Mem. Fac. Sci., Shimane Univ.,
15, pp. 1–7 Dec. 20, 1981

On Homogeneous Systems IV

Michihiko KIKKAWA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 5, 1981)

A construction of enveloping groups of normal subsystems of a homogeneous system is treated. It is shown that a certain class of normal subsystems of an analytic homogeneous system G has their tangent Lie triple algebras each of which is an ideal of the Lie triple algebra of G. The concept of congruence is introduced related to normal subsystems of homogeneous systems.

§ 0. Introduction

In the previous papers [3], [4] and [5], we have investigated various algebraic and differential geometric properties concerning analytic homogeneous systems, their subsystems and their tangent Lie triple algebras. These results are applicable to homogeneous Lie loops and their tangent Lie triple algebras treated by us in [1] and [2]. In [4–III], we have introduced the concept of normal subsystems of homogeneous systems as a generalization of the concept of normal subgroups of (Lie) groups. In [5], we have obtained a decomposition theorem for analytic homogeneous systems as product homogeneous systems of their normal subsystems, by using the results concerning the decomposition of Lie triple algebras into direct sums of their ideals.

In this paper, we provide a construction of enveloping groups of normal subsystems by some groups of automorphisms of a homogeneous system G (Theorem 1). By using this result we find in § 2 a sufficient condition for normal subsystems of an analytic homogeneous system G on which their tangent Lie triple algebras are ideals of the tangent Lie triple algebra $\mathfrak{G}$ of G (Theorem 2). In § 3 we show that each normal subsystem H determines an equivalence relation R in G, called a congruence of G, and that the tangent Lie triple algebra of H is an ideal of $\mathfrak{G}$ if R is an invariant subsystem of the product homogeneous system $G \times G$ (Theorem 4).

§ 1. Enveloping groups of normal subsystems

Let (G, η) be an abstract homogeneous system with a base point e (cf. [3], [4]). The multiplication xy at e defined by

$$(1.1) \qquad xy = \eta(e, x, y) \quad \text{for} \quad x, y \in G$$

satisfies the following

$$(1.2)\qquad xe=ex=x,\quad x^{-1}x=x\,x^{-1}=e,\quad L_x^{-1}=L_{x^{-1}}$$

and

$$(1.3)\qquad L_{x,y}e=e,\quad L_{x,y}(zw)=(L_{x,y}z)\,(L_{x,y}w)$$

for $x, y, z, w\in G$, where $x^{-1}=\eta(x, e, e)$ is the inverse of x, L_x is the left translation by x and $L_{x,y}=L_{xy}^{-1}L_xL_y$ is the left inner mapping at e. This multiplication defines a homogeneous loop (cf. [1]) if and only if each of the right translations is a permutation of G. Since each displacement $\eta(x, y)\colon z\to\eta(x, y, z)$ satisfies $\eta(x, y)^{-1}=\eta(y, x)$, (1.2) implies

$$(1.4)\qquad \eta(e, x^{-1})=\eta(x, e),\quad \eta(x^{-1}, e)=\eta(e, x).$$

Set $\lambda_{x,y}=\eta(y, e)\eta(x, y)\eta(e, x)$ for $x, y\in G$. Then the following relations are obtained immediately from the definition of the homogeneous system (G, η) and (1.4):

$$(1.5)\qquad L_{x,y}=\lambda_{x,xy}=\lambda_{y,x^{-1}},\quad \lambda_{x,y}^{-1}=\lambda_{y,x}.$$

Let Λ_e denote the left inner mapping group of the multiplication at e, i.e., Λ_e is a group of permutations of G generated by all of the left inner mappings $\{L_{x,y};\ x, y\in G\}$, which is a subgroup of the group of automorphisms of the multiplication at e. By (1.5) Λ_e can be regarded as a subgroup of $A_e(\eta)$ generated by the set $\{\lambda_{x,y};\ x, y\in G\}$, where $A_e(\eta)$ is the isotropy subgroup of the automorphism group of (G, η) at e. If K_e is a subgroup of $A_e(\eta)$ containing Λ_e, then the product $A=G\times K_e$ is a group, called an *enveloping group* of G by K_e, under the multiplication

$$(1.6)\qquad (x, \alpha)\,(y, \beta)=(x\alpha(y),\ L_{x,\alpha(y)}\alpha\beta)$$

for $(x, \alpha), (y, \beta)\in G\times K_e$. The identity element of A is $(e, 1)$, and $(x, \alpha)^{-1}=(\alpha^{-1}(x^{-1}), \alpha^{-1})$.

Now, assume that H is a normal subsystem of G (cf. [4–III]), i.e., H is a subsystem of G satisfying

$$(1.7)\qquad \eta(xH, yH, zH)=\eta(x, y, z)H\qquad \text{for}\quad x, y, z\in G,$$

where $xH=\eta(H, x, H)$. By Lemma 1 of [4–III] H is an invariant subsystem of G, i.e.,

$$(1.8)\qquad \eta(x, y)xH=yH\qquad \text{for}\quad x, y\in G.$$

In the following, a base point e is chosen in H and the multiplication xy is always regarded as defined at e.

LEMMA 1. (1) $\eta(x, y)zH=\eta(x, y, z)H$,

(2) $L_{x,y}H=\lambda_{x,y}H=H$,

(3) $xH=L_xH,\ (xH)(yH)=(xy)H$

for $x, y, z\in G$.

PROOF. (2) is an immediate consequence of (1.8). (3) is obtained from $(xH)(yH)=\eta(e, xH, yH)\subset\eta(e, x, y)H=xyH$ and $xyH=\eta(e, x)yH\subset\eta(e, xH, yH)$ by using (1). We prove (1): From $\eta(e, x)yH\subset xyH$ and $yH=\eta(H, y, H)$ we get $\eta(xH, xy, xH)\subset\eta(H, xy, H)$, which implies

$$(1.9)\qquad \eta(xH, z, xH)\subset zH \quad \text{for} \quad x, z\in G.$$

Operating $\eta(e, x^{-1})$ on the both side of (1.9) we have $\eta(H, x^{-1}z, H)\subset\eta(x^{-1}H, x^{-1}z, x^{-1}H)$, which implies

$$(1.10)\qquad zH\subset\eta(xH, z, xH) \quad \text{for} \quad x, z\in G.$$

From (1.9) and (1.10) we get the equality

$$(1.11)\qquad zH=\eta(xH, z, xH) \quad \text{for} \quad x, z\in G.$$

Now, by using (1.11) and (1.8) we have (1) as follows;

$$\begin{aligned}\eta(x, y)zH&=\eta(x, y)\eta(xH, z, xH)\\&=\eta(yH, \eta(x, y, z), yH)\\&=\eta(x, y, z)H.\end{aligned}$$

q.e.d.

Let $\Lambda_e(H)$ denote the subgroup of Λ_e generated by the set $\{\lambda_{x,h};\ x\in G,\ h\in H\}$, which can be also regarded as the subgroup generated by the set $\{\lambda_{h,x};\ h\in H,\ x\in G\}$ or by the set of left inner mappings $\{L_{x,h};\ x\in G,\ h\in H\}$, since $\lambda_{x,h}^{-1}=\lambda_{h,x}$ and $L_{x,h}=\lambda_{h,x^{-1}}$.

LEMMA 2. *If each element of the group* K_e *in the construction of the enveloping group of* G *preserves the normal subsystem* H, *then* $\Lambda_e(H)$ *is a normal subgroup of* K_e.

PROOF. For $\alpha\in K_e$ and $L_{x,h}\in\Lambda_e(H)$ we have

$$\begin{aligned}\alpha\lambda_{x,h}&=\alpha\eta(h, e)\eta(x, h)\eta(e, x)\\&=\eta(\alpha(h), e)\eta(\alpha(x), \alpha(h))\eta(e, \alpha(x))\\&=\lambda_{\alpha(x),\alpha(h)}\alpha.\end{aligned}$$

q.e.d.

LEMMA 3. $\sigma(xH)=xH$ *for any* $\sigma\in\Lambda_e(H)$ *and* $x\in G$.

PROOF. If x and y belong to G and h belongs to H, then we get

$$\begin{aligned}\lambda_{y,h}(xH)&=\eta(h, e)\eta(y, h)\eta(e, y)\,(xH)\\&\subset\eta(H, H)\eta(yH, H)\eta(H, yH)\,(xH)\\&=\eta(e, e)\eta(y, e)\eta(e, y)xH \quad \text{by (1) of Lemma 1}\\&=xH.\end{aligned}$$

In the same way we have $\lambda_{h,y}(xH)\subset xH$. Since $\lambda_{y,h}^{-1}=\lambda_{h,y}$ we have $\sigma(xH)=xH$ for all $\sigma\in A_e(H)$. q.e.d.

About enveloping groups of normal subsystems we have the following;

THEOREM 1. *Let H be a normal subsystem of a homogeneous system (G, η) and K_e be a subgroup of $A_e(\eta)$ as above. Assume that each element of K_e preserves the subsystem H. If K'_e is a normal subgroup of K_e containing $A_e(H)$ and preserving each xH, $x\in G$, then $B=H\times K'_e$ is a normal subgroup of the enveloping group $A=G\times K_e$ of G by K_e.*

PROOF. For any elements $(x, \alpha)\in A$ and $(h, \sigma)\in B$, the G-component of $(x, \alpha)(h, \sigma)(x, \alpha)^{-1}$ in A is equal to $(x\alpha(h))\,(L_{x,\alpha(h)}\alpha\sigma\alpha^{-1}(x^{-1}))$. By Lemma 3 and the assumption of the theorem we have $L_{x,\alpha(h)}\alpha\sigma\alpha^{-1}(x^{-1})\in x^{-1}H$. Then, from (3) of Lemma 1 we obtain

$$(x\alpha(h))\,(L_{x,\alpha(h)}\alpha\sigma\alpha^{-1}(x^{-1}))=(xh')\,(x^{-1}k')=k\in H,$$

for some h', $k'\in H$. By using these elements, the K_e-component of $(x, \alpha)(h, \sigma)(x, \alpha)^{-1}$ is expressed as $L_{xh',x^{-1}k'}L_{x,h'}\alpha\sigma\alpha^{-1}$. Since $L_{xh',x^{-1}k'}=\lambda_{xh',(xh')(x^{-1}k')}=\lambda_{xh',k}$ holds by (1.5), we have $(x, \alpha)(h, \sigma)(x, \alpha)^{-1}\in H\times K'_e=B$. q.e.d.

§2. Normal subsystems of analytic homogeneous systems

In this section we assume that (G, η) is an analytic homogeneous system on a connected analytic manifold G and we consider analytic subsystems and analytic mappings and automorphisms in G. The notations and results concerning analytic homogeneous systems and their tangent Lie triple algebras are referred to [4] and [5].

Now, let H be a connected normal subsystem of (G, η) and choose a base point e in H. Denoting by K_e the closure of the left inner mapping group A_e in the affine transformation group of the canonical connection of (G, η), we can construct a Lie group $A=G\times K_e$ as an enveloping group of G considered in §1, and in this case we can regard G as the reductive homogeneous space A/K_e of K. Nomizu [7] with the canonical connection of the 2nd kind, under the canonical decomposition of the Lie algebra $\mathfrak{A}$ of A as $\mathfrak{A}=\mathfrak{G}+\mathfrak{K}$, where $\mathfrak{K}$ is the Lie algebra of K_e and $\mathfrak{G}$ is the tangent Lie triple algebra of G at e. Let $A_e(H)$ be the subgroup of the (analytic) automorphism group of (G, η) defined in §1. By Lemma 2 and Lemma 3, $A_e(H)$ is a normal subgroup of the Lie group K_e and it preserves each xH, $x \in G$. Denote by $L_e(H)$ the normal subgroup of K_e consisting of all elements $\sigma \in K_e$ preserving each xH, $x \in G$.

THEOREM 2. *Let H be a normal subsystem of an analytic homogeneous system (G, η), $A_e(H)$ and $L_e(H)$ be (abstract) normal subgroups of the Lie group $K_e = \overline{A_e}$ defined above with respect to the base point e. Assume that there exists a Lie subgroup K'_e of K_e such that $A_e(H) \subset K'_e \subset L_e(H)$. Then, the tangent Lie triple algebra $\mathfrak{H}$ of H is an ideal of the tangent Lie triple algebra $\mathfrak{G}$ of G.*

PROOF. From Theorem 1 it follows that the enveloping group $B = H \times K'_e$ of H by K'_e is a normal subgroup of the enveloping group $A = G \times K_e$ and, as a matter of fact, B is a Lie subgroup of A. Let $\mathfrak{B}$ and $\mathfrak{K}'$ be the Lie subalgebras of $\mathfrak{A}$ corresponding to the Lie subgroups B and $K'_e \cong \{e\} \times K'_e$, respectively. Then, $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ and it is decomposed as $\mathfrak{B} = \mathfrak{H} + \mathfrak{K}'$, i.e., $\mathfrak{H}$ is the $\mathfrak{G}$-component and $\mathfrak{K}'$ is the $\mathfrak{K}$-component of $\mathfrak{B}$, respectively. Therefore, we get

$$(2.1) \qquad [\mathfrak{G}, \mathfrak{H}] \subset \mathfrak{B} \quad \text{and} \quad [\mathfrak{K}', \mathfrak{G}] \subset \mathfrak{B} \cap \mathfrak{G} = \mathfrak{H}.$$

Since the bracket operation of $\mathfrak{A}$ is defined by

$$[X, Y] = XY + D(X, Y),\ [U, X] = UX \qquad \text{for} \quad X, Y \in \mathfrak{G},\ U \in \mathfrak{K},$$

where XY and $D(X, Y)$ are respectively the bilinear multiplication and the inner derivation of the Lie triple algebra $\mathfrak{G}$, (2.1) implies $\mathfrak{G}\mathfrak{H} \subset \mathfrak{H}$, $D(\mathfrak{G}, \mathfrak{H}) \subset \mathfrak{K}'$ and $D(\mathfrak{G}, \mathfrak{H})\mathfrak{G} \subset [\mathfrak{K}', \mathfrak{G}] \subset \mathfrak{H}$. Thus, it is shown that the Lie triple subalgebra $\mathfrak{H}$ is an ideal of the Lie triple algebra $\mathfrak{G}$ of G. q.e.d.

REMARK 1. If H is a closed normal subsystem of G, then $\overline{A_e(H)} \subset L_e(H)$ and the theorem holds for $K'_e = \overline{A_e(H)}$, that is, the theorem above involves the result in the previous half of Theorem 3 in [4–III].

REMARK 2. Suppose that G is a homogeneous system of a Lie group with the identity element e. Then, the subsystem H containing e is normal if and only if H is a normal subgroup of G. In this case, $A_e = K_e = \{1\}$ is the trivial Lie subgroup and $\mathfrak{G}$ is reduced to Lie algebra. Therefore, for the homogeneous system of a Lie group G, the theorem is reduced to the well-known theorem: The Lie algebra of a normal Lie subgroup of G is an ideal of the Lie algebra $\mathfrak{G}$ of G. Thus, we see that the theorem contains the results for non closed normal subsystems.

§3. Congruences of homogeneous systems

Let (G, η) be again an abstract homogeneous system. A *congruence* $R \subset G \times G$ of G is an equivalence relation in G such that $(\eta(x, y, z), \eta(x', y', z')) \in R$ whenever

(x, x'), (y, y') and (z, z') belong to R, i.e., R is a subsystem of the product homogeneous system $G\times G$.

THEOREM 3. *Let (G, η) be a homogeneous system. If $R\subset G\times G$ is a congruence of G, then each congruence class H in G is a normal subsystem of G. Conversely, if H is a normal subsystem of G, then $R=\{(x, x')\in G\times G;\ x'\in xH\}$ is a congruence of the homogeneous system G.*

PROOF. Suppose that R is a congruence of G and let H be a congruence class containing an element e. Then, for any $h, k, m\in H$ we get $(e, \eta(h, k, m))=(\eta(e, e, e), \eta(h, k, m))\in R$ and we see that H is a subsystem of G. Moreover, for any $x'=\eta(h, x, k)$ in $xH=\eta(H, x, H)$ we have $(x, x')=(\eta(e, x, e), \eta(h, x, k))\in R$ and we see that xH is contained in the congruence class of x. On the other hand, if $(x, x')\in R$, then $(\eta(x, e, x), \eta(x, e, x'))\in R$ and $\eta(x, e, x')\in H$ since $e=\eta(x, e, x)$. Thus we see that the congruence class of x is xH, and obtain $\eta(xH, yH, zH)\subset\eta(x, y, z)H$. If $w\in\eta(x, y, z)H$, then $(\eta(x, y, z), w)\in R$ which implies that $\eta(y, x, w)$ is contained in zH, i.e., $w\in\eta(x, y)zH\subset\eta(xH, yH, zH)$. Thus we have $\eta(xH, yH, zH)=\eta(x, y, z)H$ for any $x, y, z\in H$. It is easy to show the converse part of the theorem and we omit the proof of it. q.e.d.

Now, we assume that (G, η) is a regular analytic homogeneous system (cf. [4–I]). By Theorem 3 above, we see that any normal subsystem H of G determines a congruence R of homogeneous system. From Theorem 2 we obtain the following:

THEOREM 4. *Let (G, η) be a regular analytic homogeneous system on a connected analytic manifold G and H a normal subsystem of G. The tangent Lie triple algebra $\mathfrak{H}$ of H at e is an ideal of the tangent Lie triple algebra $\mathfrak{G}$ of G if the congruence R determined by H is an invariant (analytic) subsystem of the product homogeneous system $G\times G$.*

PROOF. Let α be an element of $A_e(\eta)$, i.e., an analytic automorphism of G leaving $e\in H$ fixed. Then, $1\times\alpha$: $G\times G\to G\times G$ is an analytic automorphism of the product homogeneous system $G\times G$, and α preserves each xH, $x\in G$, if and only if $1\times\alpha$ preserves R. Since $1\times\alpha$ is an affine transformation of the canonical connection of $G\times G$ and since $(1\times\alpha)(e, e)=(e, e)$, it is determined uniquely by its differential $d(1\times\alpha)$: $\mathfrak{G}+\mathfrak{G}\to\mathfrak{G}+\mathfrak{G}$ at (e, e), where $\mathfrak{G}$ is the tangent Lie triple algebra of G at e. By assumption, R is an invariant subsystem of $G\times G$. Hence R is an autoparallel submanifold of $G\times G$ with respect to the canonical connection (cf. [4–I]). Therefore, for $\alpha\in A_e(\eta)$, $1\times\alpha$ preserves R if and only if its differential $d(1\times\alpha)$ preserves the tangent Lie triple algebra $\mathfrak{R}$ of R at (e, e). By Lemma 3, every element of the group $A_e(H)$ preserves each xH, $x\in G$. Thus $d(1\times\alpha)$ preserves $\mathfrak{R}$ for any $\alpha\in\overline{A_e(H)}$ in the Lie group $K_e=\overline{A_e}$, i.e., $\overline{A_e(H)}$ is a Lie subgroup of K_e contained in L_e. The hypothesis of Theorem 2 is satisfied by $K'_e=\overline{A_e(H)}$ and the theorem follows. q.e.d.

REMARK 3. O. Loos [8] introduced the concept of congruence of symmetric spaces as follows: An equivalence relation R in a symmetric space G is a *congruence* if R is a symmetric subspace of the product symmetric space $G\times G$. If a regular homogeneous system (G, η) is symmetric, i.e., the map S_x: $G\to G$ defined by $S_x y = \eta(y, x, x)$ is an automorphism of (G, η) for each $x\in G$, then the theorem mentioned above is reduced to the results for the symmetric space G which is the part a) of proposition 2.1 in [8, p. 131]. In fact, if a congruence R of a symmetric homogeneous system (G, η) is an invariant subsystem of the product symmetric homogeneous system $G\times G$, then R defines a congruence of the symmetric space G in the sense of O. Loos.

References

[1] M. Kikkawa, Geometry of homogeneous Lie loops, Hiroshima Math. J., **5** (1975), 141–179.

[2] ———, A note on subloops of a homogeneous Lie loop and subsystems of its tangent Lie triple algebra, Hiroshima Math. J., **5** (1975), 439–446.

[3] ———, On the left translations of homogeneous loops, Mem. Fac. Lit. & Sci., Shimane Univ. Nat. Sci., **10** (1976), 19–25.

[4] ———, On homogeneous systems I, II, III, Mem. Fac. Lit. & Sci., Shimane Univ. Nat. Sci., **11** (1977), 9–17; Mem. Fac. Sci. Shimane Univ., **12** (1978), 5–13; Mem. Fac. Sci. Shimane Univ., **14** (1980), 41–46.

[5] ———, On the decomposition of homogeneous systems with nondegenerate Killing-Ricci tensor, Hiroshima Math. J. **11** (1981), to appear.

[6] ———, On the Killing-Ricci forms of Lie triple algebras, Pacific J. Math., **96** (1981), 153–161.

[7] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math., **76** (1954), 33–65.

[8] O. Loos, Symmetric Spaces I, Benjamin 1969.