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# Strictly Inversive Semigroups.[1)]

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§ 1. **Introduction.** A semigroup S is called *inversive* if it satisfies the following

(C) (1) S has an idempotent, and the totality I of idempotents of S is a subband of S.[2)]
(2) For any element x of S, there exists an element $x^*$ such that $xx^* = x^*x$ and $xx^*x = x$.

In this case, for any element x of S there exists one and only one element y such that $xy = yx$, $xyx = x$ and $yxy = y$. In fact, let $y = xx^*x^*xx^*$.
Then, we have

$$xy = x(xx^*x^*xx^*) = xx^*,$$
$$yx = (xx^*x^*xx^*)x = xx^*,$$
$$xyx = x(xx^*x^*xx^*)x = x$$
and $$yxy = (xx^*x^*xx^*)x(xx^*x^*xx^*) = xx^*x^*xx^* = y.$$

Next, suppose that there exist $y_1$ and $y_2$ such that $xy_1 = y_1x$, $xy_1x = x$, $y_1xy_1 = y_1$, $xy_2 = y_2x$, $xy_2x = x$ and $y_2xy_2 = y_2$. Since $xy_1x = x$ and $xy_2x = x$, we have $xy_1 = xy_2xy_1 = y_2xy_1x = y_2x = xy_2$. Hence, $y_1 = y_1xy_1 = y_2xy_1 = y_2xy_2 = y_2$.

Such a y is called the *relative inverse* of x, and denoted by $x^{-1}$. Now, let $S(e) = \{x : xx^{-1} = e\}$ for each element e of I. Then, it is easy to see that each S(e) is a subgroup of S and S is the class sum (i.e. the disjoint sum) of all S(e) (A.H. Clifford [1] has shown that a semigroup satisfying the condition (2) of (C), which is called *a semigroup admitting relative inverses*, is the class sum of subgroups). Therefore, inversive semigroups are not too far away from groups. Actually, as a special case, the author has proved in [6] that if I is a rectangular band then all S(e) are isomorphic to each other and S is isomorphic to the direct product of S(e) and I (an inversive semigroup in which the totality of idempotents is a rectangular subband is called *an (R)-inversive semigroup*). Further, in this case S satisfies the following (C.1):

(C.1) If $xx^{-1} = e$ and if f is an idempotent such that $f \leqq e$ (i.e. $fe = ef = f$), then $fx = xf$.

By *a strictly inversive semigroup*, we shall mean an inversive semigroup satisfying the condition (C.1). As is stated above, an (R)-inversive semigroup is strictly inversive. More generally, it is easy to see that a semigroup M which is isomorphic to the direct product of a group and a band (hereafter, we shall call such a semigroup M *a B-group*) is strictly inversive. However, the converse is not true.

That is,

*an inversive semigroup is not necessarily a B-group, even if it is strictly inversive.*

In fact, this can be seen from the following example : Consider the semigroup K defined by the following multiplication table.

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1) An abstract of this paper has appeared in Proc. Japan Acad., 39, 100—106 (1963).

2) A semigroup in which every element is an idempotent is called a band.

It is easy to see that K is strictly inversive. However, K is not a B-group, since K consists of 7 elements and is neither a group nor a band.

| | a | b | c | d | e | f | g |
|---|---|---|---|---|---|---|---|
| a | a | b | c | d | e | f | g |
| b | b | c | a | e | f | d | g |
| c | c | a | b | f | d | e | g |
| d | a | b | c | d | e | f | g |
| e | b | c | a | e | f | d | g |
| f | c | a | b | f | d | e | g |
| g | g | g | g | g | g | g | g |

From the above-mentioned result, it follows that the class of B-groups is properly contained in the class of strictly inversive semigroups. The main purpose of this paper is to present a structure theorem for strictly inversive semigroups, and some relevant matters. Particularly, in § 5 we shall also present necessary and sufficient conditions for an inversive semigroup to be isomorphic to some special subdirect product of a group and a band.

§ **2. The structure of strictly inversive semigroups.** Let G be a semigroup. If there exist a band $\Omega$ and a collection $\{G_\alpha : \alpha \in \Omega\}$ of subsemigroups of type $\mathfrak{T}$ such that

(i) $G = \cup \{G_\alpha : \alpha \in \Omega\}$,

(ii) $G_\beta \cap G_\gamma = \phi$ for $\beta \neq \gamma$

and (iii) $G_\beta G_\gamma \subset G_{\beta\gamma}$,

then we shall say that *G is a band* $\Omega$ *of semigroups* $G_\alpha$ *of type* $\mathfrak{T}$. In this sense, the following results follow from the papers [1],[2] of A.H. Clifford:

( I ) A semigroup admitting relative inverses is a semilattice of completely simple semigroups without zero.

( II ) A semigroup G admitting relative inverses is a band of groups if and only if $Gba = Gba^2$ and $abG = a^2bG$ for any elements a,b of G.

Next, we shall define some special inversive semigroups. Let S be an inversive semigroup, and I the subband consisting of all idempotents of S. Then, S is called *normal, left normal, right normal, commutative, rectangular, left singular, right singular* or *trivial* respectively, if it satisfies the following corresponding identity

(N) xyzw=xzyw, (L.N) xyz=xzy, (R.N) xyz=yxz, (C) xy=yx, (R) xyz=xz, (L.S) xy=x, (R.S) xy=y or (T) x=y.

Moreover, S is said to be *(N)-inversive, (L.N)-inversive, (R.N)-inversive, (C)-inversive, (R)-inversive, (L.S)-inversive, (R.S)-inversive* or *(T)-inversive* respectively, if I satisfies the above-mentioned corresponding identity (N), (L.N), (R.N), (C), (R), (L.S), (R.S) or (T).

**Remark.** A band is of course an inversive semigroup. The structure of bands satisfying one of the above-mentioned identities has been completely determined by the author [6], N. Kimura [3] and N. Kimura and the author [5]. Further, it is also clear that any (T)-inversive semigroup is a group and a trivial inversive semigroup is a trivial band.

Under these definitions we have the following theorem, which is a special case of the above-mentioned result ( I ).

Theorem 1. *An inversive semigroup S is expressible as a semilattice of (R)-inversive semigroups. That is, there exist a semilattice* $\Gamma$ *and a collection* $\{S_\gamma : \gamma \in \Gamma\}$ *of (R)-inversive subsemigroups such that*

(i) $S=\cup\{S_\gamma : \gamma\in\Gamma\}$,

(ii) $S_\alpha \cap S_\beta = \phi$ *for* $\alpha \neq \beta$

*and* (iii) $S_\alpha S_\beta \subset S_{\alpha\beta}$.

*Further,* $\Gamma$ *is determined uniquely up to isomorphism, and accordingly so are the* $S_\gamma$.

Proof. From the above-mentioned result (I) of A.H. Clifford, an inversive semigroup S is a semilattice $\Gamma$ of completely simple semigroups without zero ; that is,

$$\text{(A)}\begin{cases} \text{(i)} & S=\cup\{S_\gamma : \gamma\in\Gamma\}, \\ \text{(ii)} & S_\alpha \cap S_\beta = \phi \text{ for } \alpha\neq\beta \\ \text{and (iii)} & S_\alpha S_\beta \subset S_{\alpha\beta}, \end{cases}$$

where each $S_\gamma$ is a completely simple semigroup without zero.

Let $E_\gamma$ be the totality of idempotents of $S_\gamma$. Then, $E_\gamma$ is a subband of $S_\gamma$. Now, let e,f be two elements of $E_\gamma$. Since $S_\gamma$ is completely simple, efee=eefe=efe implies efe=e. Hence, $E_\gamma$ is rectangular. According to A.H. Clifford [1], a completely simple semigroup without zero is a semigroup admitting relative inverses. Therefore, $S_\gamma$ is an (R)-inversive semigroup, and hence S is a semilattice $\Gamma$ of (R)-inversive semigroups.

Next, suppose that there exists another decomposition of S into a semilattice of (R)-inversive semigroups, say

$$\text{(B)}\begin{cases} \text{(i)} & S=\cup\{S^*_\xi : \xi\in\Gamma^*\}, \\ \text{(ii)} & S_\zeta \cap S^*_\tau = \phi \text{ for } \zeta\neq\tau \\ \text{and (iii)} & S^*_\zeta\ S^*_\tau \subset S^*_{\zeta\tau}, \end{cases}$$

where each $S^*_\xi$ is an (R)-inversive semigroup and $\Gamma^*$ is a semilattice.

Let $E^*_\xi$ be the totality of idempotents of $S^*_\xi$. Then, $E^*_\xi$ is a rectangular subband of $S^*_\xi$. Let I be the subband of idempotents of S.

Then

$$\begin{cases} \text{(i)} & I=\cup\{E_\gamma : \gamma\in\Gamma\}, \\ \text{(ii)} & E_\alpha \cap E_\beta = \phi \text{ for } \alpha\neq\beta, \\ \text{(iii)} & E_\alpha E_\beta \subset E_{\alpha\beta} \end{cases}$$

and

$$\begin{cases} \text{(i)} & I=\cup\{E^*_\xi : \xi\in\Gamma^*\}, \\ \text{(ii)} & E^*_\zeta \cap E^*_\tau = \phi \text{ for } \zeta\neq\tau, \\ \text{(iii)} & E^*_\zeta\ E^*_\tau \subset E^*_{\zeta\tau}. \end{cases}$$

are semilattice decompositions of I into rectangular bands. According to D. MacLean [4], such a decomposition of I is unique. Therefore, we can assume that $\Gamma=\Gamma^*$ and $E_\gamma=E^*_\gamma$ for each $\gamma\in\Gamma$. Now, since two decompositions (A) and (B) are different, there exists $\alpha\in\Gamma$ such that $S_\alpha \neq S^*_\alpha$. Hence, there exists $\beta\in\Gamma(\beta\neq\alpha)$ such that $S^*_\alpha\cap S_\beta\neq\phi$ or $S_\alpha\cap S^*_\beta\neq\phi$. If $S^*_\alpha\cap S_\beta\ni x$, then $x^{-1}\in S^*_\alpha\cap S_\beta$. Hence, $xx^{-1}\in S^*_\alpha\cap S_\beta$, and hence $xx^{-1}\in E_\alpha\cap E_\beta$. Similarly, $S_\alpha\cap S^*_\beta\neq\phi$ implies $E_\alpha\cap E_\beta\neq\phi$. This is a contradiction. Hence, such a decomposition of S is unique.

Now, for a strictly inversive semigroup we have

Lemma 1. *If* $aa^{-1}=e$ *and* $bb^{-1}=f$, *then* $(ab)^{-1}=eb^{-1}a^{-1}f$ *and* $(ab)(ab)^{-1}=ef$.

Proof. Let a,b be any elements of a strictly inversive semigroup. Let $aa^{-1}=e$ and $bb^{-1}=f$. Then,

$abeb^{-1}a^{-1}f=abfefb^{-1}a^{-1}f=afefbb^{-1}a^{-1}f=aefefea^{-1}f=efefeaa^{-1}f=efefef=ef$,

$eb^{-1}a^{-1}fab = eb^{-1}a^{-1}efeab = eb^{-1}a^{-1}aefeb = eb^{-1}fefefb = eb^{-1}bfefef = efefef = ef,$

$eb^{-1}a^{-1}fef = eb^{-1}a^{-1}efef = eb^{-1}a^{-1}ef = eb^{-1}a^{-1}f,$

$efeb^{-1}a^{-1}f = efefb^{-1}a^{-1}f = efb^{-1}a^{-1}f = eb^{-1}a^{-1}f,$

$abef = abfef = afefb = aefefb = aefb = ab$

and $efab = efeab = aefeb = aefefb = aefb = ab$.

Thus, $(ab)^{-1} = eb^{-1}a^{-1}f$ and $(ab)(ab)^{-1} = ef$.

Using this lemma, we obtain the following theorem as a special case of the above-mentioned result (II).

Theorem 2. *Let S be an inversive semigroup, and I the subband consisting of all idempotents of S. Then, S is expressible as a band of groups if and only if S is strictly inversive. Further, in this case such a decomposition is uniquely determined, and it is the decomposition such that*

(i) $S = \cup \{S(e) : e \in I\}$,

(ii) $S(f) \cap S(h) = \phi$ *for* $f \neq h$

*and* (iii) $S(f)S(h) \subset S(fh)$,

*where* $S(e) = \{x : xx^{-1} = e\}$ *for every* $e \in I$.

Proof. Let S be strictly inversive. The relations (i),(ii) of the theorem are obvious. Next, we prove the relation (iii). Let x,y be elements of S(f) and S(h) respectively. Then, $xx^{-1} = f$ and $yy^{-1} = h$. By Lemma 1, $(xy)(xy)^{-1} = fh$. Hence, $xy \in S(fh)$. Thus, S is expressible as a band of groups.

Conversely, suppose that S is expressible as a band of groups :

(i) $S = \cup \{G_\alpha : \alpha \in B\}$,

(ii) $G_\beta \cap G_\gamma = \phi$ for $\beta \neq \gamma$

and (iii) $G_\beta G_\gamma \subset G_{\beta\gamma}$,

where B is a band and each $G_\alpha$ is a group.

Let e,f be idempotents such that $f \leqq e$. Let x be an element such that $xx^{-1} = e$. There exist $G_\delta$, $G_\zeta$ such that $G_\delta \ni e$ and $G_\zeta \ni f$. It is clear that x and $x^{-1}$ are also contained in $G_\delta$. Since $ef = fe = f$, we have $xf \in G_\zeta$ and $fx \in G_\zeta$. Since the identity element of $G_\zeta$ is f, we obtain $xf = fxf = fx$. The uniqueness of such a decomposition is obvious.

We shall call Γ in Theorem 1 *the structure semilattice* of S and $S_\gamma$ *the γ-kernel* of S. Also in this case we write $S \sim \Sigma \{S_\gamma : \gamma \in \Gamma\}$ and call it *the structure decomposition* of S.[3)]

**Remark.** If S is an inversive semigroup, then the subband I of all idempotents of S is also inversive. In this case, it is easy to see that S and I have the same structure semi-lattice. Next, we have

Lemma 2. *A (N)-inversive semigroup is strictly inversive.*

Proof. Let x be an element of S(e), and f an element snch that $f \leqq e$, where e,f are idempotents. Putting $fx(fx)^{-1} = u$ and $xf(xf)^{-1} = v$, we have $fu = u$, $ue = eu = u$, $vf = v$ and $ev = ve = v$.

Now, $fx = fxu = fxefue = fxeufe = fxf$ and $xf = vxf = evfexf = efvexf = fxf$. Hence, $xf = fx$.

In particular a (C)-inversive semigroup is strictly inversive, because a semilattice satisfies normality. Therefore, by Theorem 2 a (C)-inversive semigroup is expressible as a band of groups.

Further, for (C)-inversive semigroups Theorem 2 is sharpened as follows:

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3) In particular, for the structure decomposition of a band see also N. Kimura [3].

Theorem 3. (A.H. Clifford). *A semigroup is expressible as a semilattice of groups if and only if it is (C)-inversive.*

Proof. Let S be a semigroup and I the totality of idempotents of S.
The 'only if' part : Suppose that S is expressibe as a semilattice of groups; that is

(i) $S = \cup \{S_\gamma : \gamma \in \Gamma\}$,

(ii) $S_\alpha \cap S_\beta = \phi$ for $\alpha \neq \beta$,

(iii) $S_\alpha S_\beta \subset S_{\alpha\beta}$,

where each $S_\gamma$ is a group and $\Gamma$ is a semilattice.
Let $e_\gamma$ be the identity element of $S_\gamma$. Then, $I = \{e_\gamma : \gamma \in \Gamma\}$. Take two elements $e_\alpha$, $e_\beta$ of I. Clearly, $e_\alpha e_\beta \in S_{\alpha\beta}$. Now, $e_\alpha e_{\alpha\beta} e_\alpha e_{\alpha\beta} = e_\alpha e_\alpha e_{\alpha\beta} = e_\alpha e_{\alpha\beta}$. Hence, $e_\alpha e_{\alpha\beta}$ is an idempotent of $S_{\alpha\beta}$, and hence $e_\alpha e_{\alpha\beta} = e_{\alpha\beta}$. Similarly, $e_\beta e_{\alpha\beta} = e_{\alpha\beta}$. Now, $e_\alpha e_\beta = e_\alpha e_\beta e_{\alpha\beta} = e_{\alpha\beta}$. Further, $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha$. Therefore, I is a semilattice. Since S is the class sum of groups and I is a band, it is clear that S is inversive.
The 'if' part : Suppose that S is (C)-inversive. Since a (C)-inversive semigroup is (N)-inversive, by Lemma 2 S is strictly inversive. Hence, by Theorem 2 S is expressible as a band of groups. Thus S is expressible as a semilattice of groups since I is a semilattice.

Let $\Gamma$ be a given semilattice and $S_\gamma$ be, for each $\gamma \in \Gamma$, a given group. Let S be the class sum of all $S_\gamma$.
Consider the semigroup $S(\circ)$ which consists of all elements of S and in which a multiplication $\circ$ is defined such that

(M) $\begin{cases} (1) \text{ for any } \gamma \in \Gamma,\ S_\gamma \text{ is a subsemigroup of } S(\circ);\ a_\gamma \circ b_\gamma = a_\gamma b_\gamma \text{ for any elements } a_\gamma, b_\gamma \in S_\gamma, \\ (2) \text{ for any } \alpha, \beta \in \Gamma,\ S_\alpha \circ S_\beta \subset S_{\alpha\beta}. \end{cases}$

Such a semigroup $S(\circ)$ is called *a compound semigroup* of $\{S_\gamma : \gamma \in \Gamma\}$ by $\Gamma$. The author [7] has shown that there exists at least one compound semigroup of $\{S_\gamma : \gamma \in \Gamma\}$ by $\Gamma$ for any given semilattice $\Gamma$ and any given collection $\{S_\gamma : \gamma \in \Gamma\}$ of groups $S_\gamma$. From Theorem 3, it follows that the problem of constructing all possible (C)-inversive semigroups is reduced to the problem of finding all possible compound semigroups of $\{S_\gamma : \gamma \in \Gamma\}$ by $\Gamma$ for a given semilattice $\Gamma$ and a given collection $\{S_\gamma : \gamma \in \Gamma\}$ of groups $S_\gamma$. This problem was completely solved by A.H. Clifford [1] and the author [7].[4)]

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4) Let $\{M_\xi : \xi \in L\}$ be a disjoint family of non-empty sets with a semilattice L as its index set. For any pair $(\varepsilon, \zeta)$, where $\varepsilon \geqq \zeta$, consider a mapping $\delta^\varepsilon_\zeta : M_\varepsilon \to M_\zeta$. Then a family $\Phi = \{\delta^\varepsilon_\zeta : \varepsilon \geqq \zeta,\ \varepsilon, \zeta \in L\}$ is called a *normal family* on the family (of sets) $\{M_\xi : \xi \in L\}$, if it satisfies the following two conditions:

(1) $\delta^\varepsilon_\varepsilon$ is the identity mapping for all $\varepsilon \in L$.

(2) $\delta^\varepsilon_\zeta \delta^\zeta_\tau = \delta^\varepsilon_\tau$ for $\varepsilon \geqq \zeta \geqq \tau$.

Now, let $\{S_\tau : \gamma \in \Gamma\}$ be a disjoint family of groups (commutative groups) with a semilattice $\Gamma$ as its index set. Take a homomorphism $\varphi^\alpha_\beta : S_\alpha \to S_\beta$, for each pair $(\alpha, \beta)$ of $\alpha, \beta \in \Gamma$ such that $\alpha \geqq \beta$. If the family $\{\varphi^\alpha_\beta : \alpha \geqq \beta,\ \alpha, \beta \in \Gamma\}$ of all $\varphi^\alpha_\beta$ is a normal family on the family $\{S_\gamma : \gamma \in \Gamma\}$, then such a system $\{\varphi^\alpha_\beta : \alpha \geqq \beta,\ \alpha, \beta \in \Gamma\}$ is called a *transitive system* of homomorphisms induced by $\{S_\gamma : \gamma \in \Gamma\}$.

A.H. Clifford [1] and the author [7] have proved the following theorem:

Theorem. *Every compound semigroup (commutative compound semigroup) $S(\circ)$ (where $S = \cup \{S_\gamma : \gamma \in \Gamma\}$) of $\{S_\gamma : \gamma \in \Gamma\}$ by $\Gamma$ is found as follows. Let $\{\varphi^\alpha_\beta : \alpha \geqq \beta,\ \alpha, \beta \in \Gamma\}$ be a transitive system of homomorphisms induced by $\{S_\gamma : \gamma \in \Gamma\}$. Then S becomes a compound semigroup (commutative compound semigroup) of $\{S_\gamma : \gamma \in \Gamma\}$ by $\Gamma$ if multiplication $\circ$ therein is defined by the following*

*(P)* $a \circ b = \varphi^\alpha_{\alpha\beta}(a) \varphi^\beta_{\alpha\beta}(b)$ *for* $a \in S_\alpha$, $b \in S_\beta$.

Next, we shall introduce a special kind of subdirect product,[5] which is called the *spined product*.

Let $S_1$, $S_2$ be inversive semigroups having $\Gamma$ as their structure semilattices, and $S_1 \sim \Sigma \{S_1^\gamma : \gamma \in \Gamma\}$ and $S_2 \sim \Sigma \{S_2^\gamma : \gamma \in \Gamma\}$ be the structure decompositions of $S_1$ and $S_2$. Then, the set $S = \cup \{S_1^\gamma \times S_2^\gamma : \gamma \in \Gamma\}$ becomes a subdirect product of $S_1$ and $S_2$. Such an S is called the spined product of $S_1$ and $S_2$ with respect to $\Gamma$, and denoted by $S_1 \bowtie S_2$ $(\Gamma)$. We sometimes omit $(\Gamma)$, if there is no confusion.

Under this definition, we have the following main theorem.

Theorem 4. (Structure theorem) *Let S be a strictly inversive semigroup having $\Gamma$ as its structure semilattice. Let I be the subband consisting of all idempotents of S. Then, S is isomorphic to $C \bowtie I$ $(\Gamma)$ for some (C)-inversive semigroup C having $\Gamma$ as its structure semilattice. Conversely, if C and I are a (C)-inversive semigroup and a band having $\Gamma$ as their structure semilattices, then $C \bowtie I$ $(\Gamma)$ is a strictly inversive semigroup. In other words, a semigroup is isomorphic to the spined product of a (C)-inversive semigroup and a band if and only if it is strictly inversive.*

Proof. The first half of the theorem : Let S be a strictly inversive semigroup and $S \sim \Sigma \{S_\gamma : \gamma \in \Gamma\}$ its structure decomposition. Let I be the subband consisting of all idempotents of S. Let $E_\gamma$ be the totality of all idempotents of $S_\gamma$. The structure decomposition of I is clearly $I \sim \Sigma \{E_\gamma : \gamma \in \Gamma\}$.

Now, we introduce a relation R in S as follows :

$$x\ R\ y \text{ if and only if } x,y \in S_\gamma \text{ for some } \gamma \in \Gamma \text{ and } xy^{-1} \in E_\gamma.$$

Then, R is a congruence on S. In fact, it is proved as follows.

(1) *$x\ R\ x$ for any $x \in S$.* Obvious from the fact that each $S_\gamma$ is (R)-inversive.

(2) *$x\ R\ y$ implies $y\ R\ x$.* Since $x\ R\ y$, there exists $S_\gamma$ such that $x,y \in S_\gamma$ and $xy^{-1} \in E_\gamma$. Let $xx^{-1}=e$, $yy^{-1}=f$ and $xy^{-1}=h$. Then, $eyx^{-1}f=(xy^{-1})^{-1}=h$. Hence, $fefyx^{-1}efe=fhe=fe \in E_\gamma$. Thus, $yx^{-1} \in E_\gamma$. Hence, $y\ R\ x$.

(3) *$x\ R\ y$, $y\ R\ z$ implies $x\ R\ z$.* Since $x\ R\ y$ and $y\ R\ z$, there exists $S_\gamma$ such that $x,y,z \in S_\gamma$ and $xy^{-1}$, $yz^{-1} \in E_\gamma$. Now, let $xx^{-1}=e$, $yy^{-1}=f$ and $zz^{-1}=h$.

Then,

$$xy^{-1}yz^{-1}=xfz^{-1}=xefhz^{-1}=xehz^{-1}=xz^{-1}.$$

Since $xy^{-1}yz^{-1} \in E_\gamma$, we have $xz^{-1} \in E_\gamma$.

(4) *$x\ R\ y$ implies $cx\ R\ cy$ and $xc\ R\ yc$ for any $c \in S$.* Since $x\ R\ y$, there exists $S_\alpha$ such that $x,y \in S_\alpha$ and $xy^{-1} \in E_\alpha$. Let c be an element of S. Then, c is contained in some kernel of S, say $S_\beta$. It is clear that cx, cy are elements of $S_{\beta\alpha}$. Let $xx^{-1}=e$, $yy^{-1}=f$, $cc^{-1}=h$ and $xy^{-1}=k$.

Then,

$$cx(cy)^{-1}=cxhy^{-1}c^{-1}f=cxehfefy^{-1}c^{-1}f=cehfexfy^{-1}c^{-1}f=cehfexy^{-1}c^{-1}f$$
$$=chehfkhc^{-1}f=(he)(hfk)(hf) \in E_{\alpha\beta}.$$

Hence, $cx\ R\ cy$. Similarly, we can prove the relation $xc\ R\ yc$.

Consequently, R is a congruence on S. Therefore, we can consider the factor semigroup S/R of S mod R. We denote the congruence class containing x by $\tilde{x}$, and put $\{\tilde{x}_\gamma : x_\gamma \in S_\gamma\} = G_\gamma$. Then, it is easy to see that each $G_\gamma$, $\gamma \in \Gamma$, is a group. Let $\tilde{x}_\alpha$, $\tilde{x}_\beta$ be elements of $G_\alpha$ and

5) For definition of a subdirect product, see G. Birkhoff, *Lattice Theory*, p. 91.

$G_\beta$ respectively. Clearly, $\tilde{x}_\alpha \tilde{x}_\beta = \widetilde{x_\alpha x_\beta}$. Since $x_\alpha x_\beta \in S_{\alpha\beta}$, $\widetilde{x_\alpha x_\beta}$ is an element of $G_{\alpha\beta}$. Therefore, $G_\alpha G_\beta \subset G_{\alpha\beta}$. This means that S/R is the semilattice $\Gamma$ of the groups $G_\gamma$; i.e. $S/R \sim \Sigma\{G_\gamma; \gamma \in \Gamma\}$.

Next, consider the spined product $S/R \bowtie I\ (\Gamma)$; i.e.

$$S/R \bowtie I\ (\Gamma) = \cup\{G_\gamma \times E_\gamma:\ \gamma \in \Gamma\}.$$

Define a mapping $\varphi$ of S into $S/R \bowtie I\ (\Gamma)$ as follows:

$$\varphi(x) = (x,\ xx^{-1}),\ x \in S.$$

Then, it is easily proved that $\varphi$ is an isomorphism of S onto $S/R \bowtie I\ (\Gamma)$. The second half of the theorem: Let C and I be a (C)-inversive semigroup and a band having $\Gamma$ as their structure semilattices. Since both C and I are strictly inversive, $C \times I$ is also strictly inversive and $C \bowtie I\ (\Gamma)$ is inversive. Since any inversive subsemigroup of a strictly inversive semigroup is strictly inversive and since $C \bowtie I\ (\Gamma)$ is an inversive subsemigroup of $C \times I$, $C \bowtie I\ (\Gamma)$ is also strictly inversive.

From Theorem 4, the problem of determining the structure of strictly inversive semigroups is reduced to the problems of determining the structures of (C)-inversive semigroups and bands. As stated above, the formar was completely solved by Theorem 3, A.H. Clifford [1] and the author [7]. The latter was also partially solved by several papers. Particularly, the structure of normal bands (hence, of course, the structure of left normal, right normal, rectangular, commutative, left singular, right singular or trivial bands) was completely determined by the author [6], N. Kimura [3] and the author and N. Kimura [5].[6)]

§ 3. **Applications for (N)-inversive semigroups.** Since a P-inversive semigroup, where P is (N), (L.N), (R.N), (C), (R), (L.S), (R.S) or (T), is (N)-inversive and since any (N)-inversive semigroup is strictly inversive, we have the following corollaries for P-inversive semigroups as special cases of Theorem 4.

Corollary 1. *A semigroup is isomorphic to the spined product of a (C)-inversive semigroup and a normal (left normal, right normal) band if and only if it is (N)-((L.N)-, (R.N.)-) inversive.*

Proof. Obvious.

Corollary 2. *A semigroup is isomorphic to the direct product of a group and a rectangular (left singular, right singular) band if and only if it is (R)-((L.S)-, (R.S)-) inversive (see also the author [6]).*

Proof. The 'only if' part of the corollary is clear.

The 'if' part of the corollary: Let S be an (R)-((L.S)-, (R.S)-) inversive semigroup, and I the subband consisting of all idempotents of S. Then, each of the structure semilattices

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6) N. Kimura and the author [5] has proved the following two theorems:

Theorem. *Let $\Gamma$ be a semilattice, let $\{S_\gamma: \gamma \in \Gamma\}$ be a disjoint family of non-empty sets with $\Gamma$ as its index set, and let $S = \cup\{S_\gamma : \gamma \in \Gamma\}$. Let $\Phi = \{\varphi^\alpha_\beta:\ \alpha \geqq \beta,\ \alpha, \beta \in \Gamma\}$ be a normal family of mappings on the family $\{S_\gamma:\ \gamma \in \Gamma\}$. Then S becomes a left (right) normal band whose structure decomposition is $S \sim \Sigma \{S_\gamma: \gamma \in \Gamma\}$ under the multiplication defined by*

$$ab = \varphi^{\alpha}_{\alpha\beta}(a)\ \ (ab = \varphi^{\beta}_{\alpha\beta}(b))$$

*for $a \in S_\alpha$, $b \in S_\beta$.*

*Further every left (right) normal band is constructed by this way.*

Theorem. *A band is normal if and only if it is isomorphic to the spind product of a left normal band and a right normal band.*

of S and I consists of a single element. Hence S/R in the proof of Theorem 4 is a group, and $S/R \bowtie I = S/R \times I$.

Corollary 3. *A semigroup is isomorphic to the spined product of a commutative inversive semigroup and a normal (left normal, right normal) band if and only if it is normal (left normal, right normal) inversive semigroup.*

Proof. The 'only if' part of the corollary is clear.

Let S be a normal inversive semigroup. To prove the 'if' part, we need only to show that S/R in the proof of Theorem 4 is commutative. Let $\tilde{x},\tilde{y}$ be elements of S/R. Let $xx^{-1}=e$ and $yy^{-1}=f$. Then, $xy(yx)^{-1}=xyfx^{-1}y^{-1}e=xx^{-1}yy^{-1}e=efe$ by the normality of S. Hence $xy=yx$, i.e. $xy=yx$. Thus, S/R is commutative.

**Remark.** For construction of commutative inversive semigroups, see the author [7],[4] for that of normal, left normal or right normal bands, see N. Kimura and the author [5].[6]

Corollary 4. *A rectangular (left singular, right singular) inversive semigroup is a rectangular (left singular, right singular) band.*

§ **4. Group-semilattices.** Let G be a group, and $\Gamma$ a semilattice. Let $\{G_\gamma:\gamma\in\Gamma\}$ be a collection of subgroups $G_\gamma$ of G such that

$$(1)\quad G=\cup\{G_\gamma:\gamma\in\Gamma\}$$

$$\text{and}\quad (2)\quad G_\alpha\supset G_\beta \text{ for } \alpha\leqq\beta.$$

Let $S=\Sigma_{\gamma\in\Gamma} G_\gamma$, where $\Sigma$ denotes the class sum of sets. If $x\in G$ is an element of $G_\gamma$, then we denote x by $(x,\gamma)$ when we regard x as an element of $G_\gamma$ in S. Now, S becomes a semigroup under the multiplication $\circ$ defined by the following

$$(M)^*\ (x,\alpha)\circ(y,\beta)=(xy,\alpha\beta).$$

That is, S is compound semigroup of $\{G_\gamma:\gamma\in\Gamma\}$ by $\Gamma$, and accordingly a (C)-inversive semigroup. We shall call such an S *a group-semilattice* of G, and denote it by $\{G_\gamma|\Gamma, G\}$. Moreover, in this case we shall call G *the basic group* of S. Now, let I be a band whose structure decomposition is $I\sim\Sigma\{I_\gamma:\gamma\in\Gamma\}$. Then, we can consider the spined product of S and I with respect to $\Gamma$, because S and I have the same structure semilattice $\Gamma$.

As a connection between subdirect products of G and I and the spined product of S and I, we have

Theorem 5. *The spined product of a group-semilattice of G and a band I is isomorphic to an inversive subdirect product of G and I.*[7] *Conversely, any inversive subdirect product of a group G and a band I is isomorphic to the spined product of a group-semilattice of G and I.*

Proof. Let S be an inversive subdirect product of a group G and a band I. Let $I\sim\Sigma\{E_\gamma:\gamma\in\Gamma\}$ be the structure decomposition of I. Let $S(E_\gamma)=\{x: x\in G, e\in E_\gamma, (x,e)\in S\}$. Then, $S(E_\gamma)$ is a subgroup of G and the structure decomposition of S is $S\sim\Sigma\{S(E_\gamma)\times E_\gamma:\gamma\in\Gamma\}$. Now, let $\alpha,\beta$ be elements of $\Gamma$ such that $\alpha\leqq\beta$. Take an element of $S(E_\beta)$, say x. Then, $(x,e)\in S(E_\beta)\times E_\beta$ for each $e\in E_\beta$. For $(1,f)\in S(E_\alpha)\times E_\gamma$, we have $(1,f)(x,e)=(x,fe)\in S(E_\alpha)\times E_\alpha$. Hence, $x\in S(E_\alpha)$. This implies $S(E_\beta)\subset S(E_\alpha)$. Accordingly, S is isomorphic to the spined product of the group-semilattice $\{S(E_\gamma)|\Gamma, G\}$ and the band I. Conversely, let S be the spined product of a group-semilattice $\{S_\gamma|\Gamma, G\}$ and a band I. Let $I\sim\Sigma\{E_\gamma:\gamma\in\Gamma\}$ be the structure decomposition of I. Then, S is clearly inversive and

7) Let D be a subdirect product of G and I. Then, D is clearly a semigroup. If D is an inversive semigroup, then D is called *an inversive subdirect product* of G and I.

the structure decomposition of S is $S \sim \Sigma\{S_\gamma \times E_\gamma : \gamma \in \Gamma\}$. Since $S_\gamma \times E_\gamma \subset G \times E_\gamma$ for every $\gamma \in \Gamma$ if we regard every element of $S_\gamma$ as an element of G, we have $\sum_{\gamma \in \Gamma} S_\gamma \times E_\gamma \subset G \times I$. Hence, S is isomorphic to an inversive subdirect product of the group G and a band I.

Corollary 5. *An inversive semigroup is isomorphic to a subdirect product of a group G and a band I if and only if it is isomorphic to the spined product of a group-semilattice of G and I.*

Proof. Obvious from Theorem 5.

**Remark.** Let G be a group, and I a band whose structure decomposition is $I \sim \Sigma\{I_\gamma : \gamma \in \Gamma\}$. Then, the direct product of G and I is isomorphic to the spined product of $\{G_\gamma \mid \Gamma, G\}$ and I, where $G_\gamma = G$ for all $\gamma$, and vice versa.

By Theorem 5 and its remark, the connection between direct products, subdirect products and spined products is somewhat clarified. These results will be used in the next paragraph.

## § 5. Necessary and sufficient conditions for an inversive semigroup to be isomorphic to the spined product of a group-semilattice and a band.

The following is a well-known result: If $C_1$, $C_2$ are congruences on an algebraic system A such that

(S.1) $C_1 \cap C_2 = 0$,

then A is isomorphic to a subdirect product of $A/C_1$ and $A/C_2$.

Further, if $C_1$, $C_2$ are permutable congruences and if they satisfy (S.1) and

(S.2) $C_1 \cup C_2 = 1$,

then A is isomorphic to the direct product of $A/C_1$ and $A/C_2$.

Using Theorem 5, its corollary and the result above, we have

Theorem 6. *An inversive semigroup S is isomorphic to the spined product of a group-semilattice and a band if and only if the following relations $R_1$, $R_2$ are congruences on S:*

*(1) $a R_1 b$ if and only if $ab^{-1}$ and $ba^{-1}$ are idempotents.*

*(2) $a R_2 b$ if and only if $aa^{-1} = bb^{-1}$.*

*Further, if S is isomorphic to the spined product of a group-semilattice L and a band B, then the basic group of L and the band B are isomorphic to $S/R_1$ and $S/R_2$ respectively. Accordingly, in this case S is also isomorphic to a subdirect product of $S/R_1$ and $S/R_2$.*

Proof. Let S be an inversive semigroup, and I the subband consisting of all idempotents of S.

The first half of the theorem: I. The 'if' part. Let $R_1$, $R_2$ be the relations on S defined by (1), (2) of the theorem. Assume that $R_1$, $R_2$ are congruences on S. We shall show that $R_1 \cap R_2 = 0$. Suppose that there exist x and y such that $x R_1 y$ and $x R_2 y$. Then, both $xy^{-1}$ and $yx^{-1}$ are idempotents and $xx^{-1} = yy^{-1}$. Put $xx^{-1} = e$, and $\{t : tt^{-1} = e\} = S(e)$. Then, S(e) is a group and contains x,y. Hence, $xy^{-1} = e$, and hence $x = y$. Thus, $R_1 \cap R_2 = 0$. Therefore, S is isomorphic to a subdirect product of $S/R_1$ and $S/R_2$. On the other hand, it is easy to see that $S/R_1$ is a group and $S/R_2$ is isomorphic to I. Accordingly, by the corollary to Theorem 5 S is isomorphic to the spined product of a group-semilattice of the group $S/R_1$ and the band $S/R_2$.

II. The 'only if' part. Assume that S is the spined product of a group-semilattice $\{S_\gamma \mid \Gamma, G\}$ and a band B. Then, B is clearly isomorphic to I. Hence, we can assume that $B = I$. Now, the structure decomposition of S is $S \sim \Sigma\{S_\gamma \times E_\gamma : \gamma \in \Gamma\}$, where $E_\gamma$ is the $\gamma$-kernel of I. For any $x \in G$, there exists $S_\gamma$ such that $S_\gamma \ni x$ (such an $S_\gamma$ is not necessarily unique).

we denote x by $(x,\gamma)$ when we regard it as an element of $S_\gamma$. Then, (1) and (2) in the theorem are paraphrased as follows:

(1)′ $((x,\alpha),\alpha)R_1((y,\beta),\beta)$ if and only if $xy^{-1}=1$ and $yx^{-1}=1$.

(2)′ $((x,\alpha),\alpha)R_2((y,\beta),\beta)$ if and only if $\alpha=\beta$.

Now, it is easy to see that $R_1$ and $R_2$ are congruences on S and $S/R_1$, $S/R_2$ are isomorphic to G and I respectively.

The second half of the theorem: Obvious from I and II.

**Remark.** In Theorem 6, let I be the subband consisting of all idempotents of S. Then, it is also easily seen from the proof of Theorem 6 that $S/R_2$ is isomorphic to I.

Corollary 6. *An inversive semigroup S is a B-group if and only if $R_1$, $R_2$ are permutable congruences on S and satisfy the condition*

*(S.2)* $R_1 \cup R_2 = 1$.

Proof. Obvious from the corollary to Theorem 5, Theorem 6 and the definitions of $R_1$ and $R_2$.

Moreover, Theorem 6 is paraphrased as follows :

Theorem 7. *Let S be an inversive semigroup, and I the subband consisting of all idempotents of S. Then, S is isomorphic to the spined product of a group-semilattice and a band if and only if it satisfies the following (C.1) and (C.2) :*

*(C.1) S is strictly inversive.*

*(C.2) For any* $e \in I$, $ab \in I$ *if and only if* $aeb \in I$.

Proof. To prove the theorem, we need only to show that $R_1, R_2$ are congruences on S if and only if S satisfies (C.1) and (C.2). If $R_1$, $R_2$ are congruences on S, then S is isomorphic to a subdirect product of a group and a band. Hence, S satisfies (C.1) and (C.2). Conversely, suppose that S satisfies the conditions (C.1) and (C.2). Since S is strictly inversive, S is expressible as a band of groups. Hence, it is clear that $R_2$ is a congruence on S. Next, we shall show that $R_1$ is also a congruence on S. At first, it is clear that $R_1$ is reflexive and symmetrical. Let $xR_1y$ and $yR_1z$. Then, $xy^{-1}$, $yz^{-1}$, $yx^{-1}$, $zy^{-1} \in I$. Let $yy^{-1}=f \in I$. Since $xy^{-1}$, $yz^{-1} \in I$, we have $xy^{-1}\,yz^{-1} \in I$, i.e. $xfz^{-1} \in I$. Hence $xz^{-1} \in I$. Similarly, $zx^{-1} \in I$. Accordingly $xR_1z$. Next, let $xR_1y$ and let c be any element of S. Then, $xy^{-1}$, $yx^{-1} \in I$. Putting $cc^{-1}=e$ and $yy^{-1}=f$, we have $cx(cy)^{-1}=cxey^{-1}c^{-1}f=chc^{-1}f$, where $xey^{-1}=h \in I$. Hence, $cx(cy)^{-1}=cehec^{-1}f=ehecc^{-1}f=ehef \in I$. Similarly, we can prove the relation $cy(cx)^{-1} \in I$. Therefore $cxR_1cy$. Similarly, we can also prove that $xR_1y$ implies $xcR_1yc$ for any element c of S. Thus, $R_1$ is a congruence on S.

Corollary 7. *Let S be an inversive semigroup, and I the subband consisting of all idempotents of S. Then, S is a B-group if and only if it satisfies the conditions (C.1), (C.2) and the following (C.3) :*

*(C.3) For any* $a,b \in S$, *there exist x,y such that* $aa^{-1}=xx^{-1}$, $bb^{-1}=yy^{-1}$ *and* $xb^{-1}$, $ya^{-1}$, $bx^{-1}$, $ay^{-1} \in I$.

Proof. Obvious from the corollary to Theorem 6, Theorem 7 and the definitions of $R_1$ and $R_2$.

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