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Biharmonic Green Function of an Infinite Network

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We discuss the existence and some properties of the biharmonic Green function of an infinite network. Our results are very analogous to that of a Riemannian manifold. Some potential-theoretic characterizations of the network are also given by means of the biharmonic Green function.

§ 1. Harmonic Green functions

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive function on Y. The quartet $N=\{X, Y, K, r\}$ is called an infinite network if the graph $\{X, Y, K\}$ is connected, locally finite and has no self-loop. For notation and terminology, we mainly follow [2] and [3].

Let $L(X)$ be the set of all real functions on X and $L^+(X)$ be the subset of $L(X)$ which consists of non-negative functions. For $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ of u is defined by

$$(1.1) \qquad \Delta u(x) = -\sum_{y\in Y} K(x, y)r(y)^{-1} \sum_{z\in X} K(z, y)u(z).$$

A function $u \in L(X)$ is called harmonic, superharmonic, subharmonic or biharmonic on a set A according as $\Delta u(x)=0$, $\Delta u(x)\leq 0$, $\Delta u(x)\geq 0$ or $\Delta^2 u(x)=\Delta(\Delta u)(x)=0$ on A, respectively. Let us put

$$\boldsymbol{SHP}(N)=\{u \in L^+(X);\ \Delta u\leq 0 \text{ on } X\},$$

$$\boldsymbol{H}(N)=\{u \in L(X);\ \Delta u=0 \text{ on } X\}.$$

For each finite subnetwork $N'=\langle X', Y'\rangle$ of N, denote by $nb(N')$ the finite subnetwork $\langle nb(X'), nb(Y')\rangle$ of N defined by $nb(X')=\cup\{X(x);\ x \in X'\}$ and $nb(Y')=\{y \in Y;\ e(y)\subset nb(X')\}$. Let us put $b(X')=nb(X')-X'$ and $b(Y')=nb(Y')-Y'$.

Similarly to [3; Lemma 2.1], we can prove the following minimum principle:

LEMMA 1.1. *Let $N'=\langle X', Y'\rangle$ be a finite subnetwork of N. If u is superharmonic on X' and if $m=\min\{u(x);\ x \in b(X')\}$, then $u(x)\geq m$ on X' and the equality holds only if $u=m$ on X'.*

COROLLARY. *If u is subharmonic on X' and if $M=\max\{u(x);\ x \in b(X')\}$, then*

$u(x)\leq M$ on X' and the equality holds only if $u=M$ on X'.

We have

THEOREM 1.1. *If $u\in \boldsymbol{H}(N)$ and $\sum_{x\in X} u(x)^2<\infty$, then $u=0$ on X.*

PROOF. For any $\varepsilon>0$, there exists a finite subnetwork $N'=\langle X', Y'\rangle$ of N such that $\sum_{x\in X-X'} u(x)^2<\varepsilon^2$. Then $|u(x)|<\varepsilon$ on $X-X'$. By Lemma 1.1 and its Corollary, we have $|u(x)|<\varepsilon$ on X. By the arbitrariness of ε, we conclude that $u=0$ on X.

For a finite subnetwork $N'=\langle X', Y'\rangle$ of N, the harmonic Green function g'_a of N' with pole at $a\in X'$ is defined by

$$(1.2)\qquad \Delta g'_a(x)=-\varepsilon_a(x) \quad \text{on} \quad X',$$

$$(1.3)\qquad g'_a(x)=0 \quad \text{on} \quad X-X',$$

where $\varepsilon_a(x)=0$ if $x\neq a$ and $\varepsilon_a(a)=1$.

The unique existence and some fundamental properties of g'_a were studied in [3]. Note that $g'_a(x)=g'_x(a)$ for each $a, x\in X'$.

Let $\{N_n\}$ be an exhaustion of N and let $g_a^{(n)}$ be the harmonic Green function of N_n with pole at a. Then $g_a^{(n)}\leq g_a^{(n+1)}$ and the limit g_a of $\{g_a^{(n)}\}$ exists and either $g_a\in \boldsymbol{SHP}(N)$ or $g_a=\infty$. Note that g_a does not depend on the choice of an exhaustion of N and that $g_a=\infty$ if and only if $N\in O_G$, i.e., $\boldsymbol{SHP}(N)$ consists only of constant functions. In case $g_a\in \boldsymbol{SHP}(N)$, we call g_a the harmonic Green function of N with pole at a. We have $\Delta g_a(x)=-\varepsilon_a(x)$ and $g_a(x)=g_x(a)$ for each $a, x\in X$.

We prepare

LEMMA 1.2. *If $v\in \boldsymbol{SHP}(N)$ and $\Delta(v-g_a)(x)\leq 0$ on X, then $g_a(x)\leq v(x)$ on X.*

PROOF. Let $g_a^{(n)}$ be the same as above. Since $\Delta g_a^{(n)}=\Delta g_a$ on X_n, we have $\Delta(v-g_a^{(n)})(x)\leq 0$ on X_n, where $N_n=\langle X_n, Y_n\rangle$. Since $v(x)-g_a^{(n)}(x)\geq 0$ on $b(X_n)$, we see by Lemma 1.1 that $g_a^{(n)}(x)\leq v(x)$ on X_n, so that $g_a(x)\leq v(x)$ on X.

We give a discrete analog of Harnack's principle:

LEMMA 1.3. *Let $N'=\langle X', Y'\rangle$ be a finite subnetwork of N and let $a, b\in X'$. Then there exists a positive constant $\beta=\beta(a, b)$ which satisfies $\beta^{-1}u(b)\leq u(a)\leq \beta u(b)$ for all $u\in L^+(X)$ such that $\Delta u(x)\leq 0$ on X'.*

PROOF. Let us put

$$t(x, z)=\sum_{y\in Y}|K(x, y)K(z, y)|\,r(y)^{-1} \qquad \text{for} \quad x\neq z,$$

$$t(z, z)=0,$$

$$t(z)=\sum_{y\in Y}|K(z, y)|r(y)^{-1}.$$

Then we have

$$\Delta u(z) = -t(z)u(z) + \sum_{x \in X} t(x, z)u(x).$$

We may assume that $a \neq b$. There exists a set $\{x_i;\ i=1,\dots, n\}$ in X' such that $x_{i+1} \in X(x_i)$ for $i=0, 1,\dots, n$ with $x_0 = a$ and $x_{n+1} = b$. Let $u \in L^+(X)$ and $\Delta u(x) \leq 0$ on X'. Then

$$t(x_i)u(x_i) \geq \sum_{x \in X} t(x, x_i)u(x) \geq t(x_{i+1}, x_i)u(x_{i+1}),$$

$$t(x_{i+1})u(x_{i+1}) \geq \sum_{x \in X} t(x, x_{i+1})u(x) \geq t(x_i, x_{i+1})u(x_i).$$

Thus we have $\beta_1 u(b) \leq u(a) \leq \beta_2 u(b)$, where

$$\beta_1 = \prod_{i=0}^{n} t(x_{i+1}, x_i)/t(x_i) \quad \text{and} \quad \beta_2 = \prod_{i=0}^{n} (x_{i+1})/t(x_i, x_{i+1}).$$

Taking $\beta = \max(\beta_2, \beta_1^{-1})$, we see easily that β has the desired property.

Corollary. *For any $a, b \in X$, there exists a positive constant β such that $\beta^{-1}u(b) \leq u(a) \leq \beta u(b)$ for all $u \in \boldsymbol{SHP}(N)$.*

§2. Biharmonic Green functions

Let $N' = \langle X', Y' \rangle$ be a finite subnetwork of N. We define the biharmonic Green function q'_a of N' with pole at $a \in X'$ by

(2.1) $\quad \Delta^2 q'_a(x) = \varepsilon_a(x) \quad$ on $\quad X'$,

(2.2) $\quad \Delta q'_a(x) = 0 \quad$ on $\quad b(X')$,

(2.3) $\quad q'_a(x) = 0 \quad$ on $\quad X - nb(X')$.

We shall prove

Theorem 2.1. *There exists a unique biharmonic Green function q'_a of a finite subnetwork N' of N with pole at a.*

Proof. First we prove the uniqueness of q'_a. Let q'_a and q''_a be biharmonic Green functions of N' with pole at a and put $u = q'_a - q''_a$ and $v = \Delta u$. Then $\Delta v = 0$ on X' and $v = 0$ on $b(X')$, so that $v = 0$ on X' by the minimum principle. Since $\Delta u = 0$ on $nb(X')$ and $u = 0$ on $X - nb(X')$, we conclude by the minimum principle that $u = 0$ on $nb(X')$. In order to prove the existence of q'_a, let g'_a and $\bar{g}'_a$ be the harmonic Green functions of N' and $nb(N')$ with pole at a respectively. Taking

$$u(x) = \sum_{z \in X'} g'_a(z)\bar{g}'_z(x),$$

we easily see that $u(x)=0$ on $X-nb(X')$,

$$\Delta u(x)=\sum_{z\in X'} g'_a(z)[\Delta \bar{g}'_z(x)]=-g'_a(x)$$

on $nb(X')$. It follows that $\Delta u(x)=0$ on $b(X')$ and that $\Delta^2 u(x)=-\Delta g'_a(x)=\varepsilon_a(x)$ on X'.

Let $\{N_n\}$ $(N_n=\langle X_n, Y_n\rangle)$ be an exhaustion of N with $a\in X_1$ and let $g_a^{(n)}$ and $\bar{g}_a^{(n)}$ be harmonic Green functions of N_n and $nb(N_n)$ with pole at a respectively. Then the biharmonic Green function $q_a^{(n)}$ of N_n with pole at a is given by

$$q_a^{(n)}(x)=\sum_{z\in X_n} g_a^{(n)}(z)\bar{g}_z^{(n)}(x). \tag{2.4}$$

Since $g_a^{(n)}\leq g_a^{(n+1)}$ and $\bar{g}_z^{(n)}\leq \bar{g}_z^{(n+1)}$ on X, we have $q_a^{(n)}\leq q_a^{(n+1)}$ on X, so that $q_a(x)=\lim_{n\to\infty} q_a^{(n)}(x)$ exists for each $x\in X$. It is easily seen that this function q_a does not depend on the choice of an exhaustion of N. Since $q_a^{(n)}$ is superharmonic on X_n, we have either $q_a\in \boldsymbol{SHP}(N)$ or $q_a=\infty$ (cf. [3; Lemma 2.4]).

We have

THEOREM 2.2. *Let a, $b\in X$. Then $q_a=\infty$ if and only if $q_b=\infty$.*

PROOF. Let $\{N_n\}$ $(N_n=\langle X_n, Y_n\rangle)$ be an exhaustion of N. There exists n_0 such that a, $b\in X_{n_0}$. By Lemma 1.3, we can find a positive constant β such that

$$\beta^{-1}g_z^{(n)}(b)\leq g_z^{(n)}(a)\leq \beta g_z^{(n)}(b)$$

for each $z\in X_n$ $(n\geq n_0)$. Thus we have for $n\geq n_0$

$$\beta^{-1}q_b^{(n)}(x)\leq q_a^{(n)}(x)\leq \beta q_a^{(n)}(x),$$

and hence $\beta^{-1}q_b(x)\leq q_a(x)\leq \beta q_b(x)$ on X. This shows our assertion.

DEFINITION 2.1. We write $N\in O_\Gamma$ if there exists $a\in X$ such that $q_a=\infty$. In case $N\notin O_\Gamma$, we call q_a the biharmonic Green function of N with pole at a.

By our definition, the biharmonic Green function q_a of N is given by

$$q_a(x)=\sum_{z\in X} g_a(z)g_z(x). \tag{2.5}$$

Clearly, $q_a(b)=q_b(a)$ for all a, $b\in X$. Furthermore q_a has the following properties:

$$\Delta q_a(x)=-g_a(x) \quad \text{on} \quad X, \tag{2.6}$$

$$\Delta^2 q_a(x)=\varepsilon_a(x) \quad \text{on} \quad X. \tag{2.7}$$

It is clear that $O_G\subset O_\Gamma$. We show by the following example that $O_G\neq O_\Gamma$.

EXAMPLE 2.1. Let J be the set of all non-negative integers and take $X=\{x_n; n\in J\}$ and $Y=\{y_{n+1}; n\in J\}$. Define K by $K(x_n, y_{n+1})=-1$ and $K(x_{n+1}, y_{n+1})=1$

for each $n \in J$ and $K(x, y)=0$ for any other pair. For a strictly positive function r on Y, $N=\{X, Y, K, r\}$ is an infinite network. Let us put $c_k=\sum_{n=k}^{\infty} r(y_n)$. Then we have $g_{x_m}(x_k)=c_{m+1}$ if $0 \leq k \leq m$ and $g_{x_m}(x_k)=c_{k+1}$ if $k \geq m+1$ (cf. [3; Example 3.1]). Note that $N \in O_G$ if and only if $c_1=\infty$. We have by (2.5)

$$q_{x_0}(x_m)=\sum_{k=0}^{m} c_{k+1}c_{m+1}+\sum_{k=m+1}^{\infty} c_{k+1}^2 .$$

Thus $N \notin O_\Gamma$ if and only if $\sum_{m=1}^{\infty} c_m^2<\infty$. Let us now take $r(y_n)=n^{-1/2}-(n+1)^{-1/2}$. Then $c_k=k^{-1/2}$. Hence $q_{x_0}(x_m)=\infty$ for all $m \in J$. Thus $N \in O_\Gamma - O_G$.

As a characterization of the condition $N \in O_\Gamma$, we have

THEOREM 2.3. *The condition $N \notin O_\Gamma$ holds if and only if any one of the following conditions is fulfilled:*

(a) $\sum_{x \in X} g_a(x)^2<\infty$ *for some* $a \in X$.

(b) There exists $v \in \boldsymbol{SHP}(N)$ *such that* $0<\sum_{x \in X} v(x)^2<\infty$.

PROOF. The condition $N \notin O_\Gamma$ implies (a) and (b) by (2.5) with $x=a$. Assume that there exists $v \in \boldsymbol{SHP}(N)$ such that $0<\sum_{x \in X} v(x)^2<\infty$. Then $v \notin \boldsymbol{H}(N)$ by Theorem 1.1, so that there exists $a \in X$ such that $\Delta v(a)=-t<0$. By Lemma 1.2, we have $g_a \leq v/t$ on X, and hence condition (a) holds. Next we assume that condition (a) holds. Then

$$q_a(a)=\sum_{z \in X} g_a(z) g_z(a)=\sum_{z \in X} g_a(z)^2<\infty$$

by (2.5), i.e., $q_a \neq \infty$. Thus $N \notin O_\Gamma$.

Let us put

$$\boldsymbol{QP}(N)=\{u \in L^+(X);\ \Delta u(x)=-1 \text{ on } X\}$$

and denote by O_{QP} the set of all infinite networks for which $\boldsymbol{QP}(N)$ is the empty set.

We have

THEOREM 2.4. $O_\Gamma \subset O_{QP}$.

PROOF. We proved in [4; Theorem 3.1] that $N \in O_{QP}$ if and only if $\sum_{x \in X} g_a(x) =\infty$ for all $a \in X$. Suppose that $N \notin O_{QP}$. Then there exists $a \in X$ such that $\sum_{x \in X} g_a(x) <\infty$. It follows that $\sum_{x \in X} g_a(x)^2<\infty$, so that $N \notin O_\Gamma$ by Theorem 2.3.

The relation $O_{QP} \neq O_\Gamma$ is shown by the following

EXAMPLE 2.2. Let X, Y and K be the same as in Example 2.1 and let us take $r(y_n)=n^{-1}-(n+1)^{-1}$. Then $c_k=k^{-1}$, and hence $N \notin O_\Gamma$. We have

$$\sum_{m=0}^{\infty} g_{x_0}(x_m) = \sum_{m=0}^{\infty} c_{m+1} = \infty,$$

so that $N \in O_{QP}$.

§ 3. Some properties of q_a

For $u \in L(X)$, the Dirichlet integral $D(u)$ of u is defined by

$$D(u) = \sum_{y \in Y} r(y)^{-1}[\sum_{z \in X} K(z, y)u(z)]^2.$$

Denote by $\boldsymbol{D}(N)$ the set of all $u \in L(X)$ such that $D(u) < \infty$ and by $L_0(X)$ the set of all $u \in L(X)$ with finite support $Su = \{x \in X; u(x) \neq 0\}$. Note that $\boldsymbol{D}(N)$ is a Hilbert space with respect to the norm $\|u\| = [D(u) + u(x_0)^2]^{1/2}$ $(x_0 \in X)$. Denote by $\boldsymbol{D}_0(N)$ the closure of $L_0(X)$ in $\boldsymbol{D}(N)$ with respect to the norm.

For $\mu \in L^+(X)$, the harmonic Green potential $G\mu$ and the harmonic Green potential energy $G(\mu, \mu)$ are defined by

$$G\mu(a) = \sum_{x \in X} g_a(x)\mu(x),$$

$$G(\mu, \mu) = \sum_{a \in X} [G\mu(a)]\mu(a).$$

Let us put

$$M(G) = \{\mu \in L^+(X); G\mu \in L(X)\},$$

$$E(G) = \{\mu \in L^+(X); G(\mu, \mu) < \infty\}.$$

Note that $\mu \in L^+(X)$ belongs to $M(G)$ if and only if there exists $x_1 \in X$ such that $G\mu(x_1) < \infty$.

We prepare

LEMMA 3.1. *Let* $\mu \in L^+(X)$. *If* $G\mu \in \boldsymbol{D}(N)$, *then* $\mu \in E(G)$, $G\mu \in D_0(N)$ *and* $D(G\mu) = G(\mu, \mu)$.

PROOF. Consider an exhaustion $\{N_n\}$ $(N_n = \langle X_n, Y_n \rangle)$ of N and define $\mu_n \in L^+(X)$ by $\mu_n(x) = \mu(x)$ on X_n and $\mu_n(x) = 0$ on $X - X_n$. Put $u = G\mu$ and $u_n = G\mu_n$. Since $(u, g_a^{(n)}) = \sum_{z \in X} \mu(z) g_a^{(n)}(z)$ and $\|g_a^{(n)} - g_a\| \to 0$ as $n \to \infty$, we have $(u, g_a) = u(a)$ and

$$D(u_n) = G(\mu_n, \mu_n) \leq G(\mu, \mu_n) = (u, u_n) \leq [D(u)]^{1/2}[D(u_n)]^{1/2}$$

by [3; Lemma 5.3], so that $G(\mu_n, \mu_n) \leq D(u) < \infty$. It follows that $G(\mu, \mu) \leq D(u) < \infty$, i.e., $\mu \in E(G)$. The rest of our assertion follows from [3; Lemma 5.4].

We have

THEOREM 3.1. *Assume that* $N \notin O_\Gamma$. *Then* $q_a \in \boldsymbol{D}(N)$ *if and only if* $q_a \in \boldsymbol{D}_0(N)$.

PROOF. Since $\boldsymbol{D}_0(N) \subset \boldsymbol{D}(N)$, we have only to prove the "only if" part. Assume that $D(q_a) < \infty$. Then we have by (2.5)

$$q_a(x) = \sum_{z \in X} g_a(z) g_x(z) = G\mu(x)$$

with $\mu = g_a$. Thus $q_a \in \boldsymbol{D}_0(N)$ by Lemma 3.1.

THEOREM 3.2. *Assume that $N \notin O_\Gamma$. Then the following three conditions are equivalent*:
(i) $q_a \in \boldsymbol{D}_0(N)$.
(ii) $q_a \in M(G)$.
(iii) $g_a \in E(G)$.

If any one of conditions (i), (ii) *and* (iii) *is fulfilled, then*

$$D(q_a) = \sum_{x \in X} q_a(x) g_a(x) = G(g_a, g_a). \tag{3.1}$$

PROOF. Assume that $q_a \in \boldsymbol{D}_0(N)$. Since q_a is superharmonic, we have by [3; Corollary of Theorem 5.2]

$$D(q_a) = -\sum_{x \in X} [\Delta q_a(x)] q_a(x) = \sum_{x \in X} q_a(x) g_a(x) = Gq_a(a) < \infty,$$

so that $q_a \in M(G)$. Thus (i) implies (ii). Assume that $q_a \in M(G)$. Then

$$G(g_a, g_a) = \sum_{z \in X} [\sum_{x \in X} g_z(x) g_a(x)] g_a(z)$$
$$= \sum_{z \in X} q_a(z) g_a(z) = Gq_a(a) < \infty,$$

so that $g_a \in E(G)$. Thus (ii) implies (iii). Now we assume that $g_a \in E(G)$. Then we have $q_a = Gg_a \in \boldsymbol{D}_0(N)$ and $D(q_a) = G(g_a, g_a)$ by [3; Lemma 5.4]. Thus (iii) implies (i).

By this theorem and Lemma 1.3, we have

COROLLARY. *If $q_a \in \boldsymbol{D}_0(N)$ for some $a \in X$, then $q_b \in \boldsymbol{D}_0(N)$ for all $b \in X$.*

We show by the following example that $q_a \in \boldsymbol{D}_0(N)$ does not hold in general.

EXAMPLE 3.1. Let X, Y and K be the same as in Example 2.1 and consider an infinite network $N = \{X, Y, K, r\}$. Put $c_k = \sum_{n=k}^{\infty} r(y_n)$. Then we have by Example 2.1 and (3.1)

$$G(g_{x_0}, g_{x_0}) = \sum_{m=0}^{\infty} [c_{m+1}^2 \sum_{k=0}^{m} c_{k+1} + c_{m+1} \sum_{k=m+1}^{\infty} c_{k+1}^2].$$

If $\sum_{k=1}^{\infty} c_k < \infty$, then $\sum_{k=1}^{\infty} c_k^2 < \infty$ and $G(g_{x_0}, g_{x_0})$ is finite, and hence $q_{x_0} \in \boldsymbol{D}_0(N)$ by Theorem 3.2. If $c_k = k^{-(1+t)/2}$ $(0 < t \leq 1/3)$, then we have $N \notin O_\Gamma$ by Example 2.1 and

$$G(g_{x_0}, g_{x_0}) \geq \sum_{m=1}^{\infty} m^{-(1+t)/2} [\sum_{k=m+1}^{\infty} k^{-(1+t)}]$$

$$\geq \sum_{m=1}^{\infty} (m+1)^{-(1+t)/2}(m+1)^{-t}/t$$

$$=(1/t) \sum_{m=1}^{\infty} (m+1)^{-(1+3t)/2} = \infty,$$

so that $q_{x_0} \notin \boldsymbol{D}_0(N)$.

As for the biharmonic Green potential and the biharmonic Green potential energy of μ, we can easily verify

THEOREM 3.3. *Let* $\mu \in L^+(X)$ *and* $v = G\mu$. *Then*

(3.2) $$\sum_{x \in X} q_a(x)\mu(x) = \sum_{z \in X} g_a(z)v(z) = Gv(a).$$

(3.3) $$\sum_{a \in X} \sum_{x \in X} q_a(x)\mu(x)\mu(a) = \sum_{z \in X} [G\mu(z)]^2.$$

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