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AN INJECTIVE CPS-TRANSLATION FOR THE EXTENSIONAL λ-CALCULUS

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ABSTRACT. We give a syntactical proof to the statement that a novel CPS-translation with surjective pairing is injective for the extensional λ-calculus. The result itself might be preliminary, since the source language (the extensional λ-calculus) of the translation is a sublanguage of the target language (λ-calculus with surjective pairing). However this paper shows that there exists a nontrivial injection from the extensional λ-calculus into the λ-calculus with surjective pairing. In this sense our result can be regarded as an extension of Plotkin, i.e., a call-by-value simulation of call-by-name λ-calculus with η-rule (extensionality). Moreover, the method presented here can be naturally extended to the case of the extensional $\lambda\mu$-calculus which is defined from the extensional λ-calculus together with control operators.

1. INTRODUCTION

Parigot [13, 14] introduced the $\lambda\mu$-calculus from the viewpoint of classical logic, and established an extension of the Curry-Howard isomorphism [10, 7, 12]. From the motivation of a universally computational point of view, we investigate type free $\lambda\mu$-calculus [2, 5].

In terms of a category of continuations, it is proved that for any $\lambda\mu$-theory a continuation semantics of $\lambda\mu$-calculus is sound and complete by Hofmann and Streicher [9]. Selinger [16] proposed the control category to establish an isomorphism between call-by-name and call-by-value $\lambda\mu$-calculi with conjunction and disjunction types. In Streicher and Reus [17], the category of negated domains is applied for a model of type free $\lambda\mu$-calculus. They remarked that the traditional CPS-translation[1] naïvely based on Plotkin [15] cannot validate η-rule. All of the work [9, 16, 17] introduced a novel CPS-translation which requires, at least, products as a primitive notion, so that η-rule can be validated by the use of surjective pairing, as observed in [4].

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[1] CPS stands for *continuation-passing style.*

Towards a model theoretical investigation of type free $\lambda\mu$-calculus, as a preliminary step we show that the novel CPS-translation with surjective pairing is injective. It is syntactically proved that the CPS-translation is sound and complete for the extensional λ-calculus. Here the extensionality means that the λ-calculus has not only β-rule but also η-rule.

As a corollary the injective CPS-translation reveals a Church-Rosser fragment of the λ-calculus with surjective pairing, which is not Church-Rosser as proved by Klop [1]. Along the line of Plotkin [15], this work can also be regarded as a call-by-value simulation of call-by-name λ-calculus with η-rule (extensionality). It is remarked that the completeness in [15] has been proved by the essential use of the Church-Rosser property of the target calculus (call-by-value λ-calculus). However our target calculus is not Church-Rosser as stated above. In order to define an inverse translation and prove the completeness, we introduce a context-free grammar which describes the image of the CPS-translation.

Although this paper handles only type free λ-calculus, our main theorem is still valid under typed λ-calculus. Moreover, the syntactical method presented in this paper can be naturally extended to the case of the extensional $\lambda\mu$-calculus [13, 14] which is defined from the extensional λ-calculus together with control operators.

2. CPS-Translation of λ-calculus into λ-calculus with surjective pairing

We show a preliminary result that the novel CPS-translation is sound and complete for the extensional λ-calculus. The definitions of terms and reduction rules are respectively given to the extensional λ-calculus denoted by Λ and the extensional λ-calculus with surjective pairing denoted by $\Lambda^{\langle\rangle}$.

Definition 1 (λ-calculus Λ).

$$\Lambda \ni M \quad ::= \quad x \mid \lambda x.M \mid MM$$

(β): $(\lambda x.M_1)M_2 \to M_1[x := M_2]$
(η): $\lambda x.Mx \to M$ *if* $x \notin FV(M)$

Definition 2 (λ-calculus with surjective pairing $\Lambda^{\langle\rangle}$).

$$\Lambda^{\langle\rangle} \ni M \quad ::= \quad x \mid \lambda x.M \mid MM \mid \langle M, M\rangle \mid \pi_1(M) \mid \pi_2(M)$$

(β): $(\lambda x.M_1)M_2 \to M_1[x := M_2]$
(η): $\lambda x.Mx \to M$ *if* $x \notin FV(M)$
(π): $\pi_i\langle M_1, M_2\rangle \to M_i$ $(i = 1, 2)$
(sp): $\langle \pi_1(M), \pi_2(M)\rangle \to M$

The term $M_1[x := M_2]$ denotes the result of substituting M_2 for the free occurrences of x in M_1. $FV(M)$ stands for the set of free variables in M. The one step reduction relation is denoted by $\to_R$ where R consists of (β), (η), $(\beta) + (\eta)$, $\lambda^{\langle\rangle}(= (\beta) + (\eta) + (\pi) + (\text{sp}))$, etc. We write $\to_R^+$ and $\to_R^*$ to denote the transitive closure and the reflexive and transitive closure of $\to_R$, respectively. We employ the notation $=_R$ to indicate the symmetric, reflexive and transitive closure of $\to_R$. The binary relation $\equiv$ denotes the syntactic identity under renaming of bound variables.

It is noted that the rule of (η) implies the extensional equality of functions, i.e., $f =_{\beta\eta} g$ if $fM =_{\beta\eta} gM$ for any $M \in \Lambda$. Suppse $fM =_{\beta\eta} gM$ for any $M \in \Lambda$. Then we have $fx =_{\beta\eta} gx$ for a fresh variable x, and we also have $\lambda x.fx =_{\beta\eta} \lambda x.gx$. Hence, an application of (η) gives $f =_{\beta\eta} g$.

For a CPS-translation, we assume that $\Lambda^{\langle\rangle}$ has two kinds of variables denoted by x and a.

Definition 3 (CPS-translation : $\Lambda \to \Lambda^{\langle\rangle}$)**.**

(i): $[\![x]\!] = x$
(ii): $[\![\lambda x.M]\!] = \lambda a.(\lambda x.[\![M]\!])(\pi_1 a)(\pi_2 a)$
(iii): $[\![M_1 M_2]\!] = \lambda a.[\![M_1]\!]\langle[\![M_2]\!], a\rangle$

Example 1. *It is instructive to calculate the following where $m, n \geq 0$:*

$$[\![\lambda x_1 \dots x_m.xM_1 \cdots M_n]\!] \;\to_\beta^+\; \lambda a.x\langle[\![M_1]\!], \dots, \langle[\![M_n]\!], \pi_2^m a\rangle \dots\rangle$$
$$[x_1 := \pi_1 a, x_2 := \pi_1(\pi_2 a), \dots, x_m := \pi_1(\pi_2^{m-1} a)]$$

Proposition 1 (Soundness)**.** *Let $M_1, M_2 \in \Lambda$. If we have $M_1 \to_{\beta\eta} M_2$ then $[\![M_1]\!] \to_{\lambda\langle\rangle}^+ [\![M_2]\!]$.*

Proof. By induction on the derivation of $M_1 \to_{\beta\eta} M_2$. We show some of the base cases.

Case of: (β)

$$\begin{aligned}
[\![(\lambda x.M_1)M_2]\!] &= \lambda a.[\![\lambda x.M_1]\!]\langle[\![M_2]\!], a\rangle \\
&= \lambda a.(\lambda a'.(\lambda x.[\![M_1]\!])(\pi_1 a')(\pi_2 a'))\langle[\![M_2]\!], a\rangle \\
&\to_\beta \lambda a.(\lambda x.[\![M_1]\!])(\pi_1\langle[\![M_2]\!], a\rangle)(\pi_2\langle[\![M_2]\!], a\rangle) \\
&\to_\pi^+ \lambda a.(\lambda x.[\![M_1]\!])[\![M_2]\!]a \\
&\to_\beta \lambda a.[\![M_1]\!][x := [\![M_2]\!]]a \;=\; \lambda a.[\![M_1[x := M_2]]\!]a \\
&\to_\eta [\![M_1[x := M_2]]\!]
\end{aligned}$$

Case of: (η) where $x \notin FV(M)$

$$\begin{aligned}
[\![\lambda x.Mx]\!] &= \lambda a.(\lambda x.[\![Mx]\!])(\pi_1 a)(\pi_2 a) \\
&= \lambda a.(\lambda x.\lambda a'.[\![M]\!]\langle x, a'\rangle)(\pi_1 a)(\pi_2 a) \\
&\to_\beta^+ \lambda a.[\![M]\!]\langle\pi_1 a, \pi_2 a\rangle \\
&\to_{\mathrm{sp}} \lambda a.[\![M]\!]a \\
&\to_\eta [\![M]\!]
\end{aligned}$$

□

It is remarked that Proposition 1 holds true even under the restricted form V, i.e., the call-by-value computation as follows:

$$V \;::=\; x \mid \lambda x.M \mid \langle V, V\rangle \mid \pi_1(V) \mid \pi_2(V)$$

(β_v)**:** $(\lambda x.M)V \to M[x := V]$

(η_v): $\lambda x.Vx \to V$
(π_v): $\pi_i\langle V_1, V_2\rangle \to V_i$ $(i = 1, 2)$
(sp_v): $\langle \pi_1 V, \pi_2 V\rangle \to V$

Hence this work can be regarded as a call-by-value simulation of call-by-name λ-calculus with η-rule.

2.1. **Universe of the translation.** We will give a definition of the inverse translation to each element of the universe of the CPS-translation:

$$\mathit{Univ}_\lambda \stackrel{\mathrm{def}}{=} \{P \in \Lambda^{\langle\rangle} \mid [\![M]\!] \to^*_{\lambda^{\langle\rangle}} P \text{ for some } M \in \Lambda\}$$

Every element in the universe will be generated by the following context-free grammar:

$$\begin{array}{rcl} \mathcal{R} & ::= & x \mid \pi_1\mathcal{K} \mid (\lambda x.\mathcal{R})\mathcal{R} \mid \lambda a.\mathcal{R}\mathcal{K} \\ \mathcal{K} & ::= & a \mid \pi_2\mathcal{K} \mid \langle \mathcal{R}, \mathcal{K}\rangle \end{array}$$

Lemma 1 (Subject reduction property)**.** *The categories $\mathcal{R}$ and $\mathcal{K}$ are closed under the following reductions:*

(β_x): $(\lambda x.R_1)R_2 \to R_1[x := R_2]$
(β_a): $(\lambda a.RK_1)K_2 \to RK_1[a := K_2]$
(η_a): $\lambda a.Ra \to R$ *if* $a \notin FV(R)$
($\pi_{R,K}$): $\pi_1\langle R, K\rangle \to R$ *and* $\pi_2\langle R, K\rangle \to K$
(sp_K): $\langle \pi_1(K), \pi_2(K)\rangle \to K$

Proof. Because we have that $R_1[x := R_2] \in \mathcal{R}, K[x := R] \in \mathcal{K}$ by simultaneous induction on the structures of R_1 and K; and that $R[a := K] \in \mathcal{R}, K_1[a := K_2] \in \mathcal{K}$ similarly. □

Proposition 2. *$\mathit{Univ}_\lambda \subseteq \mathcal{R}$, i.e., Univ_λ is generated by $\mathcal{R}$.*

Proof. From definition 3, we have $[\![M]\!] \in \mathcal{R}$ for any $M \in \Lambda$. Moreover, from Lemma 1, $\mathcal{R}$ and $\mathcal{K}$ are closed under the reductions, and hence $\mathit{Univ}_\lambda \subseteq \mathcal{R}$ is obtained. □

There uniquely exists a projection normal form by the sole use of $(\pi_{R,K})$, and the projection normal form of K is in the following form K_{nf}:

$$K_{nf} ::= \pi_2^n a \mid \langle R_{nf}, K_{nf}\rangle$$

where $n \geq 0$. For a technical reason, an occurrence of a single variable $a \in \mathcal{K}$, i.e., $\pi_2^n a$ where $n = 0$ is handled as an (sp_K)-expansion form; $\langle \pi_1 a, \pi_2 a\rangle$. Under this consideration, K can be supposed to be in the form of $\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle$ with $m \geq 0, n \geq 1$.

Lemma 2 (π-normal form)**.** *Let $m \geq 0$ and $n \geq 1$. Then every element in the universe Univ_λ is one of the following forms up to $(\pi_{R,K})$-reductions and (sp_K)-expansions:*

(1): x
(2): $\pi_1(\pi_2^i a)$ *for some* $i \geq 0$
(3): $(\lambda x.R)R_1$

(4): $\lambda a.R\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle$ *for some* $m \geq 0$ *and* $n \geq 1$

where R *and* R_i $(1 \leq i \leq m)$ *are in the form of* (1), (2), (3), *or* (4) *above.*
We call the occurrence of $(\pi_2^n a)$ *in the case of* (4) *above a tail with the variable* a.
Moreover, the following property is satisfied under renaming of bound variables:

(i): *For each* λa, *there exists a unique occurrence of the corresponding tail* $\pi_2^n a$ *for some* $n \geq 1$;

(ii): *If we have* $\pi_1(\pi_2^i a)$ *as a proper subterm where* $i \geq 0$, *then there exists the least* $n \geq 1$ *such that the condition* $i + 1 \leq n$ *holds for the tail* $\pi_2^n a$ *with the variable* a.

Proof. First obtain a projection normal form only by the use of $(\pi_{R,K})$-reductions, and then check whether the form has the condition (ii). The application of (sp_K)-expansion guarantees that (ii) holds true. □

2.2. **Inverse translation.** π-normal forms above play a role of representatives of the image $Univ_\lambda$ under the translation $[\![-]\!]$. We give the definition of the inverse translation $\natural$ to every element in

$$\pi\text{-nf}(Univ_\lambda) \stackrel{\text{def}}{=} \{\pi\text{-normal}(P) \in \Lambda^{\langle\rangle} \mid [\![M]\!] \to^*_{\lambda^{\langle\rangle}} P \text{ for some } M \in \Lambda\}.$$

That is, $P^\natural = (\pi\text{-normal}(P))^\natural$ for any $P \in Univ_\lambda$.

Definition 4 (Inverse translation $\natural : \pi\text{-nf}(Univ_\lambda) \to \Lambda$)**.**

(1): $x^\natural = x$
(2): $(\pi_1(\pi_2^i a))^\natural = a_{i+1}$ $(i \geq 0)$
(3): $((\lambda x.R)R_1)^\natural = (\lambda x.R^\natural)R_1^\natural$
(4): $(\lambda a.R\langle R_1, \ldots \langle R_m, \pi_2^n a\rangle \ldots\rangle)^\natural = \lambda a_1 \ldots a_n.R^\natural R_1^\natural \cdots R_m^\natural$ $(m \geq 0, n \geq 1)$

Lemma 3. *For any* $M \in \Lambda$, *we have* $[\![M]\!]^\natural \to^*_\eta M$.

Proof. By induction on the structure of $M \in \Lambda$. We show some of the base cases.

(i):

$$\begin{array}{rcl} [\![\lambda x.M]\!]^\natural & = & (\lambda a.(\lambda x.[\![M]\!])(\pi_1 a)(\pi_2 a))^\natural \\ & = & \lambda a_1.(\lambda x.[\![M]\!]^\natural)a_1 \\ & \to_\eta & \lambda x.[\![M]\!]^\natural \\ & \to^*_\eta & \lambda x.M \quad \text{by the induction hypothesis.} \end{array}$$

(ii):

$$\begin{array}{rcl} [\![M_1 M_2]\!]^\natural & = & (\lambda a.[\![M_1]\!]\langle [\![M_2]\!], a\rangle)^\natural \\ & = & (\lambda a.[\![M_1]\!]\langle [\![M_2]\!], \langle \pi_1 a, \pi_2 a\rangle\rangle)^\natural \\ & = & \lambda a_1.[\![M]\!]^\natural [\![M_2]\!]^\natural a_1 \\ & \to_\eta & [\![M_1]\!]^\natural [\![M_2]\!]^\natural \\ & \to^*_\eta & M_1 M_2 \quad \text{by the induction hypotheses.} \end{array}$$

□

For the variables a_i $(i \geq 1)$, the CPS-translation is naturally extended as follows:

Definition 5.

$$\begin{array}{rcl} [\![a_i]\!] & = & a_i \ (i \geq 1) \\ [\![\lambda a_i.M]\!] & = & \lambda a.(\lambda a_i.[\![M]\!])(\pi_1 a)(\pi_2 a) \end{array}$$

Lemma 4. *For any $P \in Univ_\lambda$, we have that $[\![P^\natural]\!]\theta \to^*_{\lambda\langle\rangle} P$ where $\theta = [a_1 := \pi_1 a, a_2 := \pi_1(\pi_2 a), \ldots, a_{i+1} := \pi_1(\pi_2^i a), \ldots]$.*

Proof. By induction on the structure of $P \in Univ_\lambda$. We show some of the cases.

Case: P of $(\lambda x.R)R_1$

$$\begin{array}{rcl} [\![((\lambda x.R)R_1)^\natural]\!] & = & \lambda a.(\lambda a'.(\lambda x.[\![R^\natural]\!])(\pi_1 a')(\pi_2 a'))\langle [\![R_1^\natural]\!], a\rangle \\ & \to^+_{\lambda\langle\rangle} & (\lambda x.[\![R^\natural]\!])[\![R_1^\natural]\!] \end{array}$$

Then from the induction hypotheses we have the desired property:

$$\begin{array}{rcl} [\![((\lambda x.R)R_1)^\natural]\!]\theta & \to^+_{\lambda\langle\rangle} & (\lambda x.([\![R^\natural]\!]\theta))([\![R_1^\natural]\!]\theta) \\ & \to^*_{\lambda\langle\rangle} & (\lambda x.R)R_1 \end{array}$$

Case: P of $\lambda a.R\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle$

$$\begin{array}{rcl} [\![\lambda a.R\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle]\!] & = & [\![\lambda a_1 \ldots a_n.R^\natural R_1^\natural \ldots R_m^\natural]\!] \\ & \to^+_\beta & \lambda a.[\![R^\natural]\!]\langle [\![R_1^\natural]\!], \ldots, \langle [\![R_m^\natural]\!], \pi_2^n a\rangle \ldots\rangle \\ & & \quad [a_1 := \pi_1 a, a_2 := \pi_1(\pi_2 a), \ldots, a_n := \pi_1(\pi_2^{n-1} a)] \end{array}$$

Now the use of the induction hypotheses gives what we need:

$$\begin{array}{rcl} [\![\lambda a.R\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle]\!]\theta & \to^+_\beta & \lambda a.([\![R^\natural]\!]\theta)\langle([\![R_1^\natural]\!]\theta), \ldots, \langle([\![R_m^\natural]\!]\theta), \pi_2^n a\rangle \ldots\rangle \\ & \to^*_{\lambda\langle\rangle} & \lambda a.R\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle \end{array}$$

□

Lemma 5. *Let $R, R_1, \ldots, R_n \in \mathcal{R}$.*

(1): $R^\natural[x := R_1^\natural] = (R[x := R_1])^\natural$

(2): $(R[b := \langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle])^\natural$
$= R^\natural[b_1 := R_1^\natural, \ldots, b_m := R_m^\natural, b_{m+1} := a_{n+1}, b_{m+2} := a_{n+2}, \cdots]$
under the simultaneous substitution where $m \geq 0$ and $n \geq 1$.

Proof. By straightforward induction on the structure of R. We show the base case for (2).

Case of: $i + 1 \leq m$

$$\begin{array}{rcl} (\pi_1(\pi_2^i b)[b := \langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle])^\natural & = & (\pi_1(\pi_2^i\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle))^\natural \\ & = & R_{i+1}^\natural = (\pi_1(\pi_2^i b))^\natural[b_{i+1} := R_{i+1}^\natural] \end{array}$$

Case of: $i+1>m$

$$
\begin{aligned}
(\pi_1(\pi_2^i b)[b := \langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle])^\natural &= (\pi_1(\pi_2^i\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle))^\natural \\
&= (\pi_1(\pi_2^{n+i-m} a))^\natural = a_{n+i-m+1} \\
&= (\pi_1(\pi_2^i b))^\natural[b_{m+1} := a_{n+1}, \ldots] \\
&= b_{i+1}[b_{m+1} := a_{n+1}, \ldots] = a_{i+1-m+n}
\end{aligned}
$$

□

Proposition 3 (Completeness)**.** *Let $P, Q \in Univ_\lambda$.*

(1): *If $P \to_{\beta_x} Q$ then $P^\natural \to_\beta Q^\natural$.*
(2): *If $P \to_{\beta_a} Q$ then $P^\natural \to_\beta^+ Q^\natural$.*
(3): *If $P \to_{\eta_a} Q$ then $P^\natural \to_\eta Q^\natural$.*
(4): *If $P \to_{\pi_{R,K}} Q$ then $P^\natural \equiv Q^\natural$.*
(5): *If $P \to_{\mathrm{sp}_K} Q$ then $P^\natural \to_\eta^* Q^\natural$.*

Proof. By induction on the derivations. We show one case for (2), and other cases are straightforward.

Let K be $\langle S_1, \ldots, \langle S_q, \pi_2^p b\rangle \ldots\rangle$ with $q \geq 0, p \geq 1$, and K' be $\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle$ with $m \geq 0, n \geq 1$. Let θ be $[b := K']$. Now we prove the case P of $\lambda a.(\lambda b.RK)K'$:

$$(\lambda a.(\lambda b.RK)K')^\natural = \lambda a_1 \ldots a_n.(\lambda b_1 \ldots b_p.R^\natural S_1^\natural \cdots S_q^\natural)R_1^\natural \cdots R_m^\natural$$

Case of: $p+1 \leq m$

$$
\begin{aligned}
(\lambda a.(\lambda b.RK)K')^\natural &\to_\beta^+ \lambda a_1 \ldots a_n.(R^\natural S_1^\natural \cdots S_q^\natural)[b_1 := R_1^\natural, \ldots, b_p := R_p^\natural]R_{p+1}^\natural \cdots R_m^\natural \\
&= \lambda a_1 \ldots a_n.(R^\natural S_1^\natural \cdots S_q^\natural) \\
&\quad [b_1 := R_1^\natural, \ldots, b_m := R_m^\natural, b_{m+1} := a_{m+1}, b_{m+2} := a_{m+2}, \ldots]R_{p+1}^\natural \cdots R_m^\natural \\
&\qquad \text{since none of } b_{p+1}, b_{p+2}, \ldots \text{ appears in } R^\natural, S_1^\natural, \ldots, S_q^\natural \\
&= \lambda a_1 \ldots a_n.((R\theta)^\natural (S_1\theta)^\natural \cdots (S_q\theta)^\natural)R_{p+1}^\natural \cdots R_m^\natural \quad \text{by Lemma 5} \\
&= (\lambda a.R\theta\langle S_1\theta, \ldots, \langle S_q\theta, \langle R_{p+1}, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle\rangle \ldots\rangle)^\natural \\
&= (\lambda a.R\theta\langle S_1\theta, \ldots, \langle S_q\theta, \pi_2^p\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle\rangle \ldots\rangle)^\natural \\
&= (\lambda a.RK[b := K'])^\natural
\end{aligned}
$$

Case of: $p+1 > m$

$$
\begin{aligned}
(\lambda a.(\lambda b.RK)K')^\natural &\to_\beta^+ \lambda a_1 \ldots a_n.\lambda b_{m+1} \ldots b_p.(R^\natural S_1^\natural \cdots S_q^\natural)[b_1 := R_1^\natural, \ldots, b_m := R_m^\natural] \\
&= \lambda a_1 \ldots a_n a_{n+1} \ldots a_{p-m+n}.R^\natural S_1^\natural \cdots S_q^\natural \\
&\qquad [b_1 := R_1^\natural, \ldots, b_m := R_m^\natural, b_{m+1} := a_{n+1}, b_{m+2} := a_{n+2}, \ldots] \\
&= \lambda a_1 \ldots a_{n+p-m}.(R\theta)^\natural (S_1\theta)^\natural \cdots (S_q\theta)^\natural \quad \text{by Lemma 5} \\
&= (\lambda a.R\theta\langle S_1\theta, \ldots, \langle S_q\theta, \pi_2^{n+p-m}(a)\rangle \ldots\rangle)^\natural \\
&= (\lambda a.R\theta\langle S_1\theta, \ldots, \langle S_q\theta, \pi_2^p\langle R_1, \ldots, \langle R_m, \pi_2^n a\rangle \ldots\rangle\rangle \ldots\rangle)^\natural \\
&= (\lambda a.RK[b := K'])^\natural
\end{aligned}
$$

□

Now we can establish our main theorem (equational correspondence between Λ and $Univ_\lambda \subseteq \Lambda^{\langle\rangle}$).

Theorem 1. **(i):** *Let $M_1, M_2 \in \Lambda$. $M_1 =_{\beta\eta} M_2$ if and only if $[\![M_1]\!] =_{\lambda\langle\rangle} [\![M_2]\!]$.*
(ii): *Let $P_1, P_2 \in Univ_\lambda$. $P_1 =_{\lambda\langle\rangle} P_2$ if and only if $P_1^\natural =_{\beta\eta} P_2^\natural$.*

Proof. **(i):** From Propositions 1 and 3 and Lemma 3.
(ii): From Propositions 1 and 3 and Lemma 4.

□

Let $[M] \stackrel{\text{def}}{=} \{N \in \Lambda \mid [\![M]\!] =_{\lambda\langle\rangle} [\![N]\!]\}$ for $M \in \Lambda$. The theorem above means that we have $N_1 =_{\beta\eta} N_2$ for any $N_1, N_2 \in [M]$.

Corollary 1. *Let $[\![\Lambda]\!] \stackrel{\text{def}}{=} \{[\![M]\!] \in \Lambda^{\langle\rangle} \mid M \in \Lambda\}$. $[\![\Lambda]\!]$ is a Church-Rosser subset, in the sense that if $P \to^*_{\lambda\langle\rangle} P_1$ and $P \to^*_{\lambda\langle\rangle} P_2$ where $P, P_1, P_2 \in [\![\Lambda]\!]$ then there exists some $Q \in [\![\Lambda]\!]$ such that $P_1 \to^*_{\lambda\langle\rangle} Q$ and $P_2 \to^*_{\lambda\langle\rangle} Q$.*

3. Concluding Remarks

(1): $\Lambda^{\langle\rangle}$ is not Church-Rosser by Klop [1].
As stated in Corollary 1, Theorem 1 reveals that

$$[\![\Lambda]\!] \stackrel{\text{def}}{=} \{[\![M]\!] \in \Lambda^{\langle\rangle} \mid M \in \Lambda\}$$

is a confluent fragment, in the sense that if $P_1 =_{\lambda\langle\rangle} P_2$ for $P_1, P_2 \in [\![\Lambda]\!]$ then there exists some $P \in [\![\Lambda]\!]$ such that $P_1 \to^*_{\lambda\langle\rangle} P$ and $P_2 \to^*_{\lambda\langle\rangle} P$.

(2): There exists a one-to-one correspondence between $\Lambda^{\langle\rangle}$ and C-monoids[2] by Lambek and Scott [11] and by Curien [3]:
(C_{Ass}): $(x \circ y) \circ z = x \circ (y \circ z)$
(C_{Idl}): $1 \circ x = x$
(C_{Idr}): $x \circ 1 = x$
(C_{Fst}): $\pi_1 \circ \langle x, y\rangle = x$
(C_{Snd}): $\pi_2 \circ \langle x, y\rangle = y$
(C_{SP}): $\langle \pi_1 \circ x, \pi_2 \circ x\rangle = x$
(C_{App}): $\texttt{App} \circ \langle \texttt{Cur}(x) \circ \pi_1, \pi_2\rangle = x$
($C_{\text{S}\Lambda}$): $\texttt{Cur}(\texttt{App} \circ \langle x \circ \pi_1, \pi_2\rangle) = x$
Theorem 1 implies that there also exists a nontrivial injection from Λ into C-monoids.

(3): Even in the case of typed λ-calculus, the novel CPS-translation works well as a negative translation from proofs of intuitionistic logic consisting of $\Rightarrow$ (implication) into those of that consisting of $\Rightarrow$ and $\wedge$ (conjunction). $\neg A$ (negation of A) is defined by $A \Rightarrow \bot$ where $\bot$ is treated as an arbitrary proposition letter. From the Curry-Howard isomorphism [10], formulae are regarded as types and proofs are as terms or programs. Here the judgement $\Gamma \vdash M : A$ says that M is a proof of the formula A under the set of

[2] According to [11], C stands for Curry, Church, combinatory or cartesian.

assumptions Γ. The inference rules of typed $\Lambda^{\langle\rangle}$ is given as follows, and those of typed Λ is defined by typed $\Lambda^{\langle\rangle}$ without $(\wedge I)$ nor $(\wedge E)$:

$$\frac{x\!:\!A \in \Gamma}{\Gamma \vdash x : A}$$

$$\frac{\Gamma, x\!:\!A_1 \vdash M : A_2}{\Gamma \vdash \lambda x.M : A_1 \Rightarrow A_2}\ (\Rightarrow I) \qquad \frac{\Gamma \vdash M_1 : A_1 \Rightarrow A_2 \quad \Gamma \vdash M_2 : A_1}{\Gamma \vdash M_1 M_2 : A_2}\ (\Rightarrow E)$$

$$\frac{\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \langle M_1, M_2\rangle : A_1 \wedge A_2}\ (\wedge I) \qquad \frac{\Gamma \vdash M : A_1 \wedge A_2}{\Gamma \vdash \pi_i(M) : A_i}\ (\wedge E)$$

Proposition 4. $\Gamma \vdash M : A$ *in typed* Λ *if and only if* $\Gamma^k \vdash [\![M]\!] : A^k$ *in typed* $\Lambda^{\langle\rangle}$*, where formulae (types) are embedded as follows:*

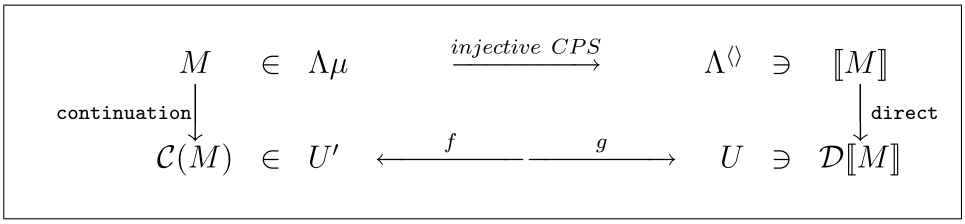
$$\begin{cases} (A_1 \Rightarrow A_2)^k &= \neg(A_1^k \wedge A_2^*); \\ A^k &= \neg\neg A \quad \text{if } A \text{ is atomic; and} \\ A^* &= B \quad \text{where } \neg B \equiv A^k. \end{cases}$$

We remark that the embedding A^k is essentially equivalent to the Gödel-Gentzen negative translation, since we have $\neg(A_1^k \wedge A_2^*) \Leftrightarrow (A_1^k \Rightarrow A_2^k)$ in the so-called minimal logic. This observation can be applied to prove the only-if part of the proposition above.

(4): A recursive domain safisfying

$$U \cong U \times U \cong [U \to U]$$

gives a model of the λ-calculus with surjective pairing [8]. The domain $U \cong U \times U \cong [U \to U]$ can provide continuation denotational semantics of the extensional $\lambda\mu$-calculus as well. From a natural extension of Theorem 1 the completeness of the continuation denotational semantics of the $\lambda\mu$-calculus depends on that of the direct denotational semantics of $\Lambda^{\langle\rangle}$. See also [6] for a formal relation, via continuous functions f and g, between the continuation denotational semantics $\mathcal{C}(-)$ of the $\lambda\mu$-calculus and the CPS-translation followed by the direct denotational semantics $\mathcal{D}(-)$ of $\Lambda^{\langle\rangle}$:



where $U' = [U \times U \to U]$.

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