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S^1-Actions on Sphere Bundles over Spheres

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In this paper, we shall construct compact Lie group actions on total spaces of orientable sphere bundles over spheres. All actions considered in this paper preserve bundle structures, that is, each element of groups gives a bundle map. The author intends to construct actions on all sphere bundles over S^n for $n \leqq 8$. For $n > 8$, actions on S^k-bundles over S^n are given for the case of $k \geqq n$ and other particular n, k. Thus we can conclude that these bundle spaces have positive degrees of symmetry.

In section 1, we construct actions on S^3-bundles over S^4 and S^7-bundles over S^8. By means of reductions of structure groups, we can give S^1-actions on S^k-bundles over S^n for $k \geqq n$. Using well-known results from the homotopy theory of spheres and rotation groups, we construct actions on S^k-bundles over S^n for $k < n \leqq 8$ in section 2. In the last section, we construct actions on S^{4s-1}-bundles over S^{4s} of types $B_{l,0}$, $B_{0,l}$ and $B_{\varepsilon k, mk}$, where $\varepsilon = 1$ if s is odd, $\varepsilon = 2$ if s is even, $m = (2s-1)!/2$ and k is an integer.
The technique used in this paper is quite homotopical and actually elementary. Our results are essentially due to the computations of M. A. Kervaire, [3] and we shall use it frequently in this paper.

§1. Some general results

Consider an S^k-bundle over S^n. Denote by $B^{(n,k)}$ the total space. According to §18 in [6], the space is obtained from the disjoint union of $D_1^n \times S^k$ and $D_2^n \times S^k$ by identifying each point (x, y) in the boundary of $D_1^n \times S^k$ with $(x, \chi(x)(y))$ which is considered as a point in the boundary of $D_2^n \times S^k$, where D^n denotes an n-disk and χ the characteristic map of S^{n-1} into the rotation group $SO(k+1)$. Thus our construction can be obtained by the construction of actions on $D^n \times S^k$ which are compatible with identifications.

We adopt notations similar to 22.3 and 22.6 in [6], and define homomorphisms $\rho: S^3 \to SO(3)$ by $\rho(q)(q') = qq'q^{-1}$, where $q \in S^3$, $q' \in S^2$, and $\sigma: S^3 \to SO(4)$ by $\sigma(q)(q') = qq'$, where $q, q' \in S^3$. Then we have a bundle equivalence $\varphi: S^3 \times SO(3) \to SO(4)$, which is given by $\varphi(q_1, \rho(q_2)) = \sigma(q_1) \cdot \rho(q_2)$, where $q_1, q_2 \in S^3$. The map ρ is a double covering. Thus we obtain an isomorphism $\varphi_*: \pi_i(S^3) + \pi_i(S^3) \to \pi_i(SO(4))$ for each $i \geqq 2$, which is given by $\varphi_*(\alpha, \beta) = j_*^{(4,3)}\rho^*(\alpha) + \sigma_*(\beta)$, where $j^{(n,n-1)}$ denotes the inclusion map $SO(n-1) \to SO(n)$. We denote by ρ_4 and σ_4 the elements $j_*^{(4,3)}\rho_*(\iota_3)$ and $\sigma_*(\iota_3)$ respectively, where ι_3 is the generator of $\pi_3(S^3)$. Then the group $\pi_3(SO(4))$ is generated by ρ_4 and σ_4. First we have

THEOREM 1. *Any S^3-bundle over S^4 admits an S^3-action and any S^7-bundle over S^8 yields a G_2-action.*

PROOF. Let $B_{m,n}^{(4,3)}$ be the bundle space having $m\rho_4+n\sigma_4$ as the homotopy class of the characteristic map, where m, n are integers. Then $B_{m,n}^{(4,3)}$ is obtained by the identification $(x, y)\equiv(x, x^{m+n}yx^{-m})$. Define an action of S^3 on $D_i^4\times S^3$ for $i=1, 2$ by $q(x, y)=(qxq^{-1}, qyq^{-1})$, where $q\in S^3$ and $(x, y)\in D_i^4\times S^3$. By the equality

$$(qxq^{-1})^{m+n}(qyq^{-1})(qxq^{-1})^{-m}=q(x^{m+n}yx^{-m})q^{-1},$$

the action is compatible with the identification. Then the space $B_{m,n}^{(4,3)}$ admits an S^3-action which gives a bundle map for each $q\in S^3$.

Let ρ: $S^7\to SO(7)\subset SO(8)$ and σ: $S^7\to SO(8)$ be maps defined by $\rho(x)(y)=xy\bar{x}$ and $\sigma(x)(y)=xy$, where x, $y\in S^7$ and the multiplication in S^7 is that of Cayley numbers. Denote by ρ_8 and σ_8 the homotopy classes represented by ρ and σ respectively. Then by [8], the homotopy group $\pi_7(SO(8))\approx Z+Z$, the direct sum of infinite cyclic groups, is generated by ρ_8 and σ_8. Let $B_{m,n}^{(8,7)}$ be the bundle space having $m\rho_8+n\sigma_8$ as the homotopy class of the characteristic map, where m, n are integers. Then the space $B_{m,n}^{(8,7)}$ is obtained by the identification $(x, y)\equiv(x, x^{m+n}yx^{-m})$. The exceptional group G_2 acts on the algebra C of Cayley numbers as a subgroup of the orthogonal group $O(7)$ and satisfies the relation $g(x\cdot y)=g(x)\cdot g(y)$ for each x, $y\in C$ and $g\in G_2$. Thus we have $g(x^{m+n}yx^{-m})=(g(x))^{m+n}(g(y))(g(x))^{-m}$. Hence the action on $D^8\times S^7$, which is ginen by $g(x, y)=(g(x), g(y))$, is compatible with the identification. Then we have proved the theorem.

Actions on S^2-bundles over S^2 were investigated in [5]. Next we consider S^k-bundles over S^n for $k\geqq n>2$.

THEOREM 2. *Any S^k-bundle over S^n admits an S^1-action for $k\geqq n>2$.*

PROOF. Consider the case $k\geqq n+1$. Since $k-n+1\geqq 2$ and $k+1-(k-n+1)=n$, the group $\pi_{n-1}(V_{k+1,k-n+1})$ is trivial. By the exact sequence

$$\pi_{n-1}(SO(n))\longrightarrow\pi_{n-1}(SO(k+1))\longrightarrow\pi_{n-1}(V_{k+1,k-n+1})=0,$$

the characteristic map χ: $S^{n-1}\to SO(k+1)$ can be reduced to a map from S^{n-1} into $SO(n)$ and we have the composition of the inclusion maps i: $SO(n)\times SO(2)\to SO(n)\times SO(k-n+1)\to SO(k+1)$. Thus the circle group $SO(2)$ acts on each fibre as a bundle map.

When $k=n$ and n is odd, we have the exact sequence

$$\pi_{n-1}(SO(n-1))\longrightarrow\pi_{n-1}(SO(n+1))\longrightarrow\pi_{n-1}(V_{n+1,2}).$$

Since $\pi_{n-1}(V_{n+1,2})\approx Z$ and it follows from R. Bott [1] that $\pi_{n-1}(SO(n+1))\approx Z_2$ or 0, the characteristic map χ: $S^{n-1}\to SO(n+1)$ is reducible to a map χ': $S^{n-1}\to SO(n-1)$. Thus we obtain an S^1-action.

The case $n \equiv 2$ mod 8: By the long exact sequence,

$$\longrightarrow \pi_{n-1}(SO(n-1)) \longrightarrow \pi_{n-1}(SO(n+1)) \approx Z_2 \longrightarrow \pi_{n-1}(V_{n+1,2}) \approx Z_2 \longrightarrow$$
$$\pi_{n-2}(SO(n-1)) \approx Z_2 + Z_2 \longrightarrow \pi_{n-2}(SO(n+1)) \approx Z_2 \longrightarrow \pi_{n-2}(V_{n+1,2}) = 0,$$

we have also a reduction of the structure group $SO(n+1)$ into $SO(n-1)$.

The case $n \equiv 6$ mod 8: Since $\pi_{n-1}(SO(n+1)) = 0$, any bundle is trivial.

The case $n \equiv 4s$ and $s \geqq 3$: From the exact sequence

$$\longrightarrow \pi_{n-1}(SO(n-1)) \approx Z \longrightarrow \pi_{n-1}(SO(n+1)) \approx Z \longrightarrow \pi_{n-1}(V_{n+1,2}) \approx Z_2 \longrightarrow$$
$$\pi_{n-2}(SO(n-1)) \approx Z_2 \longrightarrow \pi_{n-2}(SO(n+1)) = 0,$$

we have an isomorphism $j_*^{(n+1,n)} \circ j_*^{(n,n-1)}: \pi_{n-1}(SO(n-1)) \to \pi_{n-1}(SO(n+1))$.

The case $n = 4, 8$: By the epimorphisms $j_*^{(5,4)}: \pi_3(SO(4)) \to \pi_3(SO(5))$, $j_*^{(9,8)}: \pi_7(SO(8)) \to \pi_7(SO(9))$, together with the proof of Theorem 1, we obtain required actions.

§2. Low dimensional sphere bundles over spheres

First we have

PROPOSITION 3. *Any S^1-bundle over S^2 admits a torus action.*

PROOF. Since $\pi_1(SO(2)) \approx Z$, there is a bundle B_m corresponding to each integer m, which is obtained from the disjoint union $D_1^2 \times S^1 \cup D_2^2 \times S^1$ by the identification $(x, y) \equiv (x, x^m y)$ for each $(x, y) \in S^1 \times S^1$. For an element (ρ, θ) of the torus T^2, we define an action on $D_i^2 \times S^1$, $i = 1, 2$, by

$$(\rho, \theta)(x, y) = \begin{cases} (\rho x, \theta y), & (x, y) \in D_1^2 \times S^1, \\ (\rho x, \rho^m \theta y), & (x, y) \in D_2^2 \times S^1. \end{cases}$$

In fact, these two actions are compatible with the idetification. Thus we have a torus action on B_m.

Since any S^k-bundle over S^3 is trivial, we have an $SO(4) \times SO(k+1)$-action on the bundle space. Further any S^1-bundle over S^4 is also trivial. Since $\pi_3(SO(3)) \approx Z$, there is an S^2-bundle $B_m^{(4,2)}$ over S^4 for each integer m. By Theorem 1, $B_m^{(4,2)}$ is an S^3-invariant subbundle of $B_{m,0}^{(4,3)}$.

Any S^k-bundle over S^5 is a product bundle for $k = 1$ and $k > 4$.

PROPOSITION 4. *Any S^k-bundle over S^5 admits an S^1-action for $k = 2, 3$ and 4.*

PROOF. Let $\eta_3: S^4 \to S^3$ be an essential map. Then $\pi_4(SO(3)) \approx Z_2$ is generated by the homotopy class $\{\rho \circ \eta_3\}$, where ρ is the homomorphism in §1. The Hopf fibering $h: S^3 \to S^2$ is a principal S^1-bundle and the map h is S^1-invariant. The suspension $\eta_3: S^4 \to S^3$ is also S^1-invariant with respect to the suspended action on S^4. The space $B^{(5,2)}$ is obtained by the identification $(x, y)=(x, \rho\circ\eta_3(x)(y))$. Define an S^1-action on $D_i^5 \times S^2$, $i=1, 2$, by $c(x, y)=(c(x), y)$, where $c(x)$ is the suspended action of c on S^4. Since $(c(x), \rho\circ\eta_3(c(x))(y)) \equiv (c(x), \rho\circ\eta_3(x)(y))$, we have an S^1-action on $B^{(5,2)}$.

The group $\pi_4(SO(4)) \approx Z_2+Z_2$ is generated by the homotopy classes $\{\rho\circ\eta_3\}$ and $\{\sigma\circ\eta_3\}$, where σ is the homomorphism in § 1. There are three non trivial S^3-bundles over S^5 obtained by the identifications $(x, y) \equiv (x, (\rho\circ\eta_3(x))(y)(\rho\circ\eta_3(x))^{-1})$, $(x, y) \equiv (x, (\rho\circ\eta_3(x))(y))$ and $(x, y) \equiv (x, (\rho\circ\eta_3(x))^2(y)(\rho\circ\eta_3(x))^{-1})$. The action on $D^5 \times S^3$ which is defined by $c(x, y)=(c(x), y)$ gives an S^1-action on these bundles.

The group $\pi_4(SO(5)) \approx Z_2$ is generated by the homotopy class $\{j^{(5,4)}\circ\rho\circ\eta_3\}$. By the above construction we obtain an S^1-action on the S^4-bundle over S^5. Thus we have proved the proposition.

PROPOSITION 5. *Any S^k-bundle over S^6 admits an S^1-action for $k \leqq 5$.*

PROOF. When $k=1$, any bundle is trivial. By the isomorphism $\rho^*: \pi_5(S^3) \to \pi_5(SO(3))$, the non trivial S^2-bundle over S^6 is obtained from the disjoint union $D_1^6 \times S^2 \cup D_2^6 \times S^2$ by the identification $(x, y) \equiv (x, \rho\circ\eta_3\circ\eta_4(x)(y))$, where $\eta_4: S^5 \to S^4$ is an essential map. Therefore, similarly to the case of $B^{(5,2)}$, we have an S^1-action on $B^{(6,2)}$.

Since $\pi_5(SO(4)) \approx \pi_5(SO(3))+\pi_5(S^3)$ is generated by the homotopy classes $\{\rho\circ\eta_3\circ\eta_4\}$ and $\{\sigma\circ\eta_3\circ\eta_4\}$, we can construct an S^1-action on $B^{(6,3)}_{\varepsilon_1,\varepsilon_2}$, where $\varepsilon_i=1$ or 0 $(i=1, 2)$.

By the exact sequence

$$\pi_5(SO(4)) \longrightarrow \pi_5(SO(5)) \approx Z_2 \longrightarrow \pi_5(S^4) \approx Z_2 \longrightarrow \pi_4(SO(4)) \approx Z_2+Z_2 \longrightarrow$$
$$\pi_4(SO(5)) \approx Z_2 \longrightarrow \pi_4(S^4) \approx Z,$$

the characteristic map $\chi: S^5 \to SO(5)$ can be reduced to a map of S^5 into $SO(4)$. Then an S^1-action on $B^{(6,3)}_{\varepsilon_1,\varepsilon_2}$ gives an action on $B^{(6,4)}$.

From the exact sequence $\pi_6(S^6) \to \pi_5(SO(6)) \to \pi_5(SO(7))=0$, we see that the group $\pi_5(SO(6)) \approx Z$ is generated by the homotopy class of the characteristic map χ for the tangent bundle of S^6. The space $B^{(6,5)}_m$ is obtained by the identification $(x, y) \equiv (x, (\chi(x)^m(y))$. By Satz of 6.4 in [2], the space admits the diagonal $O(5)$-action. Thus we have proved the proposition.

PROPOSITION 6. *Any S^k-bundle over S^7 admits an S^1-action for $k \leqq 6$.*

PROOF. By the isomorphism $\rho_*: \pi_6(S^3) \to \pi_6(SO(3))$, any characteristic map $\chi: S^6 \to SO(3)$ is given by $(\rho\circ\nu_3(x))^m$ for some integer m mod 12, where ν_3 is a representative of the generator of $\pi_6(S^3)$. On the other hand, since the sequence $\pi_7(S^7) \to \pi_6(Sp(1))$

$\to\pi_6(Sp(2))$ is exact and $\pi_6(Sp(2))$ is isomorphic to the stable group $\pi_2(O)=0$, ν_3 is homotopic to the characteristic map $T_2''\colon S^6\to S^3$ given by $T_2''(q_0, q_1)=(1-2q_0\cdot(1+q_1)^{-2}\bar{q}_0)$, where q_0, q_1 are quaternion numbers and $|q_0|^2+|q_1|^2=1$, Re $q_1=0$ (cf. 24.11 in [6]). Now we have the equalities $(1+q_1)^{-2}=(1-2q_1+q_1^2)/|1+q_1|^2$ and $cq_1^2\bar{c}=(cq_1\bar{c})(cq_1\bar{c})=q_1^2$ for any complex number c of absolute value 1. Define an S^1-action on S^6 by $c(q_0, q_1)=(q_0c, q_1)$. The map T_2'' is S^1-invariant. Thus we have an S^1-action on $B_m^{(7,2)}$ for each integer m mod 12.

Using the splitting $\varphi_*\colon \pi_6(SO(4))\approx\pi_6(SO(3))+\pi_6(S^3)$, we can construct an S^1-action on $B_{m,n}^{(7,3)}$ for each pair of integers m, n.

Since $\pi_6(SO(5))=\pi_6(SO(6))=\pi_6(SO(7))=0$, any S^k-bundle over S^7 is trivial for $k=4$, 5 and 6. Hence we have proved the proposition.

To construct S^1-actions on sphere bundles over S^8, we refer to the table of $\pi_7(SO(k))$, $2\leqq k\leqq 7$.

k	2	3	4	5	6	7
$\pi_7(SO(k))$	0	Z_2	Z_2+Z_2	Z	Z	Z.

Then we have

PROPOSITION 7. *Any S^k-bundle over S^8 admits an S^1-action for $k=2$, 3 and an $SU(2)$-action for $k=4$, 5 and 6.*

PROOF. The generator of $\pi_7(SO(3))\approx\pi_7(S^3)\approx Z_2$ is represented by the composite map $\nu_3\circ\eta_6\colon S^7\to S^6\to S^3$, where η_6 is the 4-fold suspension of the Hopf map $h\colon S^3\to S^2$. The space S^7 admits an S^1-action, by which the map η_6 is S^1-invariant. Then we have an S^1-action on $B^{(8,2)}$ as same as in the case of $B_m^{(7,2)}$. By the isomorphism $\varphi_*\colon \pi_7(SO(4))\approx\pi_7(SO(3))+\pi_7(S^3)$, we obtain also an S^1-action on $B_{m,n}^{(8,3)}$, where m, $n=0$ or 1.

In the case of $k=4$, 5 and 6, we shall start from the bundle space $B_{m,0}^{(8,7)}$ having the characteristic map $m\rho\colon S^7\to SO(8)$ which appeared in Theorem 1, where $\rho\colon S^7\to SO(7)\subset SO(8)$ is the map in the proof of Theorem 1. We can see that $B_m^{(8,6)}$ is a G_2-invariant subbundle of $B_{m,0}^{(8,7)}$ for the relation $G_2\subset O(7)$. By the exact sequence

$$\pi_7(SO(6))\longrightarrow\pi_7(SO(7))\longrightarrow\pi_7(S^6)\approx Z_2\longrightarrow\pi_6(SO(6))=0,$$

the homotopy class of the characteristic map $2m\rho_7$ can be reduced to the homotopy class $\{m\rho_6\}$, where ρ_6 is a representative of the generator of $\pi_7(SO(6))\approx Z$. Therefore the bundle $B_m^{(8,5)}$ is a $G_2\cap O(6)$-invariant subbundle of $B_{2m}^{(8,6)}$. Finally, by the exact sequence

$$\pi_7(SO(5))\longrightarrow\pi_7(SO(6))\longrightarrow\pi_7(S^5)\approx Z_2\longrightarrow\pi_6(SO(5))=0,$$

and the relation $G_2\cap O(5)=SU(2)$ (cf. [9]), we see that $B_m^{(8,4)}$ is an $SU(2)$-invariant subbundle of $B_{2m}^{(8,5)}$. Thus we have proved the proposition.

§3. S^{4s-1}-bundles over S^{4s}, $s\geqq 3$

Throughout this section, we assume that $n=4s$. Denote by ρ_{n-1} the generator of the infinite cyclic group $\pi_{n-1}(SO(n-1))$. The next lemma is well known, but let us give a proof, because we shall use it later.

LEMMA (*see* [4]). *The group* $\pi_{n-1}(SO(n))\approx Z+Z$ is generated by elements ρ_n *and* τ_n, *where* ρ_n *is the image of the generator* $\rho_{n-1}\in\pi_{n-1}(SO(n-1))$ *by the homomorphism induced from the inclsusion map* $j^{(n,n-1)}: SO(n-1)\to SO(n)$ and τ_n is the homotopy class of the characteristic map of the tangent bundle of S^n.

PROOF. By the exact sequence

$$\pi_{n-1}(SO(n-1))\longrightarrow\pi_{n-1}(SO(n+1))\approx Z\longrightarrow\pi_{n-1}(V_{n+1,2})\approx Z_2\longrightarrow$$
$$\pi_{n-2}(SO(n-1))\approx Z_2\longrightarrow\pi_{n-2}(SO(n+1))=0,$$

we have an isomorphism $j_*^{(n+1,n)}\circ j_*^{(n,n-1)}: \pi_{n-1}(SO(n-1))\to\pi_{n-1}(SO(n+1))$. By the exact sequence

$$\pi_n(SO(n+1))=0\longrightarrow\pi_n(S^n)\approx Z\xrightarrow{\Delta}\pi_{n-1}(SO(n))\longrightarrow\pi_{n-1}(SO(n+1))\longrightarrow 0,$$

and the relation $\Delta(\iota_n)=\tau_n$, we have the lemma.

Now we denote by $B_{l,m}$ the total space of S^{n-1}-bundle over S^n having the homotopy class $l\rho_n+m\tau_n$ of the characteristic map, where m, n are integers.

PROPOSITION 8. *The spaces* $B_{l,0}$ *and* $B_{0,m}$ *yields* S^1*-actions for arbitrary* l, m.

PROOF. By Satz of 6.4 in [2], $B_{0,m}$ *yields an* $O(n-1)$-action. The long exact sequence

$$\pi_{n-1}(SO(n-2))\longrightarrow\pi_{n-1}(SO(n-1))\longrightarrow\pi_{n-1}(S^{n-2})\longrightarrow\pi_{n-2}(SO(n-2))\longrightarrow$$
$$\pi_{n-2}(SO(n-1))\longrightarrow\pi_{n-2}(S^{n-2}),$$

is equal to the exact sequence

$$Z\longrightarrow Z\longrightarrow Z_2\longrightarrow Z_4\longrightarrow Z_2\longrightarrow Z.$$

Hence the homomorphism $j_*^{(n-1,n-2)}: \pi_{n-1}(SO(n-2))\to\pi_{n-1}(SO(n-1))$ is in fact an isomorphism. Denote by ρ_{n-2} the preimage of ρ_{n-1}. By the proof of the lemma $j_*^{(n,n-1)}: \pi_{n-1}(SO(n-1))\to\pi_{n-1}(SO(n))$ is a monomorphism. Then we have the monomorphism $j_*^{(n,n-1)}\circ j_*^{(n-1,n-2)}: \pi_{n-1}(SO(n-2))\to\pi_{n-1}(SO(n))$, where the image is generated by ρ_n. Therefore the characteristic map of the bundle $B_{l,0}$ can be reduced to a map of S^{n-1} into $SO(n-2)$. By the inclusion map i: $SO(n-2)\times SO(2)\to SO(n)$, we have the required action.

Next we discuss some reducibility of characteristic maps of S^{n-1} into unitary groups.

PROPOSITION 9. *The bundle space* $B_{\varepsilon k, k(2s-1)!/2}$ *yields an* S^1*-action for an arbitrary integer* k, *where* $\varepsilon=1$ *if* s *is odd,* $\varepsilon=2$ *if* s *is even.*

PROOF. Consider the commutative diagram (cf. [7]),

$$\begin{array}{ccc} \pi_{4s-1}(U(2s)) & \longrightarrow & \pi_{4s-1}(U(2s+1)) \\ \downarrow{\scriptstyle (j_{2s})_*} & & \downarrow{\scriptstyle (j_{2s+1})_*} \\ \pi_{4s-1}(SO(4s)) & \longrightarrow & \pi_{4s-1}(SO(4s+2)), \end{array}$$

where each homomorphism in the diagram is induced by obvious inclusion maps. Denote by μ_{2s} and μ_{2s+1} the generators of $\pi_{4s-1}(U(2s))$ and $\pi_{4s-1}(U(2s+1))$ respectively. By the Bott isomorphism of $\pi_{4s-1}(\mathbf{O}/\mathbf{U})$ onto $\pi_{4s}(\mathbf{O})$ for the stable homotopy groups and the exact sequence of the stable homotopy groups associated to the fibering $\mathbf{U}\to\mathbf{O}\to\mathbf{O}/\mathbf{U}$,

$$\pi_{4s-1}(\mathbf{U}) \longrightarrow \pi_{4s-1}(\mathbf{O}) \longrightarrow \pi_{4s-1}(\mathbf{O}/\mathbf{U}) \longrightarrow \pi_{4s-2}(\mathbf{U})=0,$$

we have $(j_{2s+1})^*\mu_{2s+1}=\varepsilon\rho_{4s+2}$, where $\varepsilon=1$ if s is odd, $\varepsilon=2$ if s is enen, and ρ_{4s+2} is the image of ρ_{4s} under the map $j_*^{(4s+2,4s+1)}\circ j_*^{(4s+1,4s)}$. By Corollary of 23.5 in [6], the kernel of $j_*^{(4s+1,4s)}$ is generated by τ_{4s}. Thus $(j_{2s})_*\mu_{2s}=\varepsilon\rho_{4s}+a\tau_{4s}$ for some integer a. Consider the exact sequence $\pi_{4s-1}(SO(4s))\to\pi_{4s-1}(S^{4s-1})\to\pi_{4s-2}(SO(4s-1))\approx Z_2\to\pi_{4s-2}(SO(4s))=0$, and the exact sequence associated to the fibering $U(2s-1))\to U(2s)\to S^{4s-1}$, $\pi_{4s-1}(U(2s))\to\pi_{4s-1}(S^{4s-1})\to\pi_{4s-2}(U(2s-1))\approx Z_{(2s-1)!}\to\pi_{2s-2}(U(2s))$. Let p: $SO(4s)\to S^{4s-1}$ and p': $U(2s)\to S^{4s-1}$ be the projections. Then we have $p\circ j_{2s}=p'$, $p'_*(\mu_{2s})=(2s-1)!\tau_{4s-1}$ and $(j_{2s})_*\mu_{2s}=\varepsilon\rho_{4s}+((2s-1)!/2)\tau_{4s}$.

Appendix. Using differential topology, M. Davis has given an example of an $O(4s-1)$-action on $B_{2l,1}$, where l is an arbitrary integer (cf. Examples of actions on manifolds almost diffeomorphic to $V_{n+1,2}$, Springer, Lecture Notes in Math. 298 (1972)).

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