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Note on the Construction of Regular $*$-Semigroups

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To Kentaro MURATA
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In this paper, we shall show how a general regualr $*$-semigroup can be constructed from a fundamental regular $*$-semigroup and a certain partial groupoid.

§ 1. Introduction

A regular $*$-semigroup is a regular semigroup S equipped with a unary operation$*$: $S \to S$ satisfying the following three axioms:

(1) $xx^*x = x$ for $x \in S$,

(2) $(x^*)^* = x$ for $x \in S$,

(3) $(xy)^* = y^*x^*$ for $x, y \in S$,

(see [1]).

An element x of S is called *a projection* if $x^2 = x$ and $x^* = x$. Hereafter, we shall call a unary operation$*$: $S \to S$ satisfying (1)–(3) above *a $*$-operation* in S. Let S be a regular semigroup, and E_S the set of idempotents of S. A subset F of E_S is called *a p-system* if

(1) for any $a \in S$, there exists a unique inverse a^* of a such that both aa^* and a^*a are contained in F,

(2) $a^*Fa \subset F$ for any $a \in S$, where $*$ is a unary operation determined by (1),

(3) $F^2 \subset E_S$.

In the previous paper [2], it has been shown that a regular semigroup becomes a regular $*$-semigroup if and only if it has at least one p-system. In this paper, all the notations and terminology should be referred to [2] and [3], unless otherwise stated.

§ 2. $*$-regular product

Let Γ be a fundamental regular $*$-semigroup, and E_Γ the set of idempotents of Γ. Let $*$ be a $*$-operation in Γ. Let F_Γ be the set of projections of Γ with respect to the

$*$-operation $*$. Of course, $F_\Gamma \subset E_\Gamma$[1]. Let $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ be a disjoint sum of groups $\{S_\lambda: \lambda\in E_\Gamma\}$ such that

(C.0) (1) M is a partial groupoid, and each S_λ is a subgroup of M,

(2) if $\alpha, \beta\in E_\Gamma$, $\alpha\beta\in E_\Gamma$, $x\in S_\alpha$ and $y\in S_\beta$, then xy is well defined in M and $xy\in S_{\alpha\beta}$; that is, $S_\alpha S_\beta\subset S_{\alpha\beta}$,

(3) if $\alpha_1, \alpha_2,\dots, \alpha_n\in E_\Gamma$, $\alpha_1\alpha_2\dots\alpha_n\in E_\Gamma$ and $x_i\in S_{\alpha_i}$ for all $i=1, 2,\dots, n$, then all the possible products of $x_1, x_2,\dots, x_n$ (associated with the binary operation in M) taken in this order take the same value (element) contained in $S_{\alpha_1\alpha_2\dots\alpha_n}$.[2] We denote it by $x_1x_2\dots x_n$.

(4) if $\lambda, \tau\in F_\Gamma$ (accordingly, of course $\lambda\tau\in E_\Gamma$), then $e_\lambda e_\tau=e_{\lambda\tau}$, where e_α is the identity of S_α.

Put $\cup\{S_\tau: \tau\in F_\Gamma\}=N$. Of course, N is a partial subgroupid of M. A mapping $\sigma: N\to N$ is called *a local endomorphism* (ℓ-endomorphism) on N if it satisfies the following:

(C.1) For any $\tau\in F_\Gamma$, $S_\tau\sigma\subset S_\xi$ for some $\xi\in F_\Gamma$; and $\sigma|S_\tau$ (the restriction of σ to S_τ) is a homomorphism.

The set of ℓ-endomorphisms on N forms a semigroup with respect to the resultant composition. We denote it by $\mathscr{L}\mathscr{E}(N)$.

Now, let $\psi: \Gamma\to\mathscr{L}\mathscr{E}(N)$ and $\phi: \Gamma\times\Gamma\to N$ be mappings such that

(C.2) (1) for any $\gamma\in\Gamma$ and $\tau\in F_{\Gamma'}$ $\gamma\psi=\bar\gamma$ maps S_τ into $S_{\gamma\tau(\gamma\tau)^*}$, and in particular $\bar\gamma$ maps $S_{\gamma^*\gamma}$ onto $S_{\gamma\gamma^*}$, where $*$ denotes the $*$-operation in Γ,

(2) $(\gamma, \delta)\phi=C(\gamma, \delta)\in S_{\gamma\delta(\gamma\delta)^*}$ for any $\gamma, \delta\in\Gamma$.

Assume that the family $\Delta=\{\bar\gamma; C(\gamma, \delta)\}_{\gamma,\delta\in\Gamma}$ satisfies the following:

(C.3) (1) $C(\lambda, \tau)=e_{\lambda\tau(\lambda\tau)^*}$ for all $\lambda, \tau\in F_\Gamma$, and $C(\gamma\gamma^*, \gamma)=C(\gamma, \gamma^*\gamma)=e_{\gamma\gamma^*}$ for all $\gamma\in\Gamma$,

(2) $C(\delta, \xi)^{\bar\gamma}C(\gamma, \delta\xi)=C(\gamma, \delta)C(\gamma\delta, \xi)$, where $x^{\bar\gamma}=x\bar\gamma$; further, $C(\lambda, \gamma)^{\bar\gamma^*}C(\gamma^*, \lambda\gamma)=C(\gamma^*, \lambda)C(\gamma^*\lambda, \gamma)=C(\gamma^*, \gamma)e_{\gamma^*\lambda\gamma}$ for $\lambda\in F_\Gamma$ and $\gamma\in\Gamma$,

(3) $\bar\gamma\bar\delta=\overline{\delta\gamma}\,\overline{C(\delta, \gamma)}$, where $\overline{C(\delta, \gamma)}$ is the mapping of N into N defined by $u\overline{C(\delta, \gamma)}=C(\delta, \gamma)uC(\delta, \gamma)^{-1}$ (where x^{-1} means the group inverse of x),

(4) $e_\lambda be_\lambda=b^{\bar\lambda}$ for $b\in S_\delta$, $\lambda, \delta\in F_\Gamma$ (especially, $e_\lambda b=b^{\bar\lambda}$ if $\lambda\delta\in F_\Gamma$),

(5) $C(\lambda\tau, \lambda\tau)=e_{\lambda\tau(\lambda\tau)^*}$ for $\lambda, \tau\in F_\Gamma$.

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1) F_Γ is the set $\{e\in E_\Gamma: e^*=e\}$.

2) For example, assume that $\alpha_1, \alpha_2, \alpha_3, \alpha_4\in E_\Gamma$, $\alpha_1\alpha_2\alpha_3\alpha_4\in E_\Gamma$ and $x_i\in S_i$ $(i=1, 2, 3, 4)$. Assume also that $(\alpha_1\alpha_2)\alpha_3$, $((\alpha_1\alpha_2)\alpha_3)\alpha_4$, $\alpha_1\alpha_2$, $\alpha_3\alpha_4\in E_\Gamma$ (hence $(\alpha_1\alpha_2)(\alpha_3\alpha_4)\in E_\Gamma$), then the corresponding $(x_1x_2)x_3$, $((x_1x_2)x_3)x_4$, x_1x_2, x_3x_4 and $(x_1x_2)(x_3x_4)$ are all well defined in M by (2) of (C. 0) and $((x_1x_3)x_4)x_1=(x_2x_2)(x_3x_4)$ follows from (3) of (C. 0).

In this case, $N\underset{\Delta}{\otimes}\Gamma=\{(a, \gamma): a\in S_{\gamma\gamma^*}, \gamma\in\Gamma\}$ becomes a regular $*$-semigroup under the multiplication and the $*$-operation defined as follows:

$$(a, \gamma)(b, \tau)=(ab^{\bar{\gamma}}C(\gamma, \tau), \gamma\tau),$$

$(a, \gamma)^*=(t, \gamma^*)$, where t is the element of $S_{\gamma^*\gamma}$ such that $t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1}$ (such t exists since $\bar{\gamma}$ is a mapping of $S_{\gamma^*\gamma}$ onto $S_{\gamma\gamma^*}$; and it is easy to see that t is unique).

In fact

THEOREM 2.1. (1) *$N\underset{\Delta}{\otimes}\Gamma$ is a regular semigroup having $F=\{(e_\lambda, \lambda): \lambda\in F_\Gamma\}$ as its p-system. Accordingly, $N\underset{\Delta}{\otimes}\Gamma$ is a regular $*$-semigroup* (*see* [2]).

(2) *Let $\sharp$ be the $*$-operation determined by F (see* [2]). *Then $\sharp=*$. Hence, F is the set of projections of $(N\underset{\Delta}{\otimes}\Gamma, *)$.*

(3) *The set of idemotents of $N\underset{\Delta}{\otimes}\Gamma$ is $E=\{(e_{\lambda\lambda^*}, \lambda): \lambda\in E_\Gamma\}$.*

PROOF. (1) Let (a, γ), (b, τ), $(c, \delta)\in N\underset{\Delta}{\otimes}\Gamma$. Then, $((a, \gamma)(b, \tau))(c, \delta)=(ab^{\bar{\gamma}}C(\gamma, \tau), \gamma\tau)(c, \delta)=(ab^{\bar{\gamma}}C(\gamma, \tau)c^{\overline{\gamma\tau}}C(\gamma\tau, \delta), \gamma\tau\delta)$......(A)

On the other hand, $(a, \gamma)((b, \tau)(c, \delta))=(a, \gamma)(bc^{\bar{\tau}}C(\tau, \delta), \tau\delta)=(a(bc^{\bar{\gamma}}C(\tau, \delta))^{\bar{\gamma}}\cdot C(\gamma, \tau\delta), \gamma\tau\delta)=(ab^{\bar{\gamma}}c^{\bar{\tau}\bar{\gamma}}C(\tau, \delta)^{\bar{\gamma}}C(\gamma, \tau\delta), \gamma\tau\delta)$......(B). Now, $c^{\bar{\tau}\bar{\gamma}}=c^{\overline{\gamma\tau}\,\overline{C(\gamma,\tau)}}=C(\gamma, \tau)c^{\overline{\gamma\tau}}\cdot C(\gamma, \tau)^{-1}$. Hence, $c^{\bar{\tau}\bar{\gamma}}C(\tau, \delta)^{\bar{\gamma}}C(\gamma, \tau\delta)=C(\gamma, \tau)c^{\overline{\gamma\tau}}C(\gamma, \tau)^{-1}C(\gamma, \tau)C(\gamma\tau, \delta)=C(\gamma, \tau)c^{\overline{\gamma\tau}}\cdot C(\gamma\tau, \delta)$. Thus, (A)=(B). That is, $N\underset{\Delta}{\otimes}\Gamma$ is a semigroup.

Next, $(a, \gamma)(a, \gamma)^*(a, \gamma)=(at^{\bar{\gamma}}C(\gamma, \gamma^*), \gamma\gamma^*)(a, \gamma)$ (where $t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1}=(at^{\bar{\gamma}}C(\gamma, \gamma^*)a^{\overline{\gamma\gamma^*}}C(\gamma\gamma^*, \gamma), \gamma)=(at^{\bar{\gamma}}C(\gamma, \gamma^*)a^{\overline{\gamma\gamma^*}}, \gamma)=(aa^{-1}C(\gamma, \gamma^*)^{-1}C(\gamma, \gamma^*)a, \gamma)=(a, \gamma)$. Hence, (a, γ) has an inverse in $N\underset{\Delta}{\otimes}\Gamma$. That is, $N\underset{\Delta}{\otimes}\Gamma$ is regular. Consider $F=\{(e_\lambda, \lambda): \lambda\in F_\Gamma\}$. For any $(a, \gamma)\in N\underset{\Delta}{\otimes}\Gamma$, $(a, \gamma)(a, \gamma)^*=(at^{\bar{\gamma}}C(\gamma, \gamma^*), \gamma\gamma^*)$ (where $t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1})=(aa^{-1}C(\gamma, \gamma^*)^{-1}C(\gamma, \gamma^*), \gamma\gamma^*)=(e_{\gamma\gamma^*}, \gamma\gamma^*)\in F$. On the other hand, $(a, \gamma)^*(a, \gamma)=(ta^{\bar{\gamma}^*}C(\gamma^*, \gamma), \gamma^*\gamma)$, where $t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1}$. Now, $t^{\bar{\gamma}\bar{\gamma}^*}=t^{\overline{\gamma^*\gamma}\,\overline{C(\gamma^*,\gamma)}}=C(\gamma^*, \gamma)t^{\overline{\gamma^*\gamma}}C(\gamma^*, \gamma)^{-1}=C(\gamma^*, \gamma)tC(\gamma^*, \gamma)^{-1}$. Hence, $t=C(\gamma^*, \gamma)^{-1}(a^{-1})^{\bar{\gamma}^*}(C(\gamma, \gamma^*)^{-1})^{\bar{\gamma}^*}\cdot C(\gamma^*, \gamma)=C(\gamma^*, \gamma)^{-1}a^{\bar{\gamma}^{*-1}}C(\gamma^*, \gamma)^{-1}C(\gamma^*, \gamma)$ (since $C(\gamma, \gamma^*)^{\bar{\gamma}^*}=C(\gamma^*, \gamma)$; in fact, $C(\gamma,\gamma^*)^{\bar{\gamma}^*}\cdot C(\gamma^*, \gamma\gamma^*)=C(\gamma^*, \gamma)C(\gamma^*\gamma, \gamma)$ (by (2) of (C.3)) implies $C(\gamma, \gamma^*)^{\bar{\gamma}^*}=C(\gamma^*, \gamma)=C(\gamma^*, \gamma)a^{\bar{\gamma}^{*-1}}$. Accordingly, $ta^{\bar{\gamma}^*}C(\gamma^*, \gamma)=C(\gamma^*, \gamma)^{-1}a^{\bar{\gamma}^{*-1}}a^{\bar{\gamma}^*}C(\gamma^*, \gamma)=C(\gamma^*, \gamma)^{-1}e_{\gamma^*\gamma}\cdot C(\gamma^*, \gamma)=C(\gamma^*, \gamma)^{-1}C(\gamma^*, \gamma)=e_{\gamma^*\gamma}$. Thus, $(a, \gamma)^*(a, \gamma)=(e_{\gamma^*\gamma}, \gamma^*\gamma)\in F$. Next, $((a, \gamma)^*)^*=(t, \gamma^*)^*$ (where $t^{\bar{\gamma}}=a^{-1}C(\gamma, \gamma^*)^{-1})=(d, \gamma)$ (where $d^{\bar{\gamma}^*}=t^{-1}C(\gamma^*, \gamma)^{-1}$). We obtain $t=C(\gamma^*, \gamma)^{-1}a^{\bar{\gamma}^{*-1}}$ as was shown above. Therefore, $d^{\bar{\gamma}^*}=a^{\bar{\gamma}^*}C(\gamma^*, \gamma)C(\gamma^*, \gamma)^{-1}=a^{\bar{\gamma}^*}$. Hence, $d^{\bar{\gamma}^*\bar{\gamma}}=a^{\bar{\gamma}^*\bar{\gamma}}$ implies $C(\gamma, \gamma^*)d^{\overline{\gamma\gamma^*}}C(\gamma, \gamma^*)^{-1}=C(\gamma, \gamma^*)a^{\overline{\gamma\gamma^*}}C(\gamma, \gamma^*)^{-1})$, whence $a=d$. Consequently, $((a, \gamma)^*)^*=(a, \gamma)$.

Suppose that (a, λ) is an idempotent. Since $(a, \lambda)^2=(a, \lambda)$, λ is an idempotent. Hence, there exist $\eta, \delta\in F_\Gamma$ such that $\eta\delta=\lambda$. $(a, \lambda)^2=(aa^{\bar{\lambda}}C(\lambda, \lambda), \lambda)=(aa^{\overline{\eta\delta}}e_{\lambda\lambda^*}, \lambda)$...(C). On the other hand, $a^{\bar{\delta}\bar{\eta}}=a^{\overline{\eta\delta}\,\overline{C(\eta,\delta)}}=C(\eta, \delta)a^{\overline{\eta\delta}}C(\eta, \delta)=e_{\lambda\lambda^*}a^{\bar{\lambda}}e_{\lambda\lambda^*}=a^{\bar{\lambda}}$.

Therefore, $a^{\bar{\lambda}}=e_\eta e_\delta a e_\delta e_\eta=e_\eta e_\delta e_{\eta\delta\eta} a e_{\eta\delta\eta} e_\delta e_\eta=e_{\eta\delta\eta} a e_{\eta\delta\eta}=a$. Thus, $(\mathrm{C})=(a^2 e_{\lambda\lambda^*}, \lambda)$. Since $(a, \lambda)^2=(a, \lambda)$, $a^2 e_{\lambda\lambda^*}=a$, whence $a=e_{\lambda\lambda^*}$. Therefore, the set of idempotents of $N \underset{\Delta}{\otimes} \Gamma$ is $E=\{(e_{\lambda\lambda^*}, \lambda): \lambda \in E_\Gamma\}$, (it is easy to see that each $(e_{\lambda\lambda^*}, \lambda)$ is an idempotent). Let (e_λ, λ), $(e_\tau, \tau) \in F$. Then, $(e_\lambda, \lambda)(e_\tau, \tau)=(e_\lambda e_\tau^{\bar{\lambda}} C(\lambda, \tau), \lambda\tau)=(e_\lambda e_{\lambda\tau\lambda}, \lambda\tau)=(e_{\lambda\tau(\lambda\tau)^*}, \lambda\tau) \in E$. Hence, $F^2 \subset E$. To prove that F is a p-system of $N \underset{\Delta}{\otimes} \Gamma$, we shall next show that $(a, \gamma)^*(e_\lambda, \lambda)(a, \gamma) \in F$ for $(e_\lambda, \lambda) \in F$ and $(a, \gamma) \in N \underset{\Delta}{\otimes} \Gamma$. Now, $(a, \gamma)^*(e_\lambda, \lambda)(a, \gamma)=(t, \gamma^*)(e_\lambda a C(\lambda, \gamma), \lambda\gamma)$ (where $t^{\bar{\gamma}}=a^{-1} C(\gamma, \gamma^*)^{-1})=(t e_\lambda^{\bar{\gamma}^*} a^{\bar{\gamma}^*} C(\lambda, \gamma)^{\bar{\gamma}^*} C(\gamma^*, \lambda\gamma), \gamma^*\lambda\gamma)$...... (D).

Since $(t, \gamma^*)^*=(a, \gamma)$ (as was shown above), we have $a^{\bar{\gamma}^*}=t^{-1} C(\gamma^*, \gamma)^{-1}$. Hence, $(\mathrm{D})=(t e_{\gamma^*\lambda\gamma} t^{-1} C(\gamma^*, \gamma)^{-1} C(\lambda, \gamma)^{\bar{\gamma}^*} C(\gamma^*, \lambda\gamma), \gamma^*\lambda\gamma)=(t e_{\gamma^*\lambda\gamma} t^{-1} e_{\gamma^*\lambda\gamma}, \gamma^*\lambda\gamma)$ (by (2) of (C.3)) $=(t t^{-1} e_{\gamma^*\lambda\gamma}, \gamma^*\lambda\gamma)=(e_{\gamma^*\gamma} e_{\gamma^*\lambda\gamma}, \gamma^*\lambda\gamma)=(e_{\gamma^*\lambda\gamma}, \gamma^*\lambda\gamma) \in F$. Thus, F is a p-system of $N \underset{\Delta}{\otimes} \Gamma$. It is easy to see that $\# = *$.

LEMMA 2.2. *The partial subgroupoid* $\bar{N}=\{(a, \lambda): \lambda \in F_\Gamma, a \in S_\lambda\}$ *of* $N \underset{\Delta}{\otimes} \Gamma$ *is isomorphic to the partial groupoid* N.

PROOF. Define $\psi: N \to \bar{N}$ by $a\psi=(a, \lambda)$ if $a \in S_\lambda$. It is obvious that ψ is bijective. Suppose that $a \in S_\lambda$, $b \in S_\delta$, $ab \in S_{\lambda\delta}$ and $\lambda, \delta, \lambda\delta \in F_\Gamma$. Then, $(ab)\psi=(ab, \lambda\delta)$. On the other hand, $(a\psi)(b\psi)=(a, \lambda)(b, \delta)=(ab^{\bar{\lambda}} C(\lambda, \delta), \lambda\delta)=(a e_\lambda b e_{\lambda\delta}, \lambda\delta)=(ab, \lambda\delta) \in \bar{N}$. Hence, $(a\psi)(b\psi)$ is well defined in $\bar{N}$, and $(ab)\psi=(a\psi)(b\psi)$. Conversely, suppose that $(a, \lambda)(b, \delta)$ is well defined in $\bar{N}$. Then, $(a, \lambda)(b, \delta)=(ab, \lambda\delta)$ implies that ab is well defined in N and $((a, \lambda)(b, \delta))\psi^{-1}=ab=(a, \lambda)\psi^{-1}(b, \delta)\psi^{-1}$. Hence, N is isomorphic to $\bar{N}$.

LEMMA 2.3. *Let* μ *be the maximum idempotent separating congruence on* $N \underset{\Delta}{\otimes} \Gamma$. *Then,* $N \underset{\Delta}{\otimes} \Gamma/\mu$ *is isomorphic to* Γ.

PROOF. Since Γ is a fundamental regular $*$-semigroup, it is obvious that $\mu=\{((a, \gamma), (b, \gamma)): a, b \in S_{\gamma\gamma^*}, \gamma \in \Gamma\}$. Hence, of course $N \underset{\Delta}{\otimes} \Gamma/\mu \cong \Gamma$.

Hereafter, we shall denote N above by $N_M(F_\Gamma)$, and call $N_M(F_\Gamma) \underset{\Delta}{\otimes} \Gamma$ *the $*$-regular product of* $N_M(F_\Gamma)$ *and* Γ *determined by the factor set* $\Delta=\{\bar{\gamma}, C(\gamma, \delta)\}_{\gamma,\delta \in \Gamma}$ *belonging to* $\{N_M(F_\Gamma), \Gamma\}$.

§3. A structure theorem

Next, let S be a regular $*$-semigroup and μ the maximum idempotent separating congruence. Then, $S/\mu=\Gamma$ is a fundamental regular $*$-semigroup, and the natural homomorphism $\xi: S \to S/\mu$ gives a $*$-homomorphism (see [3]) (hence, a $*$-operation $*$ in Γ can be defined by $(a\xi)^*=a^\#\xi$, where $\#$ is a $*$-operation in S). Further, it is obvious that $\lambda\xi^{-1}=S_\lambda$ is a subgroup of S for each $\lambda \in E_\Gamma$. Hence, $M=\cup\{S_\lambda: \lambda \in E_\Gamma\}$ (where E_Γ is the set of idempotents of Γ) is a partial subgroupoid of S and satisfies (C.0).

Let $N_M(F_\Gamma)=\cup\{S_\lambda: \lambda\in F_\Gamma\}$, where F_Γ is the set of projections of Γ, that is, $F_\Gamma=\{\tau\in E_\Gamma: \tau^*=\tau\}$. For any $\gamma\in\Gamma$, let $\gamma\xi^{-1}=S_\gamma$. Since $\gamma\gamma^*\in F_\Gamma$, $S_{\gamma\gamma^*}\subset N_M(F_\Gamma)$. For $\lambda\in E_\Gamma$, let e_λ be the identity of S_λ. Let x_γ be a representative of S_γ for each $\gamma\in\Gamma$, especially $x_\lambda=e_\lambda$ for each $\lambda\in E_\Gamma$. Then, clearly $S_{\gamma\gamma^*}x_\gamma\subset S_\gamma$. Conversely, for any $y\in S_\gamma$, we have $yx_\gamma^\#\in S_{\gamma\gamma^*}$, whence $yx_\gamma^\# x_\gamma\in S_{\gamma\gamma^*}x_\gamma$. Thus, $S_{\gamma\gamma^*}x_\gamma=S_\gamma$. Now, for any $y\in S_\gamma$ there exists a unique $u\in S_{\gamma\gamma^*}$ such that $ux_\gamma=y$ (the uniqueness of u is obvious).

For any $ux_\gamma\in S_\gamma$ (where $u\in S_{\gamma\gamma^*}$) and $vx_\delta\in S_\delta$ (where $v\in S_{\delta\delta^*}$), $ux_\gamma vx_\delta=wx_\gamma x_\delta$ for some $w\in S_{\gamma\delta(\gamma\delta)^*}$. Since $ux_\gamma vx_\delta x_\delta^\# x_\gamma^\#=wx_\gamma x_\delta(x_\gamma x_\delta)^\#$, it follows that $w=ux_\gamma vx_\gamma^\#$. Hence, $ux_\gamma vx_\delta=uv^{\bar\gamma}x_\gamma x_\delta$, where $v^{\bar\gamma}=x_\gamma vx_\gamma^\#$. Put $x_\gamma x_\delta=C(\gamma, \delta)x_{\gamma\delta}$, where $C(\gamma, \delta)\in S_{\gamma\delta(\gamma\delta)^*}$.

Then,

$$ux_\gamma vx_\delta=uv^{\bar\gamma}C(\gamma, \delta)x_{\gamma\delta} \quad \text{for} \quad u\in S_{\gamma\gamma^*},\ v\in S_{\delta\delta^*}. \tag{C.4}$$

Now, it is easy to verify that $\Delta=\{\bar\gamma, C(\gamma, \delta)\}_{\gamma,\delta\in\Gamma}$ satisfies the condition (C.3). Therefore, we can consider the $*$-regular product $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ determined by Δ. That is,

$$N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma=\{(a, \gamma): a\in S_{\gamma\gamma^*},\ \gamma\in\Gamma\}, \tag{C.5}$$

$$(a, \gamma)(b, \delta)=(ab^{\bar\gamma}C(\gamma, \delta), \gamma\delta),$$

$$(a, \gamma)^*=(t, \gamma^*), \text{ where } t^{\bar\gamma}=a^{-1}C(\gamma, \gamma^*)^{-1}.$$

Then,

LEMMA 3.1. *S is $*$-isomorphic to $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$.*

PROOF. Define $\psi: S\to N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$ by $x\psi=(u, \eta)$ if $x=ux_\eta$, $u\in S_{\eta\eta^*}$. It is obvious that ψ is bijective. For any $x=ux_\gamma$, $y=vx_\delta$, where $u\in S_{\gamma\gamma^*}$, $v\in S_{\delta\delta^*}$, $ux_\gamma vx_\delta=uv^{\bar\gamma}C(\gamma, \delta)x_{\gamma\delta}$. Hence, $(xy)\psi=(uv^{\bar\gamma}C(\gamma, \delta), \gamma\delta)=(u, \gamma)(v, \delta)=(x\psi)(y\psi)$. This implies that ψ is an isomorphism. Let $a=ux_\gamma$, $u\in S_{\gamma\gamma^*}$. Then, $a^\#=x_\gamma^\# u^\#=vx_{\gamma^*}$, $v\in S_{\gamma^*\gamma}$. Now, $vx_{\gamma^*}x_\gamma=x_\gamma^\# u^\# x_\gamma$ implies $vC(\gamma^*, \gamma)=x_\gamma^\# u^{-1}x_\gamma$. By (2) of (C.3), $u^{-1}=x_\gamma vC(\gamma^*, \gamma)x_\gamma^\#$, that is, $u^{-1}=v^{\bar\gamma}C(\gamma^*, \gamma)^{\bar\gamma}=v^{\bar\gamma}C(\gamma, \gamma^*)$. Since $(v, \gamma^*)=(u, \gamma)^*$, it follows that $a^\#\psi=(u, \gamma)^*=(a\psi)^*$. Hence, ψ is a $*$-isomorphism.

Summerizing the results above, the following is obtained:

THEOREM 3.2. *Let S be a regular $*$-semigroup. Then, there exist a fundamental regular $*$-semigroup Γ, a partial groupoid $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ (where each S_λ is a subgroup of M) satisfying* (C.0) *and a factor set $\Delta=\{\bar\gamma, C(\gamma, \delta)\}_{\gamma,\delta\in\Gamma}$ belonging to $\{N_M(F_\Gamma), \Gamma\}$, where $N_M(F_\Gamma)=\cup\{S_\lambda: \lambda\in F_\Gamma\}$, such that S is $*$-isomorphic to $N_M(F_\Gamma)\underset{\Delta}{\otimes}\Gamma$.*

Conversely, let Γ be a fundamental regular $*$-semigroup, and $M=\Sigma\{S_\lambda: \lambda\in E_\Gamma\}$ (where each S_λ is a subgroup of M) a partial groupoid satisfying (C.0). *Put $N_M(F_\Gamma)$*

$= \cup \{S_\lambda : \lambda \in F_\Gamma\}$, and let $\Delta = \{\bar{\gamma}, C(\gamma, \delta)\}_{\gamma,\delta \in \Gamma}$ *be a factor set belonging to* $\{N_M(F_\Gamma), \Gamma\}$. *Then, the* $*$*-regular product* $N_M(F_\Gamma) \underset{\Delta}{\otimes} \Gamma$ *is a regular* $*$*-semigroup.*

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