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Prehomomorphisms on Regular $*$-Semigroups

Teruo IMAOKA
Department of Mathematics, Shimane University, Matsue, Japan

To Kentaro MURATA
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The purpose of this paper is to study prehomomorphisms on regular $*$-semigroups which were firstly introduced in [5]. Firstly, we shall give a generalization of the natural order on a regular $*$-semigroup. Secondly, we shall discuss prehomomorphisms on regular $*$-semigroups. Finally, we shall obtain a generalization of a Preston-Vagner's representation to a regular $*$-semigroup.

§1. Natural order

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$,

(iii) $xx^*x = x$.

An idempotent e of a regular $*$-semigroup is called a *projection* if $e^* = e$. For a regular $*$-semigroup S, we denote the set of projections of S by $P(S)$. The notation and terminology are those of [1] and [2], unless otherwise stated.

Let S be a regular $*$-semigroup. For elements $a, b \in S$, let us define a relation $\leqq$ on S by

$$a \leqq b \Longleftrightarrow a = eb = bf \quad \text{for some} \quad e, f \in P(S).$$

LEMMA 1.1. *Let a and b be elements of S. Then the following statements are equivalent:*

(i) $a \leqq b$,

(ii) $aa^* = ba^*$ *and* $a^*a = b^*a$,

(iii) $aa^* = ab^*$ *and* $a^*a = a^*b$,

(iv) $a=aa^*b=ba^*a$.

PROOF. Assume that $a\leqq b$, that is, $a=eb=bf$ for some $e, f\in P(S)$. Then

$$aa^*=bffb^*=bfb^*=b(bf)^*=ba^*,$$

$$a^*a=b^*eeb=b^*eb=(eb)^*b=a^*b.$$

Thus we have (i)⇒(ii). Let $aa^*=ba^*$ and $a^*a=b^*a$. Then

$$aa^*=(aa^*)^*=(ba^*)^*=(a^*)^*b^*=ab^*,$$

$$a^*a=(a^*a)^*=(b^*a)^*=a^*(b^*)^*=a^*b.$$

Hence (ii)⇒(iii). Now assume that (iii) holds. Then

$$a=a(a^*a)=aa^*b,$$

$$a=(aa^*)a=(aa^*)^*a=(ab^*)^*a=ba^*a.$$

So (iv) holds. Since aa^* and a^*a are projections, it is obvious that (iv)⇒(i).

THEOREM 1.2. *The relation $\leqq$ on a regular $*$-semigroup S, defined above, is a partial order relation on S. Moreover, if $a\leqq b$ then $a^*\leqq b^*$.*

PROOF. Since $a=(aa^*)a=a(a^*a)$, $\leqq$ is reflexive. Let $a\leqq b$ and $b\leqq a$. By the lemma above,

$$a=b(a^*a)=b(b^*a)=b(b^*b)=b,$$

and hence $\leqq$ is anti-symmetric. Assume that $a\leqq b$ and $b\leqq c$. By the lemma above,

$$a=aa^*b=aa^*bb^*c=(ab^*)c=aa^*c,$$

$$a=ba^*a=cb^*ba^*a=c(b^*a)=ca^*a.$$

Then $a\leqq c$, and so $\leqq$ is transitive. It is obvious that $a\leqq b$ implies $a^*\leqq b^*$, and hence we have the theorem.

We call the relation $\leqq$ defined above *the natural order* on S.

COROLLARY 1.3. *The natural order on a generalized inverse $*$-semigroup S is compatible.*

PROOF. Assume that $a\leqq b$ and let c be any element of S. Then

$$(ac)^*ac=c^*(a^*a)c=c^*(b^*a)c=(bc)^*ac,$$

$$ac(ac)^*=acc^*a^*=b(a^*a)(cc^*)a^*=b(cc^*)(a^*a)a^*=bc(ac)^*.$$

Thus $ac \leqq bc$. Similarly $ca \leqq cb$, and hence we have the corollary.

§2. Prehomomorphisms

In his papers [3], [4], McAlister investigates prehomomorphisms on inverse semigroups and regular semigroups. In this section, we shall obtain basic properties on regular $*$-semigroups.

Let S and T be regular $*$-semigroups. A mapping $\phi: S \rightarrow T$ is called a $\vee$-[$\wedge$-]*prehomomorphism*, if it satisfies

(i) $(ab)\phi \leqq (a\phi)(b\phi)$,

[(i)′ $(ab)\phi \geqq (a\phi)(b\phi)$]

(ii) $(a\phi)^* = a^*\phi$,

for any $a, b \in S$.

Lemma 2.1. *Let ϕ be a $\vee$-prehomomorphism of a regular $*$-semigroup S to a regular $*$-semigroup T. Then we have the followings:*

(i) *ϕ maps an idempotent of S to an idempotent of T, then ϕ also maps a projection of S to a projection of T,*

(ii) *ϕ is isotone, that is, $a \leqq b$ implies $a\phi \leqq b\phi$,*

(iii) *ϕ preserves Green's relations, that is, if $\mathscr{K}$ is any one of Green's relations then $a \mathscr{K} b$ implies $a\phi \mathscr{K} b\phi$,*

(iv) *regular $*$-semigroups, with $\vee$-prehomomorphisms as morphisms, constitute a category.*

Proof. (i) Let e be an idempotent of S. Then $e\phi = e^2\phi \leqq e\phi e\phi$. By Lemma 1.1, $e\phi = e\phi(e\phi)^* e\phi e\phi = e\phi e\phi$.

(ii) Let $a \leqq b$. By Lemma 1.1, $a = aa^*b = ba^*a$. Then

$$a\phi = (aa^*b)\phi \leqq (aa^*)\phi b\phi.$$

On the other hand, $(aa^*)\phi \leqq a\phi(a\phi)^*$, and so $(aa^*)\phi = a\phi(a\phi)^*e$ for some $e \in P(T)$. Thus $a\phi \leqq (aa^*)\phi b\phi = a\phi(a\phi)^*e(b\phi)$. By using Lemma 1.1 again,

$$a\phi = a\phi(a\phi)^* a\phi(a\phi)^* e(b\phi) = a\phi(a\phi)^* e(b\phi).$$

Then we have $a\phi = (aa^*)\phi b\phi$. Similarly, we have $a\phi = b\phi(a^*a)\phi$. Since $(aa^*)\phi$ and $(a^*a)\phi$ are projections of T (by (i) above), we have $a\phi \leqq b\phi$.

To see (iii), it is sufficient to show that $a \mathscr{L} b$ implies $a\phi \mathscr{L} b\phi$. Assume that $a \mathscr{L} b$. Then there exist $x, y \in S$ such that $a = xb$ and $b = ya$. Then $a\phi = (xb)\phi \leqq x\phi b\phi$. By Lemma 1.1, $a\phi = a\phi(a\phi)^* x\phi b\phi$. Similarly, $b\phi = b\phi(b\phi)^* y\phi a\phi$. Hence we have $a\phi \mathscr{L} b\phi$.

Since the composition of $\vee$-prehomomorphisms is also a $\vee$-prehomomorphism, (iv) holds.

Let S be a regular $*$-semigroup. For each $a \in S$, let ϕ_a: $Sa^* \to Sa$ be a mapping defined by

$$x\phi_a = xa \qquad \text{for any} \quad x \in Sa^*.$$

It is clear that ϕ_a is an element of the symmetric inverse semigroup $\mathscr{I}_S$ on S. Let $\mathscr{M}_S = \{\phi_a : a \in S\}$, and define a product $\circ$ on $\mathscr{M}_S$ by

$$\phi_a \circ \phi_b = \phi_a \phi_{a^*abb^*} \phi_b,$$

where the product on right side is the usual product on $\mathscr{I}_S$.

THEOREM 2.2. *$\mathscr{M}_S(\circ)$ is a regular $*$-semigroup with a unary operation $(\phi_a)^* = \phi_{a^*}$. Let ϕ: $S \to \mathscr{M}_S$ be a mapping defined by $a\phi = \phi_a$. Then ϕ is a $*$-isomorphism of S onto $\mathscr{M}_S$.*

PROOF. Firstly, we shall show that $\phi_a \phi_{a^*abb^*} \phi_b = \phi_{ab}$. Since $Sbb^*a^*a \subset Sa$ and $Sa^*abb^* \subset Sb^*$,

$$\begin{aligned}\text{Dom}\,(\phi_a \phi_{a^*abb^*} \phi_b) &= Sa^*abb^* \phi_{a^*abb^*}^{-1} \phi_a^{-1} \\ &= Sa^*abb^*bb^*a^*aa^* \\ &= Sa^*abb^*a^* \\ &= Sabb^*a^*.\end{aligned}$$

Similarly, we have $\text{Ran}\,(\phi_a \phi_{a^*abb^*} \phi_b) = S(ab)^*ab$. For any x in $Sab(ab)^*$,

$$x\phi_a \phi_{a^*abb^*} \phi_b = xaa^*abb^*b = xab = x\phi_{ab}.$$

Since $\mathscr{I}_S$ is a semigroup, $\mathscr{M}_S$ is a regular $*$-semigroup. To see that ϕ is a $*$-isomorphism, it is sufficient to show that ϕ is one-to-one. Let $\phi_a = \phi_b$. Since each $\mathscr{R}$-class and each $\mathscr{L}$-class have one and only one projection, $aa^* = bb^*$ and $a^*a = b^*b$. Then

$$a = aa^*a = bb^*a = (bb^*)\phi_a = (bb^*)\phi_b = bb^*b = b.$$

Thus ϕ is one-to-one, and hence we have the theorem.

COROLLARY 2.3. *Let ψ: $S \to \mathscr{I}_S$ be a mapping defined by $a\psi = \phi_a$. Then ψ is a $\vee$-prehomomorphism.*

References

[1] Clifford, A. H. and G. B. Preston, The algebraic theory of semigroups, Math. Surveys No. 7, Amer. Math. Soc., Providence, Vol. I, 1961.

[2] Imaoka, T., Some remarks on fundamental regular *-semigroups, to appear.

[3] McAlister, D. B., ν-prehomomorphisms on inverse semigroups, Pacific J. Math. **67** (1976), 215–231.

[4] McAlister, D. B., Regular semigroups, fundamental semigroups and groups, J. Austral. Math. Soc. (Series A), **29** (1980), 475–503.

[5] McAlister, D. B. and N. R. Reilly, E-unitary covers for inverse semigroups, Pacific J. Math. **68** (1977), 161–174.