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ON COMPLEX SPHERES

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Abstract. A complex sphere is a typical example of spaces of which the index is just the half of the dimension. In this paper, we study characters of the complex sphere and characterize it by the hypersurface theory.

1. Introduction

If we consider amounts of study of differential geometry by the index k of the metric, the case of $k = 0$ (i.e., the Riemannian metric) will have a considerable amount in it. On the case $k \neq 0$ (i.e., pseudo-Riemannian metric), the case of $k = 1$ (i.e., the Lorentzian metric) will have a great amount, for the Lorentzian metric have an important role in the relativity theory.

Nevertheless, from a pure mathematical stand point of view, a minor field (i.e., the case of $k \neq 0$ and $k \neq 1$) may have fruitful results, we think. A very natural question is that: what can we say by generalizing results of Lorentzian case into general pseudo-Riemannian cases? Conversely, we can ask following question : (1) Is there some special number on the index ? (2) Is there a typical space of some special number of the index ?

The purpose of this paper is to give an answer to the last question. In this paper, we consider the complex sphere. For, its index is just the half of the number of the dimension.

2. Comple Riemannian metric

This section is devoted to recalling the complex Riemannian metric. For detail of this section, see [1].

Let V be a real 2n-dimensional vector space with a complex structure J and V^C the complexification of (V, J). The complex linear extension of J onto V^C is denoted by the same letter J. If we put

$$V^{1,0} = \{Z \in V^C : JZ = \sqrt{-1}Z\}, \quad V^{0,1} = \{Z \in V^C : JZ = -\sqrt{-1}Z\},$$

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then $V^C = V^{1,0} \oplus V^{0,1}$ and the complex conjugation

$$Z = X + \sqrt{-1}Y \longrightarrow \overline{Z} = X - \sqrt{-1}Y, \quad (X, Y \in V)$$

in V^C defines a linear isomorphism between $V^{1,0}$ and $V^{0,1}$.

Let g be a real inner product in (V, J) satisfing

$$g(JX, JY) = -g(X, Y), \quad X, Y \in V, \tag{2.1}$$

(hence the index of V is n). Then g can be extended uniquely to a symmetric complex nondegenerate bilinear form G of V^C with

$$G(\overline{Z}, \overline{W}) = \overline{G(Z, W)}, \quad G(Z, W) = -G(JZ, JW), \quad Z, W \in V^C. \tag{2.2}$$

Conversely, every symmetric complex bilinear form G on V^C satisfying (2.2) is the natural extension of a real inner product g in (V, J) satisfying (2.1).

The condition $G(Z, W) = -G(JZ, JW)$ ($Z, W \in V^C$) is equivalent to $G(Z, \overline{W}) = 0 (Z, W \in V^{1,0})$. That is, symmetric complex bilinear form satisfying (2.2) is completely determined by its values on $V^{1,0}$.

Let $Z = (z^1, \dots, z^n) \in \mathbb{C}^n$. By setting, $z^k = x^k + \sqrt{-1}y^k$, x^k, $y^k \in \mathbb{R}^n$ $(k = 1, \dots, n)$, the standard identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by

$$(z^1, \dots, z^n) \longrightarrow (x^1, \dots, x^n, y^1, \dots, y^n).$$

The canonical complex structure J of $\mathbb{R}^{2n}$ is

$$(x^1, \dots, x^n, y^1, \dots, y^n) \longrightarrow (-y^1, \dots, -y^n, x^1, \dots, x^n)$$

and the canonical inner product g of $\mathbb{R}^{2n}$ is defined by

$$\begin{aligned} g(Z, Z) &= 2Re\left((z^1)^2 + \dots + (z^n)^2\right) \\ &= 2\left((x^1)^2 + \dots + (x^n)^2 - (y^1)^2 - \dots - (y^n)^2\right). \end{aligned}$$

There is a natural one-to-one correspondence between the set of inner product of $\mathbb{R}^{2n}$, satisfying the property (2.1) with respect to the canonical complex structure J, and the homogeneous space $GL(n, \mathbb{C})/O(n, \mathbb{C})$.

Let (M, J) be a real 2n-dimensional manifold with the complex structure J. If $z^1, \dots, z^n$ are holomorphic coordinate in a coordinate neighbourhood U of $p \in M$ and $z^k = x^k + \sqrt{-1}y^k$ $(k = 1, \dots, n)$, then

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right), \quad Z_{\overline{\alpha}} = \frac{\partial}{\partial \overline{z^\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial y^\alpha}\right)$$

form a basis of $(T_p^C M)^{1,0}$ and $(T_p^C M)^{0,1}$, respectively.

Let G be a complex Riemannian metric of (M, J), i.e., G satisfies (cf. [1])

$$G(\overline{Z}, \overline{W}) = \overline{G(Z, W)}, G(JZ, JW) = -G(Z, W), \quad Z, W \in T^C M.$$

If we put $G_{AB} = G(Z_A, Z_B)$, then the above conditions are written as

$$G_{\overline{AB}} = \overline{G_{AB}}, \quad G_{\overline{\alpha}\beta} = G_{\alpha\overline{\beta}} = 0,$$

where

$$A, B = 1, \dots, n, \overline{1}, \dots, \overline{n}, \quad \alpha, \beta = 1, \dots, n.$$

The complex Riemannian metric G induces a real pseudo-Riemannian metric g of index n on (M, J).

On (M, J, G), there exists a unique linear connection D with components D^A_{BC} such that

$$D^A_{BC} = D^A_{CB}, \quad D^{\alpha}_{\beta\overline{\gamma}} = D^{\overline{\alpha}}_{\beta\gamma}, \quad D_{\alpha} G_{\beta\gamma} = 0.$$

If the Levi-Civita connection ∇ of G satisfies $\nabla J = 0$ then ∇ coincides with D and the converse is also true.

3. Complex sphere

This section is devoted to defining and finding some characters of the complex sphere.

The complex sphere $CS^n(c)$ is defined by

$$CS^n(c) = \{(z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} : (z^1)^2 + \cdots + (z^{n+1})^2 = c, \quad c \in \mathbb{C}, \quad c \neq 0\}.$$

Since $CS^n(c)$ is diffeomorphic to the tangent bundle TS^n of the sphere S^n of $\mathbb{R}^{n+1}$, $CS^n(c)$ is connected and simply connected. Moreover, $CS^n(c)$ is complete with respect to the induced connection ∇ from $\mathbb{C}^{n+1}$.

By the definition of $CS^n(c)$, we have $\sum_{i=1}^{n+1} z^i dz^i = 0$ on $CS^n(c)$. Let $\zeta = \sum_{i=1}^{n+1} z^i \partial/\partial z^i$ be a holomorphic vector field on $\mathbb{C}^{n+1}$. Then it follows that

$$\left(\sum_{i=1}^{n+1} z^i dz^i\right)(\zeta) = \sum_{i=1}^{n+1} (z^i)^2 = c \ (\neq 0),$$

$$g(W, \zeta) = \sum_{i=1}^{n+1} dz^i(W) dz^i(\zeta) = \left(\sum_{i=1}^{n+1} z^i dz^i\right)(W) = 0, \quad W \in T^{1,0}_p.$$

Hence ζ is transversal and a normal vector field on M. As $g(\zeta, \zeta) = \sum_{i=1}^{n+1} (z^i)^2 = c \ (\neq 0)$, the induced metric g on M is non-degenerate and satisfies $g(JX, JY) = -g(X, Y)$.

Next we consider some characters of $CS^n(c)$ as a complex hypersurface. Let D be the Levi-Civita connection on $\mathbb{C}^{n+1}$. Since $D_W \zeta = W, D_{\overline{W}} \zeta = 0$ for $W \in \mathfrak{X}(TM^{1,0})$, the shape operator S of $CS^n(c)$ is given by

$$S = -I \quad \text{on} \quad TM^{1,0}, \qquad S = 0 \quad \text{on} \quad TM^{0,1}. \tag{3.1}$$

Let ∇ be the induced Levi-Civita connection on $CS^n(c)$. The Gauss formula is defined by

$$D_X Y = \nabla_X Y + B(X, Y)\zeta, \quad X, Y \in \mathfrak{X}(TM^{1,0}).$$

Then we have

$$B(X, Y) = -\frac{1}{c} g(X, Y), \quad X, Y \in \mathfrak{X}(TM^{1,0}), \tag{3.2}$$

by virtue of

$$cB(X, Y) = g(D_X Y, \zeta) = -g(Y, D_X \zeta) = -g(X, Y).$$

By the equation of Gauss, the curvature tensor R of $CS^n(c)$ satisfies

$$\begin{aligned} &R(Z,W)U = B(W,U)SZ - B(Z,U)SW, \\ &R(Z,W)\overline{U} = 0, \\ &R(Z,\overline{W})U = -B(Z,U)S\overline{W}, \quad Z,W,U \in \mathfrak{X}(TM^{1,0}). \end{aligned}$$

Using (3.1) and (3.2), we have

$$\begin{aligned} &R(Z,W)U = \frac{1}{c}(g(W,U)Z - g(Z,U)W), \\ &R(Z,\overline{W})U = 0, \\ &R(\overline{Z},W)\overline{U} = \overline{R(Z,\overline{W})U} = 0, \\ &R(\overline{Z},\overline{W})\overline{U} = \overline{R(Z,W)U} = \frac{1}{c}(g(\overline{W},\overline{U})\overline{Z} - g(\overline{Z},\overline{U})\overline{W}). \end{aligned} \tag{3.3}$$

Let Ric be the Ricci tensor of $CS^n(c)$.

For a basis $\{e_1,\dots,e_n\}$ of $TM^{1,0}$ satisfying $g(e_i,e_j) = \delta_{ij}$, it follows that

$$\begin{aligned} &Ric(W,U) = \sum_{i=1}^{n} g(R(e_i,W)U,e_i) + \sum_{i=1}^{n} g(R(\overline{e_i},W)U,\overline{e_i}) = \frac{n-1}{c} g(W,U), \\ &Ric(W,\overline{U}) = 0, \\ &Ric(\overline{W},\overline{U}) = \overline{Ric(W,U)} = \frac{n-1}{\overline{c}} g(\overline{W},\overline{U}). \end{aligned} \tag{3.4}$$

Hence the Ricci operator Q of $CS^n(c)$ satisfies

$$QW = \frac{n-1}{c}W, \quad Q\overline{W} = \frac{n-1}{\overline{c}}\overline{W}. \tag{3.5}$$

The scalar curvature r of $CS^n(c)$ is given as

$$r = n(n-1)\left(\frac{1}{c} + \frac{1}{\overline{c}}\right) \tag{3.6}$$

by virture of (3.5).

Proposition 3.1. ([2]) *Let $CS^n(\sqrt{-1}a)$ $(a \in \mathbb{R})$ be a complex sphere with the radius whose square is pure imaginary. Then $CS^n(\sqrt{-1}a)$ is comformally flat and its scalar curvature is identically zero.*

Proof. By setting $c = \sqrt{-1}b$ in (3.6), it follows that $r = 0$.

From (3.3), (3.4), and (3.5), we have

$$R(X,Y)Z = \frac{1}{2n-2}(QX \wedge Y + X \wedge QY)Z, \quad X,Y,Z \in TM^C,$$

by virtue of $r = 0$. Hence $CS^n(\sqrt{-1}a)$ is conformally flat. $\square$

Remark. If we set $z^i = x^i + \sqrt{-1}y^i$ $(i = 1, \dots, n)$ then the equation $\sum_{i=1}^n (z^i)^2 = \sqrt{-1}a$ is rewritten as

$$\begin{cases} \sum_{i=1}^n ((x^i)^2 - (y^i)^2) = 0, \\ \\ \sum_{i=1}^n x^i y^i = \frac{a}{2}. \end{cases}$$

Therefore $CS^n(\sqrt{-1}a)$ is the intersection of the lightcone $\sum_{i=1}^n ((x^i)^2 - (y^i)^2) = 0$ and a "hyperboloidal" hypersurface on $\mathbb{R}^{2n+2}$. From this fact, we can see that $CS^n(\sqrt{-1}a)$ is conformally flat, as well.

Incidentally, the complex sphere $CS^n(a)$, $a \in \mathbb{R}$ is rewritten as

$$\begin{cases} \sum_{i=1}^n ((x^i)^2 - (y^i)^2) = a, \\ \\ \sum_{i=1}^n x^i y^i = 0. \end{cases}$$

4. Characterization of the complex sphere

In this section, we characterize $CS^n(\sqrt{-1}a)$, $a \in \mathbb{R}$, in the class of invariant hypersurfaces of $\mathbb{C}^{n+1}$.

Let $\mathbb{C}^{n+1}$ be the real (2n+2)-dimensional flat space with anti-Kähler structure J. By M^{2n}, we denote the invariant hypersurfadce of $\mathbb{C}^{n+1}$. Then the induced structure J and the metric g on M^{2n} satisfy

$$J^2 = -1, \quad g(JX, JY) = -g(X, Y), \quad g(JX, Y) = g(X, JY),$$

for any tangent vector fields X and Y of M^{2n}. This means that g is pseudo-Riemannian with index n.

Let D (resp. ∇) be the covariant derivative of $\mathbb{C}^{n+1}$ (resp. M^{2n}). Then the Gauss and Weingarten formulas are given as

$$D_X Y = \nabla_X Y + B(X, Y), \quad D_X N = -A^N(X) + \nabla_X^\perp N,$$

for any tangent vector fields X and Y, and normal vector field N, where $\nabla^\perp$ is the normal connection and $B(X, Y)$ (resp. A^N) is the second fundamental form (resp. shape operator) of this hypersurface.

If (M^{2n}, g, J) is anti-Kähler, i.e., it satisfies $\nabla J = 0$, then we have

$$B(JX, Y) = JB(X, Y), \quad JA^N = A^{JN} = A^N J.$$

Lemma 1. *Let V^{2n} be a 2n-dimensional metric vector space with complex Riemannian structure J. Then we can give an orthonormal basis $\{e_1, Je_1, \dots, e_n, Je_n\}$ on V^{2n}.*

Proof. Let x be a spacelike vector. Then Jx is timelike. As

$$J(\mathrm{Span}\{x, Jx\}) = \mathrm{Span}\{x, Jx\},$$

Span$\{x, Jx\}$ is a 2-dimensional subspace with Lorentzian metric.

A self-adjoint linear operator on a 2-dimensional Lorentzian space hase a matric of on of the following three types:

$$\text{(i)} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} a & 0 \\ \pm 1 & a \end{bmatrix}$$

where (i) and (ii) are given by an orthonormal basis and (iii) by a null basis.

Comparing the complex Riemannian structure J and (i), (ii) and (iii), we may write

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Hence, on $\mathrm{Span}\{x, Jx\}$ we have an orthogonal basis $\{e_1, Je_1\}$ such that $g(e_1, Je_1) = 0$.

Let e_2 be a unit vector in V^{2n} of $g(e_1, e_2) = g(Je_1, e_2) = 0$. Then, from $g(Je_1, Je_2) = -g(e_1, e_2) = 0$, we have

$$\mathrm{Span}\{e_1, Je_1\} \perp \mathrm{Span}\{e_2, Je_2\}.$$

Continuing this process, we can obtain an orthonormal basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$. □

Let $\{N_s, N_t\}$ be an orthonormal basis of the normal space of M^{2n}, where N_s (resp. N_t) is spacelike (resp. timelike). We may take $N_t = JN_s$. Then the equation of Gauss can be written as

$$\begin{aligned} 0 = & \, g(R(X,Y)Z, W) - g(A^{N_s}(Y), Z)g(A^{N_s}(X), W) \\ & + g(A^{N_t}(Y), Z)g(A^{N_t}(X), W) \\ & + g(A^{N_s}(X), Z)g(A^{N_s}(Y), W) - g(A^{N_t}(X), Z)g(A^{N_t}(Y), W), \end{aligned} \tag{4.1}$$

where $R(X,Y)Z$ is the curvature tensor of M^{2n}.

Then, for an orthonormal basis $\{e_1, Je_1, \ldots, e_n, Je_n\}$, the Ricci tensor $Ric(X,Y)$ of M^{2n} is given as

$$\begin{aligned} Ric(X,Y) = & 2(TrA^{N_s})g(A^{N_s}(X), Y) \\ & - 2(TrA^{JN_s})g(A^{JN_s}(X), Y) - 2g(A^{N_s}(X), A^{N_s}(Y)), \\ & TrA^{N_s} := \sum_{i=1}^{n} g(A^{N_s}(e_i), e_i), \quad TrA^{N_t} := \sum_{i=1}^{n} g(A^{N_t}(e_i), e_i). \end{aligned} \tag{4.2}$$

Now, assume that the shape operator A^{N_s} satisfies the following condition $(*)_\pm$

$$A^{N_s} = \lambda (I \pm J) \;\ldots\ldots\; (*)_\pm$$

where we use the identification $\sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i Je_i \longleftrightarrow {}^t(a_1, \cdots, a_n, b_1, \cdots, b_n)$ and I is the identity matrix of order $2n$.

Remark. This condition is inspired by (3.1). In fact, (3.1) is rewritten as

$$A^{\zeta} = -\frac{1}{2}\left(I - \sqrt{-1}J\right), \quad A^{\zeta} + \overline{A^{\zeta}} = -I = S.$$

Theorem . *Let M^{2n} be a simply connected, complete invariant anti-Kähler hypersurface in $\mathbb{C}^{n+1}$. If the shape operator of M^{2n} satisfies $(*)_{\pm}$, then M^{2n} is isometric to a complex sphere $CS^n\left(\pm\sqrt{-1}/(4\lambda^2)\right)$.*

Proof. By substituting $(*)_{\pm}$ into (4.2), it follows that

$$QX = \pm 4(n-1)\lambda^2 JX, \tag{4.3}$$

where Q is the Ricci operator of M^{2n}. From this equation, we can see that λ is a constant, and the scalar curvature of M^{2n} is zero.

From (4.1) and $(*)_{\pm}$, we have

$$\begin{aligned}
&g(R(X,Y)Z,W) - \frac{1}{2n-2}(g(Y,Z)g(QX,W) - g(QX,Z)g(Y,W) \\
&\quad + g(QY,Z)g(X,W) - g(X,Z)g(QY,W)) \\
&= -g(A^{N_s}(X),Z)g(A^{N_s}(Y),W) + g(A^{JN_s}(X),Z)g(A^{JN_s}(Y),W) \\
&\quad + g(A^{N_s}(Y),Z)g(A^{N_s}(X),W) - g(A^{JN_s}(Y),Z)g(A^{JN_s}(X),W) \\
&\quad \pm \frac{1}{2n-2}4(n-1)\lambda^2(g(Y,Z)g(JX,W) \\
&\quad - g(JX,Z)g(Y,W) + g(JY,Z)g(X,W) - g(X,Z)g(JY,W)) \\
&= \left(\mp 2\lambda^2 \pm 2\lambda^2\right)(g(X,Z)g(JY,W) + g(JX,Z)g(Y,W) \\
&\quad - g(Y,Z)g(JX,W) - g(JY,Z)g(X,W)) = 0
\end{aligned}$$

by virtue of (4.3). Therefore M^{2n} is conformally flat. Again from (4.3), we see that M^{2n} is locally symmetric. Applying Theorem 7.8 of Chapter VI of [3], we conclude that M^{2n} is isometric to a complex sphere $CS^n\left(\pm\sqrt{-1}/(4\lambda^2)\right)$. This completes the proof. □

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