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NOTE ON HOLOMORPHIC FOLIATIONS WITH TURBULENCES ON ELLIPTIC SURFACES

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0. Introduction

Let M be a complex projective manifold. We say that M has a foliation by curves if there exists a line bundle $\mathcal{L}$ on M and a non-zero homomorphism $i : \mathcal{L} \to T_M$, where T_M is a tangent bundle of M. If the above homomorphism $\mathcal{L} \to T_M$ is injective, then we say that a foliation is nonsingular. Let L_α be a 1-dimensional connected manifold. Then we say that L_α is a leaf of foliation $i : \mathcal{L} \to T_M$ if $M = \cup_\alpha L_\alpha$, $L_\alpha \cap L_\beta = \emptyset$ for $\alpha \neq \beta$, and for $x \in L_\alpha$ $i(\mathcal{L})_x$ is a tangent bundle of L_α at x. In this paper we consider the case in which M is a projective surface. We use a notation S instead of M.

If S is a ruled surface, that is, there exists a surjective morphism with connected fibers $\pi : S \to C$ such that any general fiber of π is $\mathbb{P}^1$, where C is a smooth projective curve, then the foliations by curves on S have been studied by Gómez-Mont ([G-M II]). Here we consider the case in which there exists a surjective morphism $\pi : S \to C$ with connected fibers such that any general fiber is an elliptic curve. We call this surface an elliptic surface over a smooth projective curve. Here we note that elliptic surfaces may have singular fibers and all types of singular fibers have been classified by Kodaira.

This paper consists of the following three parts;

(1) examples of special type of foliations on ellitpic surfaces,
(2) a family of foliations on elliptic surfaces,
(3) the existence of elliptic surfaces which have foliations.

In [B], Brunella obtained some interesting results for foliations without singularities on non-singular algebraic surfaces, and pointed out that a turbulent foliation can appear (un feuilletage tourbillonné). Here we mean by a foliation a holomorphic one, and discuss foliations on elliptic surfaces.

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Definition. A foliation $\mathcal{F}$ possibly with singularities is said to be *turbulent* if it has the following properties;

(1) it has a finite number of fibers which are $\mathcal{F}$-invariant (see [B]),
(2) other fibers are $\mathcal{F}$-transversal, that is, transversal to each leaves except for the fibers which appear in (1).

Let $\pi : S \to C$ be a basic elliptic surface on a non-singular curve (see [KoII] or Definition 3.15 of Chapter I in [FM]) and so it admits a section, and has no multiple fibers. We denote by $T_{S/C}$ the relative tangent bundle, T_S and T_C are tangent bundles of S and C respectively. Then there exists an exact sequence of sheaves,

$$0 \to T_{S/C} \to T_S \to \pi^* T_C \otimes I_X \to 0,$$

where I_X is the ideal sheaf of singular points of the singular fibers (c.f. Lemma 3.2 of Chapter 4 in [FM] and (6.2) in [FMW]). Now we consider the family of elliptic K3 surfaces over the projective line in the section 5 of [KoI], and denote by $\pi : S \to \mathbb{P}^1$ the projection for each member. Let L be the sheaf $(R^1\pi_*(\mathcal{O}_S))^{-1}$, which is a line bundle over $\mathbb{P}^1$. Then the induced bundle $\pi^*(L)$ is just the bundle $(T_{S/C})^{-1}$ (c.f. Chapter I in [FM] and 6.2 in [FMW]).

Then we have a monomorphism

$$\mathcal{O}_S \to \pi^* L \otimes T_S,$$

and a nontrivial section $S \to \pi^* L \otimes T_S$.

By making use of this fact, in Section 1 we prove

Theorem 1. *Let $\pi : S \to \mathbb{P}^1$ be a basic elliptic K3 surface over the projective line $\mathbb{P}^1$* ([Ko1])*. Then there exists a turbulent foliation on the surface S.*

Furthermore by making use of the facts in V.5 in [BPV] and [A], we prove

Theorem 2. *Let E be an elliptic curve and $\pi : S \to C$ be a principal E-bundle with trivial Chern class over a smooth curve C. Then there exists a turbulent foliation on the elliptic surface S.*

Next as in [G-MI] and [G-MII] we consider a family of foliations on an elliptic surface. Let α denote Chern class $c_1(\mathcal{L})$, and D_α the space of foliations whose Chern class of the tangent bundle is α. Then we have

Theorem 3. *Let $S \to C$ be an elliptic K3 surface in* [KoI]*. Then the space D_α is the complex projective $t-1$ space where $t \geq 14$.*

In Section 8 of [KoII], each basic member belongs to a meromorphic function $\mathcal{J} : C \to \mathbb{C}$ and the homological invariant G which is the sheaf belonging to the meromorphic function $\mathcal{J}$. Then we have

Theorem 4. *Let C be a non-singular algebraic curve. Let j be a natural number with $j \geq 2g(C) + 1$. Then there exists a meromorphic function $\mathcal{J}$ on C such that for any points $P_1, \ldots, P_r$ on C and natural numbers $m_1, \ldots, m_r$ with $j = \sum_{i=1}^r m_i$, $\mathcal{J}$ has an m_i-th pole on P_i for each i.*

By this theorem we obtain

Corollary to Theorem 4. *Let $g(C) \geq 1$. Then there exists a basic member $\pi : S \to C$ which admits a foliation.*

In Section 1 we prove Theorem 1, and Theorem 2 is proved in Section 2. In Section 3 we discuss some deformation of foliations, and Theorem 3, Theorem 4, and Corollary are proved.

1. Proof of Theorem 1

By (12.5) in [KoIII] and the section 5 in [KoI], we have

$$12\chi(\mathcal{O}) = c_2(S) = 24,$$

and by Lemma 3.18 in [FM]

$$\chi(\mathcal{O}_S) = \deg L.$$

Then we have

$$\deg L = 2 = \deg T_C.$$

Since L and T_C are line bundles over the projective line $\mathbb{P}^1$, the line bundle L is isomorphic to the tangent bundle T_C. Then by the consideration in Introduction, we obtain a non trivial section $S \to \pi^* T_C \otimes T_S$. As in 4.3 of Chapter IV in [FM] and 6 in [FM], the section is given locally by relatively prime holomorphic functions (f, g). Let $\{m\}$ and $\{M_0\}$ be transition functions of the bundles $\pi^* T_C$ and T_S respectively. Let $\{w, z\}$ be a local coordinates system of the surface S. By Euclidean-Weierstrass algorithm on a local coordinate U (c.f. Theorem 1.13 in [KoT]), we have uniquely a pair of holomorphic functions

$$\begin{pmatrix} a(w,z) \\ b(w,z) \end{pmatrix}$$

and a holomorphic function $h(z)$ such that

$$(*) \qquad (\, f(w,z), \quad g(w,z)\,) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a(w,z) \\ b(w,z) \end{pmatrix} = h(z)$$

where $h(z)$ is a convergent power series of z and the least common multiple of denominators, by the multiplication of this, appearing coefficients may be brought into $(\mathbb{C}\{z\})[w]$, this is denoted by $r(z)$ in the [KoT] above. We put $mM_0 = M$. Since $K_S = \mathcal{O}_S$, $\det M = m$. Then on a non empty intersection $U \cap U'$ of coordinate neighborhoods, we have

$$(\, f, \quad g\,)\, M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t M \begin{pmatrix} a \\ b \end{pmatrix} = (\det M) h(z) = mh(z).$$

Similarly to the relation $(*)$, on U' we have a unique pair

$$\begin{pmatrix} a'(w', z') \\ b'(w', z') \end{pmatrix}$$

and a unique holomorphic function $h'(z')$ such that

$$(\, f', \quad g'\,) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = h'$$

On the set $U \cap U'$ we have

$$(\, f, \quad g\,)\, M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = h',$$

$$(**) \qquad (\, f', \quad g'\,)\, M^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t M^{-1} \begin{pmatrix} a' \\ b' \end{pmatrix} = m^{-1} h'.$$

By definition of h and h', there exist holomorphic functions u' and v' such that

$$mh(z(w', z')) = u'h',$$

$$m^{-1}h' = v'h.$$

Then $h' = mv'h = u'v'h'$, and so $u'v' = 1$, therefore u' and v' are units. Since $h' = mv'h$, we get that

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = v'{}^t M \begin{pmatrix} a \\ b \end{pmatrix}.$$

For a non empty intersection $U \cap U' \cap U''$, we have $mh = u'h'$, $m'h' = u''h''$ and so $mm'h = \hat{u}''h''$. Then $mm'h = u'm'h' = u'u''h''$ and so $\hat{u}'' = u'u''$. Thus the family $\{u\}$ is a 1-cocycle, and it determine a line bundle ξ^{-1}. By the relation $(**)$ we have

$$\{\begin{pmatrix} a \\ b \end{pmatrix}\} \in H^0(\pi^* T_C \otimes T_S \otimes \xi).$$

Since $mh = u'h'$, $mv'h = h'$, we have $\{h\} \in H^0((\pi^* T_C \otimes \xi)^{\otimes 2})$ with the correspondence

$$H^0(\pi^* T_C \otimes T_S \otimes \xi) \ni \{\begin{pmatrix} a \\ b \end{pmatrix}\} \overset{\det}{\longmapsto} \{h\} \in H^0((\pi^* T_C \otimes \xi)^{\otimes 2}).$$

Let $\mathcal{L}'$ be the line bundle $\mathcal{O}(-\sum C_i) \otimes \xi \otimes (\pi^* T_C)^2$, where $\sum_i C_i$ denotes the sum of all singular fibers of $S \to C$. Now $(\mathcal{L}')^{-1} \otimes (\pi^* T_C \otimes \xi) = \mathcal{O}(\sum C_i) \otimes (\pi^* T_C)^{-1}$ and its degree is $j + r - 2 \geq 16 - 2 = 14$, where j and r denote the numbers of singular fibers of types I and II respectively ([KoI]). Then we get a non singular homomorphism $\hat{Y}' : \mathcal{L}' \to \pi^* T_C \otimes \xi$. In the process of the algorithm above we can see that the zeros of the section

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

and h are at most discrete sets. We denote by p the projection $\mathcal{L}' \to S$. Then for any $v \in \mathcal{L}'$ with $p(v) \in S - \{h^{-1}(\text{zeros})\}$, we may find a complex number λ_v such that $\hat{Y}'(v) = \lambda_v h(p(v))$. We put

$$S' = S - \{h^{-1}(\text{zeros}) \cup \begin{pmatrix} a \\ b \end{pmatrix}^{-1} (\text{zeros})\},$$

and

$$\hat{Y}'(v) = \lambda_v \begin{pmatrix} a \\ b \end{pmatrix} (p(v))$$

on S'. By the extension we get a non singular homomorphism $Y' : \mathcal{L}' \to \pi^* T_C \otimes T_S \otimes \xi$. We have identities

$$\begin{aligned} & H^0((\mathcal{L}')^{-1} \otimes \pi^* T_C \otimes T_S \otimes \xi) \\ & = H^0(\mathcal{O}(\sum C_i) \otimes (\pi^* T_C)^{-1} \otimes T_S) \\ & = \mathrm{Hom}(\mathcal{O}(-\sum C_i) \otimes \pi^* T_C, T_S). \end{aligned}$$

Then denoting $\mathcal{O}(-\sum C_i) \otimes \pi^* T_C$ by $\mathcal{L}$, we obtain a nonsingular homomorphism $Y : \mathcal{L} \to T_S$. We consider the composite homomorphism

$$\mathcal{L} \xrightarrow{Y} T_S \xrightarrow{D\pi} \pi^* T_C$$

and so $\mathcal{L}^{-1} \otimes \pi^* T_C = \mathcal{O}(\sum C_i)$. Then by a similar argument to the proof of Proposition 2.6 in [GMII] we have Theorem 1.

2. Proof of Theorem 2

Let $\eta \to C$ be the complex line bundle associated to the principal E-bundle $\pi : S \to C$. Then by Theorem 5, Proposition 1, and Theorem 2 in [A], we obtain the assertion that the exact sequence

$$0 \to S\times_E L(E) \to T_S/E \to T_C \to 0$$

admits a splitting if and only if the chern class $c(\eta)$ is zero, where $L(E)$ denotes the Lie algebra of the elliptic curve, and E acts on $L(E)$ by the adjoint action. Since the group E is abelian, and E acts freely on S, $S\times_E L(E) = C \times \mathbb{C}$, the trivial line bundle. Now we have

Proposition. *There exists a canonical isomorphism* $\pi^*(T_S/E) \cong T_S$.

By making use of this proposition we have a splitting of the exact sequence

$$0 \to S \times \mathbb{C} \to T_S \to \pi^* T_C \to 0.$$

Then for any complex line bundle $\mathcal{L} \to S$, we have

$$\dim H^0(\mathcal{L}^{-1} \otimes T_S) = \dim H^0(\mathcal{L}^{-1} \otimes \mathcal{O}_S) + \dim H^0(\mathcal{L}^{-1} \otimes \pi^* T_C).$$

Now we set $\mathcal{L} = \mathcal{O}(-\sum_{i=1}^b C_i) \otimes \pi^* T_C$, where b is a positive integer and $C_i = \pi^{-1}(p_i)$, $p_i \in C$, $i = 1, \dots, b$. Then $\mathcal{L}^{-1} \otimes \pi^* T_C = \mathcal{O}(\sum C_i)$ is the induced bundle of the product bundle $\otimes \xi_{p_i}$ of point bundles ξ_{p_i}, $i = 1, \dots, b$. Since $c_1(\otimes \xi_{p_i}) = b$, if $b > 2g - 2$,

$$\dim H^0(C, \otimes \xi_{p_i}) = b - (g - 1),$$

which is equal to $\dim H^0(\mathcal{O}(\sum C_i))$. Thus we obtain a turbulent foliation on S.

Therefore it is sufficient to prove Proposition. For topological vector bundles, G. Segal has stated the existence of such an isomorphism (c.f. Proposition (2.1) in [Se]). To prove Proposition in case of holomorphic vector bundles it sufficies to see them locally. For an open set U in the curve C, we denote by $p_1 : U \times E \to E$ and $p_2 : U \times E \to E$ the projections to each factor, and by $TU \hat{\oplus} (E \times L)$ the Whitney sum ${p_1}^* TU \oplus {p_2}^*(E \times L)$, where L is the Lie algebra of E. Let $\{g_{ij}\}$ and $\{h_{ij}\}$ be the transition functions of the principal bundle $S \to C$ and the tangent bundle $T_C \to C$ respectively. Then we have a local isomorphism

$$T(S|U) \cong (U \times \mathbb{C} \times E) \hat{\oplus} (U \times E \times L),$$

and on a non empty intersection $U \cap U'$, we have transformations

$$U \times \mathbb{C} \times E \ni (x, v, g) \mapsto (x, h_{ij}(x) v, g_{ij} g) \in U' \times \mathbb{C} \times E,$$

$$U \times E \times L \ni (x, g, w) \mapsto (x, g_{ij}(x) g, w) \in U' \times E \times L.$$

On the other hand the bundle $\pi^*(T_S/E) = S \times_C (T_S/E)$ restricts to

$$(U \times E) \times_U (T_S/E) \cong (E \times U \times \mathbb{C}) \hat{\oplus} (E \times U \times L).$$

Therefore we get the same transformations by transition functions, and obtain the required isomorphism.

3. Proofs of Theorem 3 and 4, and Corollary

Proof of Theorem 3. We have the exact sequence

$$H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z}),$$

and $\dim H^1(S, \mathcal{O}_S) =$ the irregularity q, which is zero by the assumption in Theorem 3. Now we have

Proposition. (*Theorem* 11 *in* [GMII]) *Assume that for any holomorphic line bundle $\mathcal{L}$ on S with Chern class $\alpha \in H^2(S, \mathbb{Z})$ the vector space $H^0(S, \mathcal{L}^{-1} \otimes T_S)$ is of constant dimension $r > 0$, then $\mathcal{D}_\alpha$ has a natural structure of a CP^{r-1}-bundle over a complex torus of dimension half the first Betti number of S.*

Since the homomorphism $c_1 : H^1(S, \mathcal{O}^*) \to H^2(S, \mathbb{Z})$ is a monomorphism, if line bundles $\mathcal{L}$ and $\mathcal{L}'$ have the same Chern classes, $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, and so $\mathcal{L}^{-1} \otimes T_S$ is isomorphic to $(\mathcal{L}')^{-1} \otimes T_S$, and the first Betti number $b_1 = 2q = 0$.

We consider the family of elliptic K3 surfaces in Section 5 of [KoI]. For each member in the family, any singular fiber is of type I_1 or II. Let j be the number of singular fibers of type I_1, i.e. $j = \nu(I_1)$, and $r = \nu(II)$. Then by (46) in [KoI] $j+2r = 24$, $0 \leq r \leq 8$, and $d = \deg L = 2$, and so $16 \leq j+r \leq 24$. Let $D = \sum_i C_i$ be the sum of all singular fibers of π and let $\mathcal{L} = \mathcal{O}(-\sum_i C_i) \otimes \pi^*(T_C)$. We put $b = \deg \mathcal{O}(\sum_i \pi(C_i))$. . Then

$$\begin{aligned} h^0(\mathcal{L}^{-1} \otimes T_S) &\geq h^0(\mathcal{L}^{-1} \otimes T_{S/C}) \\ &= h^0(D \otimes \pi^*(K_C \otimes L^{-1})) \\ &\geq b + g - 1 - d \\ &\geq 13. \end{aligned}$$

Thus we obtain Theorem 3. □

Proof of Theorem 4. Let $D = \sum_{i=1}^r m_i P_i$. Then by the assumption $h^1(D) = 0$. So by the Riemann-Roch theorem we have

$$h^0(D) = 1 - g(C) + \deg D \geq g(C) + 2 \geq 2.$$

Let $D_k = (m_k - 1)P_k + \sum_{i \neq k} m_i P_i$. Then $\deg D_k = \deg D - 1 \geq 2g(C)$. So by the Serre duality we have $h^1(D_k) = 0$. Therefore by the Riemann-Roch theorem we obtain

$$h^0(D_k) = 1 - g(C) + \deg D_k = \deg D - g(C).$$

For each k, we have $h^0(D_k) = h^0(D) - 1 \geq 1$. Here we note that $H^0(D) \supsetneq H^0(D_k)$ for each k. Hence there exists an element $\mathcal{J} \in H^0(D) \setminus \cup_{k=1}^r H^0(D_k)$. □

Proof of Corollary to Theorem 4. In Theorem 3, let $1 \leq r = j$ and so $m_i = 1$ for $i = 1, \ldots, j$. Then by Kodaira's construction (Section 8 of [KoII]), we have a basic member which has j-singular fibers of type I_1 and some singular fibers of type II, III, IV. Now singular fibers of type II and IV come from the value zero of the meromorphic function $\mathcal{J}$, and singular fibers of type III come from the value 1. It is known that the number j is equal to the number of zeros possibly with some multiplicity and to the number of 1's also ([Sp]). Now we have an estimation

$$\begin{aligned} c_2(S) = 12d &= j + 2\nu(II) + 4\nu(IV) + 3\nu(III) \\ &= j + 2(\nu(II) + \nu(IV)) + 2\nu(IV) + 3\nu(III) \\ &\leq j + 2j + 2j + 3j \\ &= 8j, \end{aligned}$$

and

$$d \leq \frac{2}{3}j.$$

Let $\mathcal{L} = \mathcal{O}(-\sum_i C_i) \otimes \pi^*(T_C)$. Then

$$h^0(\mathcal{L}^{-1} \otimes T_{S/C}) \geq b + g - 1 - d \geq j + g - 1 - d \geq j - d > 0,$$

where $d = \deg L$ and $b = \deg \mathcal{O}(\sum_i \pi(C_i))$. Thus we obtain the corollary. □

Recently, we published the article [FuMa] which is a smooth version of the present study.

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