0)} - \{\rho\}$, and let $i = 0$.

(STEP iii) If $i = p - 1$ then *go to* STEP iv. Otherwese, choose an element ρ' from $B_{\epsilon(m_{i+1})}$, and let $B_{\epsilon(m_{i+1})} \leftarrow B_{\epsilon(m_{i+1})} - \{\rho'\}$. Write $\alpha = t(m_i)$. Let $M_\alpha \leftarrow M_\alpha \cup \{\{(\epsilon(m_i), -o(m_i), \rho), (\epsilon(m_{i+1}), o(m_{i+1}), \rho')\}\}$. Set $i \leftarrow i + 1$ and $\rho \leftarrow \rho'$, and go back to the beginning of this STEP.

(STEP iv) [Now $i = p - 1$.] Choose $\rho' \in B_{\epsilon(m_0)}$, and let $B_{\epsilon(m_0)} \leftarrow B_{\epsilon(m_0)} - \{\rho'\}$. Write $\alpha = t(m_{p-1})$ $(= i(m_0))$ and let $M_\alpha \leftarrow M_\alpha \cup \{\{(\epsilon(m_{p-1}), -o(m_{p-1}), \rho), (\epsilon(m_0), o(m_0), \rho')\}\}$. If $c = k$ the construction is terminated here; otherwise, set $c \leftarrow c + 1$, and go to STEP i.

Let $M = (M_\alpha \,|\, \alpha \in V(\mathbf{r}))$; this is an element of $\mathcal{M}(\mathbf{r})$ corresponding to Z.

What is the number of ways to get such M from Z? At first grance, it seems to be $\prod_{e \in E(\mathbf{r})} r_e!$; but it is not true. Since L'_c has an 'internal' symmetry $\mathbf{D}_{p,w}$ it must be divided by $d(L'_c)$ per L'_c. (Recall $d(L'_c) = |\mathbf{D}_{p,w}|$.) Moreover, since, in Z, a factor L_1 appears j_1 times, a factor L_2 appears j_2 times, and so on, the number of ways must be divided by $j_1! j_2! \cdots j_n!$. Thus, we obtain the theorem. □

Corollary. *Let $\mathbf{S}(\mathbf{r})_M = \{\sigma \in \mathbf{S}(\mathbf{r}) \,|\, M \cdot \sigma = M\}$ be the stabilizer of M. Then,*

$$|\mathbf{S}(\mathbf{r})_M| = j_1! d(L_1)^{j_1} \cdots j_n! d(L_n)^{j_n}.$$

Proof. Recall $|M \cdot \mathbf{S}(\mathbf{r})| = |\mathbf{S}(\mathbf{r})| / |\mathbf{S}(\mathbf{r})_M|$ and $|\mathbf{S}(\mathbf{r})| = \prod_{e \in E(\mathbf{r})} r_e!$. □

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