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Note on $*$-Regular Semigroups

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This paper is a continuation and a supplement to the previous paper [3]. For any element x of a $*$-regular semigroup (with $*$-operation$*$) in the sense of K. S. S. Nambooripad and F. J. C. M. Pastijn [1], there exists a unique inverse x^+ of x such that $x^*\mathscr{H}x^+$. The main purpose of this paper is to investigate several conditions for a $*$-regular semigroup S to be a special $*$-regular semigroup with respect to the unary operation $+: S\to S$.

§ 0. Introduction

A regular semigroup S equipped with a unary operation $*: S\to S$ (hereafter, we denote it by $(S, *)$) is called a *$*$-regular semigroup* if

(0.1) (1) $(x^*)^*=x$ for all $x\in S$,

(2) $(xy)^*=y^*x^*$ for all $x, y\in S$,

(3) for any $x\in S$, there exists a unique inverse x^+ such that $x^*\mathscr{H}x^+$.

In a $*$-regular semigroup $(S, *)$ it is easy to see from [1] that $(xx^+)^*=xx^+$, $(x^+x)^*=x^+x$, $(x^+)^+=x$ and $(x^*)^+=(x^+)^*$ for any element $x\in S$. It is also obvious that each L-class [R-class] contains a unique projection (that is, an idempotent e such that $e^*=e$). Let $P(S)$ and $E(S)$ be the set of all projections of $(S, *)$ and the set of all idempotents of $(S, *)$ respectively. It should be noted that $p=p^*=p^+$ for all $p\in P(S)$. We can consider the unary operation $+: S\to S$ defined by $+: x\to x^+$. This operation $+$ is called *the inverse operation* induced by $*$. The semigroup S with $+$ (we denote it by $(S, +)$) is not necessarily a $*$-regular semigroup with respect to $+$. If a $*$-regular semigroup $(S, *)$ satisfies the following condition:

(0.2) $x^*=x^+$ for all $x\in S$,

then $(S, *)$ is called *a special $*$-regular semigroup.*[1)] It is obvious that a $*$-band (that is, a $*$-regular semigroup in which every element is an idempotent) $(B, *)$ is necessarily special. In case where a (special) $*$-regular semigroup is orthodox, it is called *a (special) $*$-orthodox semigroup.*

1) In [3], we used the term "a regular $*$-semigroup" for this $(S, *)$.

§ 1. Basic properties

Let $(S, *)$ be a $*$-regular semigroup. It follows from [3] that $(S, +)$ is a special $*$-regular semigroup if

(1.1) (1) $x^+P(S)x \subset P(S)$,

(2) $P(S)^2 \subset E(S)$.

In this case, we have the following; Let $\overline{P(S)} = \{p \in E(S): p^+ = p\}$. Of course, $P(S) \subset \overline{P(S)}$. Conversely, $p \in \overline{P(S)}$ implies $p^+ = p$. Since $p^* \mathscr{H} p^+$ and both p^* and p^+ are idempotents, $p^* = p^+ = p$. Thus, $p \in P(S)$. Consequently, we have $P(S) = \overline{P(S)}$.

In a $*$-orthodox semigroup $(A, *)$, $e^+ = e^*$ for each $e \in E(A)$. For, let $e \in E(A)$. Then, e^* is also an idempotent. On the other hand, e^+ is an idempotent since every inverse of an idempotent in an orthodox semigroup is also an idempotent. Since $e^* \mathscr{H} e^+$ and an H-class contains at most one idempotent, $e^* = e^+$. However, a $*$-regular semigroup $(S, *)$ satisfying $e^+ = e^*$ for $e \in E(S)$ is not necessarily orthodox (see N. R. Reilly [2]).

LEMMA 1. *Let $(S, *)$ be a $*$-regular semigroup. The following three conditions (1)–(3) are equivalent:*

(1.2) (1) *For any $p, q \in P(S)$, $(pq)^* = (pq)^+$.*

(2) *For any $p, q \in P(S)$, $(pq)^+ = qp$.*

(3) *$P(S)^2 \subset E(S)$, that is, (pq, qp) is a regular pair for any $p, q \in P(S)$.*

PROOF. (1)$\Leftrightarrow$(2): $(pq)^+ = (pq)^* \Leftrightarrow (pq)^+ = qp$.
(2)$\Leftrightarrow$(3): $(pq)^+ = qp$ implies $pq(qp)pq = pq$. Hence, $(pq)^2 = pq$, whence $pq \in E(S)$. Conversely, let $P(S)^2 \subset E(S)$. For $p, q \in P(S)$, $H_{(pq)^*} = H_{(pq)^+}$ (where H_x denotes the H-class containing x). Now, since $pq \in E(S)$, it follows that pq is an inverse of pq. Since $H_{(pq)^*} = H_{pq} = H_{(pq)^+}$ and since there exists at most one inverse of pq in each H-class, we have $(pq)^+ = qp$.

LEMMA 2. *For a $*$-regular semigroup $(S, *)$ in which $e^* = e^+$ for $e \in E(S)$, the following conditions are equivalent:*

(1.3) (1) *Any one of (1.2).*

(2) *$P(S)^2 = E(S)$.*

(3) *For any $u \in E(S)^2$, $u^* = u^+$.*

PROOF. (2)$\Rightarrow$(1): Obvious. (1)$\Rightarrow$(2): For any $e \in E(S)$, $e = ee^+e = ee^*e = ee^*e^*e = (ee^+)(e^+e)$. Since both ee^+ and e^+e are projections, $e \in P(S)^2$. Hence,

$E(S) \subset P(S)^2$. Since $P(S)^2 \subset E(S)$ is satisfied by the assumption, we have $P(S)^2 = E(S)$. (3)$\Rightarrow$(1): Since $P(S)^2 \subset E(S)^2$, $(pq)^* = (pq)^+$ for any $p, q \in P(S)$. (1)$\Rightarrow$(3): This follows from the following theorem.

THEOREM 3. *Let $(S, *)$ be a $*$-regular semigroup. Then, the subsemigroup $(\langle E(S)\rangle, *)$*[2] *is a special $*$-regular semigroup if and only if $P(S)^2 \subset E(S)$ and $e^* = e^+$ for $e \in E(S)$.*

PROOF. It is well-known that $\langle E(S)\rangle$ is a regular semigroup and $\langle E(S)\rangle \ni x$ implies $x^* \in \langle E(S)\rangle$. Hence, $(\langle E(S)\rangle, *)$ is also a $*$-regular semigroup. The "if" part: It is obvious that $P(\langle E(S)\rangle) = P(S)$. For $e \in E(S)$, $e^+P(S)e = e^*P(S)e \subset P(S)$. For, let $e^*fe \in e^*P(S)e$ where $f \in P(S)$. Then, $e^*fee^*fe = e^+((ee^+)f)((ee^+)f)e = e^+ee^+fe$ (since $P(S)^2 \subset E(S)) = e^*fe$. Further, $(e^+fe)^* = (e^*fe)^* = e^*fe = e^*fe$. Hence, $e^+P(S)e \subset P(S)$. It is obvrious that for any $e \in E(S)$ there exists a unique $e' \in V(e)$ (the set of inverses of e) such that $ee', e'e \in P(S)$ (in fact, this e' is e^+). Therefore, $P(S)$ is a p-system in $\langle E(S)\rangle$ (see [3]). Hence, $(\langle E(S)\rangle, +)$ is special $*$-regular semigroup. If $x \in (\langle E(S)\rangle, +)$, then $x = e_1e_2 \ldots e_n$ for some $e_1, e_2, \ldots, e_n \in E(S)$. Now, $x^* = e_n^* \ldots e_2^* e_1^* = e_n^+ \ldots e_2^+ e_1^+ = x^+$. Hence, $(\langle E(S)\rangle, *)$ is a special $*$-regular semigroup.

LEMMA 4. *For a $*$-regular semigroup $(S, *)$ satisfying $P(S)^2 \subset E(S)$, the following conditions are equivalent:*

(1.4) (1) *For any $x \in S$, $x^+P(S)x \subset P(S)$.*
(2) *For any $e \in P(S)$ and any $x \in S$, $(ex)^+ = x^+e$.*
(3) *For any $e \in P(S)$ and any $x \in S$, $x^*e\mathscr{R}x^+e$.*
(4) *For any $e \in P(S)$ and any $x \in S$, $ex^*\mathscr{L}ex^+$.*
(5) *For any $e \in P(S)$ and any $x \in S$, $ex^*\mathscr{H}ex^+$.*
(6) *For any $e \in P(S)$ and any $x \in S$, $x^*e\mathscr{H}x^+e$.*

PROOF. (3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(3): Let $x^*e\mathscr{R}x^+e$ for $e \in P(S)$ and $x \in S$. Then, $ex\mathscr{L}e(x^*)^+$. Putting $y = x^*$, we have $ey^*\mathscr{L}ey^+$. Next, assume the condition (4). Then, $x^*\mathscr{H}x^+$ implies $x^*\mathscr{R}x^+$, whence $ex^*\mathscr{R}ex^+$. Therefore, $ex^*\mathscr{H}ex^+$. Next, $ex^*\mathscr{H}ex^+$ implies $xe\mathscr{H}(x^+)^*e$. Putting $x^* = y$, we have $y^*e\mathscr{H}y^+e$. It is obvious that (6)$\Rightarrow$(3). Next, we shall show (3)$\Rightarrow$(2): Since $x^*e\mathscr{L}x^+e$, $(ex)^+\mathscr{H}(ex)^* = x^*e\mathscr{H}x^+e$. Since $x^+e\mathscr{H}(ex)^+$ and since both $(ex)^+$ and x^+e are inverses of ex, we have $x^+e = (ex)^+$. (2)$\Rightarrow$(1): Let $p \in P(S)$. Then, $x^+px = (x^+p)(px) = (px)^+px \in P(S)$. Thus, $x^+P(S)x \subset P(S)$. (1)$\Rightarrow$(6): For any $p \in P(S)$, $x^+px \in P(S)$. Hence, x^+p is an inveres of px such that $(px)(x^+p) = (px)(px)^+$ and $(x^+p)(px) = (px)^+(px)$. Thus, $x^*p = (px)^*\mathscr{H}(px)^+\mathscr{H}x^+p$.

REMARK. Let $(S, *)$ be a $*$-orthodox semigroup. For $e \in E(S)$, e^* is an idem-

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2) The notation $\langle E(S)\rangle$ means the subsemigroup of S generated by $E(S)$.

potent. Since S is orthodox, $e^+ \in E(S)$. Since both e^* and e^+ are contained in the same H-class, $e^* = e^+$. Further, $e = ee^*e^*e = ee^+e^+e$. Hence, $E(S) \subset P(S)^2$. Since $P(S)^2 \subset E(S)$, we have $P(S)^2 = E(S)$. Therefore, any $*$-orthodox semigroup $(S, *)$ satisfies each one of (1.3).

§2. Main results

The following is obvious:

THEOREM 5. *An H-degenerate[3] $*$-regular semigroup $(S, *)$ is special.*
Now,

THFOREM 6. *For a $*$-regular semigroup $(S, *)$, $(S, +)$ is a special $*$-regular semigroup if and only if S satisfies one of* (1.2) *and one of* (1.4).

PROOF. The "if" part follows from (1.1). The "only if" part: Suppose that $(S, +)$ is a special $*$-regular semigroup. Let $\overline{P(S)} = \{p \in E(S): p^+ = p\}$. Then, $\overline{P(S)} = P(S)$. Since $(S, +)$ is a special $*$-regular semigroup, $x^+\overline{P(S)}x = x^+P(S)x \subset P(S)$ and $P(S)^2 = \overline{P(S)}^2 \subset E(S)$ are satisfied (see [3]). Hence, S satisfies the conditions of (1.2) and (1.4).

REMARK. Let G be a group having an element a of order 2 which is not in the center (for example, let G be the symmetric group on the set $\{1, 2, 3\}$ and let $a = (12)$). Let $*: G \to G$ be the mapping defined by $x^* = a^{-1}x^{-1}a$. Then, $x^+ = x^{-1}$ for all $x \in G$. Now, $(G, *)$ is a $*$-regular semigroup, but it is not a special $*$-regular semigroup. On the other hand, $(G, +)$ is a special $*$-regular semigroup. By this example, we have the following: These exists a $*$-regular semigroup $(S, *)$ such that $(S, +)$ is a special $*$-regular semigroup, but $(S, *)$ is not a special $*$-regular semigroup.

COROLLARY 7. *For a $*$-orthodox semigroup $(S, *)$, $(S, +)$ is a special $*$-regular semigroup if and only if S satisfies one of* (1.4).

PROOF. Since S is orthodox, $e^* = e^+$ for all $e \in E(S)$. Further, $P(S) \subset E(S)^2$ is obvious. Therefore, this follows from Theorem 6.

A semigroup T is said to be H-compatible if Green's H-relation on T is a congruence.

THEOREM 8. *For an H-compatible $*$-regular semigroup $(S, *)$, $(S, +)$ is a special $*$-regular semigroup if and only if S satisfies one of* (1.2).

PROOF. The "only if" part follows from Theorem 6. The "if" part: Since S is H-compatible, $x^*e\mathscr{H}x^+e$ is satisfied for $x \in S$ and for $e \in P(S)$ since $x^*\mathscr{H}x^+$ is always

3) A semigroup S is called H-degenerate if each H-class of S consists of a single element.

satisfied. Therefore, S satisfies the condition of Theorem 5. Hence, $(S, +)$ is a speical *-regular semigroup.

COROLLARY 9. *An H-compatible *-orthodox semigroup $(S, *)$ is a special *-orthodox semigroup with respect to +. That is, $(xy)^+ = y^+x^+$ is satisfied for any $x, y \in S$.*

REMARK. Let $S = M(G; I \times I; P)$ be a Ress $I \times I$ matrix semigroup over a group G with sandwich matrix P. Assume that $P = (p_{ij})$ satisfies $p_{ji} = p_{jj}p_{ij}^{-1}p_{ii}$ for all $i, j \in I$. Now, suppose that $S = M(G; I \times I; P)$ is a *-regular semigroup with respect to a unary operation $*: S \to S$. We can assume that for $(x)_{ij} \in S$, $(x)_{ij}^* = (y)_{ji}$ for some $y \in G$. Now, it is easy to see by simple calculation that $(x)_{ij}^+ = (p_{jj}^{-1}x^{-1}p_{ii}^{-1})_{ji}$. Therefore, the set $P(S)$ of projections of $(S, *)$ is $\{(x)_{ij}(x)_{ij}^+ : (x)_{ij} \in S\} = \{(p_{ii}^{-1})_{ii} : i \in I\}$. Now, for $(p_{ii}^{-1})_{ii}, (p_{jj}^{-1})_{jj} \in P(S)$, $(p_{ii}^{-1})_{ii}(p_{jj}^{-1})_{jj} = (p_{ii}^{-1}p_{ij}p_{jj}^{-1})_{ij} = (p_{ji}^{-1})_{ij} \in E(S)$. Since S is clearly H-compatible, it follows from the theorem above that $(S, +)$ is a special *-regular semigroup.

We also obtain the following:

THEOREM 10. *If a *-regular semigroup $(S, *)$ satisfies*

(2.1) (1) $P(S)^2 \subset E(S)$,

(2) $hE(S)h \cap E(S) \subset P(S)$ *for all* $h \in P(S)$,

*then $(S, +)$ is a special *-regular semigroup.*

PROOF. For any $x \in S$, $x^+x \in P(S)$. Put $x^+x = h$. Now, $x^+P(S)x = h(x^+P(S)x)h \subset hE(S)h \cap E(S) \subset P(S)$ since $x^+P(S)x \subset E(S)$. Hence, it follows from Theorem 6 that $(S, +)$ is a special *-regular semigroup.

COROLLARY 11. *If a *-orthodox semigroup $(S, *)$ satisfies*

(2.2) $hE(S)h \subset P(S)$ *for all* $h \in P(S)$,

*then $(S, +)$ is a special *-regular semigroup.*

PROOF. Obvious.

REMARK. If $(S, *)$ is a *-orthodox semigroup in which $E(S)$ is a normal band (that is, $(S, *)$ is a generalized inverse *-regular semigroup). Let $e \in E(S)$ and $h \in P(S)$. There exist $p, q \in P(S)$ such that $pq = e$ (in fact, take $p = ee^+$ and $q = e^+e$). Now, $(heh)^* = (hpqh)^* = hqph = hpqh = heh$. Hence, $heh \in P(S)$. Therefore, $(S, +)$ is a special *-orthodox semigroup. This gives another proof for Theorem 3.2 (and its corollary) of [3].

References

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